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New Results on Generalized Metric Spaces

Abdellaoui, M.A. [1] and Dahmani, Z. * [2]

[1] *Laboratory LPAM, Department of Mathematics and Informatics, Faculty of SEI, University of Mostaganem*
[2] *Laboratory LPAM, Faculty of SEI, UMAB, University of Mostaganem*

E-mail: zzdahmani@yahoo.fr
* *Corresponding author*

ABSTRACT

In this paper, we introduce new generalized metric spaces. We prove some of their properties. Then, we establish some other properties of the associated operators. Also, we generalize some fixed point results of the paper [A generalization of fixed point theorem in S-metric spaces. Math. Vesnik, 64(3), (2012), 258-266.].

Keywords: Generalized metric space, fixed point, S^*-metric space, contractive operator.

1. Introduction

The metric spaces theory forms an important environment for studying fixed points of single and multi-valued operators and the metric fixed point theory is important in applied sciences. Many authors have studied this important theory. For instance, in Gahler (1963), S. Gahler introduced the concepts of D^*-metric spaces. Then, B.C. Dhage Dhage (1992) introduced the D-metric spaces. To develop a new fixed point theory, Z. Mustafa and B. Sims Mustafa (2006) presented new structures of generalized metric spaces. Based on these structures, the authors in Jumaili and Yang (2012) proved some new fixed point theorems. Recently, Sh. Sedghi et al. Sedghi and Shobe (2007), Sedghi and Zhou (2007) introduced D^*-metric spaces and obtained new versions for fixed point results in D^*-metric spaces. More recently in Sedghi and Alliouche (2012), by introducing new metric space, Sh. Sedghi et al. presented excellent generalizations of fixed point theorems. Other important research papers on generalized metric spaces can be found in Binayak and Metiya (2012), Chugh and Rhoades (2010), Kumar (2013), Kumar Michra (2013), Mustafa and Awawdeh (2008), Naidu and Srinivasa Rao (2004, 2005), Sedghi and Dung (2014), Shatanawi (2010, 2011), Zoran and Radenovic (2014).

In the present paper, we introduce the notion of S^*-metric spaces and some of their properties. We prove new fixed point generalizations on complete S^*-metric spaces. Then, we establish some other properties of the associated operators under some S^*- inequalities. Also, we generalize some fixed point results in the case of contractive operators. Some of the main results of Sedghi and Alliouche (2012) can be deduced as some special cases.

2. S^*-Metric Notions

We shall present some preliminaries fact that will be used throughout this paper. We being by the following definition Shirali (2006):

Definition 2.1. *Let X be a nonempty set and let $d : X \times X \to [0, +\infty[$ be a function satisfying the following axioms: For all $x, y, z \in X$, we have*

1) $d(x, y) = 0$ *if and only if* $x = y$,

2) $d(x, y) = d(y, x)$,

3) $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X, d) is called a d-metric space.

Let us now introduce the following concepts with some illustrative examples:

Definition 2.2. *Let X be a nonempty set. We define the S'-metric on X by the function $S' : X^3 \to [0, +\infty[$, such that:*

$(1) : S'(x, y, z) \geq 0, x, y, z \in X,$

$(2) : S'(x, y, z) = 0$ *if and only if* $x = y = z,$

$(3) : S'(x, y, z) = S'(p\{x, y, z\}), x, y, z \in X,$ *where p is a permutation function,*

$(4) : S'(x, y, z) \leq S'(x, a, a) + S'(y, a, a) + S'(z, a, a), x, y, z, a \in X.$

In this case, (X, S') represents the S'-metric space.

Example 2.1. 1) *Let us consider $X = \mathbb{R}^n$. We define $S'(x, y, z) = d(x, y) + d(x, z) + d(y, z)$, where d is the ordinary metric on $\mathbb{R}^n$.*

2) *Taking $X = \mathbb{R}$, and defining $S'(x, y, z) = \frac{d(x,y)}{1+d(x,y)} + \frac{d(x,z)}{1+d(x,z)} + \frac{d(y,z)}{1+d(y,z)}$, where d is the ordinary metric on $\mathbb{R}$.*

3) *For $X = C([\alpha, \beta], \mathbb{R})$, we define $S'(f, g, h) = \sup\limits_{t\in[\alpha,\beta]} |f(t) - g(t)| + \sup\limits_{t\in[\alpha,\beta]} |f(t) - h(t)| + \sup\limits_{t\in[\alpha,\beta]} |g(t) - h(t)|$, with f, g and h are continuous functions on $[\alpha, \beta]$.*

Definition 2.3. *Let X be a nonempty set. Suppose that the function $S'' : X^n \to [0, +\infty[$ satisfies the conditions:*
For any $x_1, x_2, ..., x_n, a \in X$,

$(1) : S''(x_1, x_2, ..., x_n) \geq 0,$

$(2) : S''(x_1, x_2, ..., x_n) = 0$ *if and only if* $x_1 = x_2 = ... = x_n,$

$(3) : S''(x_1, x_2, ..., x_n) = S''(p\{x_1, x_2, ..., x_n\}),$ *where p is a permutation function,*

$(4) : S''(x_1, x_2, ..., x_n) \leq S''(x_1, a, ..., a) + S''(x_2, a, ..., a) + ... + S''(x_n, a, ..., a).$
Then this function is called an S''-metric on X, and (X, S'') is the corresponding S''-metric space.

Example 2.2. *If* $X = \mathbb{R}^n$, *then we take* $S''(x_1, x_2, ..., x_n) = \sum_{i,j=1, i\neq j}^{n-1} \|x_i - x_j\|_{\mathbb{R}^n}$, *for each* $x_1, x_2, ..., x_n \in \mathbb{R}^n$.

Remark 2.1. *In Definition 2.3, if we take* $n = 3$, *we obtain an* S'*-metric, and for* $n = 2$, *we obtain the* $d-$*ordinary metric on* X.

We also introduce:

Definition 2.4. *An* S^**-metric on nonempty set* X *is a function* $S^* : X^n \to [0, +\infty[$ *satisfying:*

$(1) : S^*(x_1, x_2, ..., x_n) \geq 0,$

$(2) : S^*(x_1, x_2, ..., x_n) = 0$ *if and only if* $x_1 = x_2 = ... = x_n$,

$(3) : S^*(x_1, x_2, ..., x_n) \leq S^*(x_1, x_1, ..., x_1, a) + S^*(x_2, x_2, ..., x_2, a) + ... + S^*(x_n, x_n, ..., x_n, a),$
for any $x_1, x_2, ..., x_n, a \in X$.
In this case, (X, S^*) *is the corresponding* S^**-metric space.*

Example 2.3. *Let us consider* $X = C([\alpha, \beta], \mathbb{R})$. *Then, we take* $S^*(f_1, ..., f_n) = \sum_{i=1}^{n-1} \sup_{t\in[\alpha,\beta]} |f_i(t) - f_{i+1}(t)|$, *where* $n = 2, 3, ...$ *and* f_i *are continuous functions on* $[\alpha, \beta]$.

Remark 2.2. *In Definition 2.4, if we take* $n = 3$, *we obtain Definition 2.1 introduced in Sedghi and Alliouche (2012).*

We prove the following identity:

Lemma 2.1. *Suppose that* (X, S^*) *is an* S^**-metric space. Then, for all* $x_1, x_2 \in X$, *we have*

$$S^*(x_1, x_1, ..., x_1, x_2) = S^*(x_2, x_2, ..., x_2, x_1). \quad (1)$$

Proof. Let $x_1, x_2 \in X$. Then by the third condition of S^*-metric, for each $a \in X$, we can write

$$S^*(x_1, x_1, ..., x_1, x_2) \leq (n-1) S^*(x_1, x_1, ..., x_1, a) + S^*(x_2, x_2, ..., x_2, a). \quad (2)$$

Taking $a = x_1$, we obtain

$$S^*(x_1, x_1, ..., x_1, x_2) \leq (n-1) S^*(x_1, x_1, ..., x_1, x_1) + S^*(x_2, x_2, ..., x_2, x_1). \tag{3}$$

Thanks to the second condition of S^*-metric in Definition 7, we get

$$S^*(x_1, x_1, ..., x_1, x_2) \leq S^*(x_2, x_2, ..., x_2, x_1). \tag{4}$$

Similarly, we have

$$S^*(x_2, x_2, ..., x_2, x_1) \leq S^*(x_1, x_1, ..., x_1, x_2). \tag{5}$$

By (4) and (5), we obtain

$$S^*(x_1, x_1, ..., x_1, x_2) = S^*(x_2, x_2, ..., x_2, x_1). \tag{6}$$

This ends the proof of the Lemma 2.1. □

Definition 2.5. *Suppose that (X, S^*) is an S^*-metric space. Taking $r > 0, x_0 \in X$, then we can define:*

$$\begin{aligned} B_{S^*}(x_0, r) &:= \{x \in X : S^*(x, x, ..., x, x_0) < r\}, \\ \overline{B_{S^*}}(x_0, r) &:= \{x \in X : S^*(x, x, ..., x, x_0) \leq r\}. \end{aligned} \tag{7}$$

Example 2.4. *Let $X = \mathbb{R}$, and $n = 3$. We define the S^*-metric on X by: $S^*(x, y, z) = |x - y| + |x - z|$. Hence, it yields that:*

$$\begin{aligned} B_{S^*}\left(1, \sqrt{2}\right) &= \left\{x \in \mathbb{R} : S^*(x, x, 1) < \sqrt{2}\right\} \\ &= \left\{x \in \mathbb{R} : |x - 1| < \sqrt{2}\right\} \\ &= \left]1 - \sqrt{2}, 1 + \sqrt{2}\right[. \end{aligned}$$

Definition 2.6. *Suppose that $A \subset X$ and (X, S^*) is an S^*-metric space. A is an open subset of X if for any x_0 of A, there exists $r > 0$, which satisfies $B_{S^*}(x_0, r) \subset A$.*

Definition 2.7. *Suppose that $A \subset X$ and (X, S^*) is an S^*-metric space. The set A is S^*-bounded if there exists $r > 0, S^*(x, x, ..., x, y) \leq r$, for each $x, y \in A$.*

Definition 2.8. *We say that the sequence $\{x_p\}_{p \in \mathbb{N}}$ of the space X is convergent to x if $S^*(x_p, x_p, ..., x_p, x) \to 0; p \to +\infty$. We write $\lim\limits_{p \to +\infty} x_p = x$.*

Definition 2.9. *We say that the sequence* $\{x_p\}_{p\in\mathbb{N}}$ *of the space* X *is a of Cauchy if for each* $\varepsilon > 0, \exists p_0 \in \mathbb{N}$, *such that for any* $p, q \geq p_0, S^*(x_p, x_p, ..., x_p, x_q) < \varepsilon$.

Definition 2.10. *The space* (X, S^*) *is a complete if its Cauchy sequences are convergent.*

We prove the following lemmas:

Lemma 2.2. *Let* (X, S^*) *be an* S^**-metric space. If* $\{x_p\}_{p\in\mathbb{N}}$ *in* X *converges to* x*, then* x *is unique.*

Proof. Let $\{x_p\}_{p\in\mathbb{N}}$ converges to x and y. Then for any $\epsilon > 0$, we can state that

$$\exists p_1 \in \mathbb{N}, \text{ for all } p \geq p_1 \Rightarrow S^*(x_p, ..., x_p, x) < \frac{\varepsilon}{2(n-1)} \tag{8}$$

and

$$\exists p_2 \in \mathbb{N}, \text{ for all } p \geq p_2 \Rightarrow S^*(x_p, ..., x_p, y) < \frac{\varepsilon}{2}. \tag{9}$$

Taking $p_0 = max\{p_1, p_2\}$, then for any $p \geq p_0$, it yields the following inequality

$$S^*(x, ..., x, y) \leq (n-1) S^*(x, ..., x, x_p) + S^*(y, ..., y, x_p) < \varepsilon. \tag{10}$$

Therefore,

$$S^*(y, ..., y, x) \leq \frac{(n-1)\varepsilon}{2(n-1)} + \frac{\varepsilon}{2} < \varepsilon. \tag{11}$$

Hence,

$$S^*(x, ..., x, y) = S^*(y, ..., y, x) = 0. \tag{12}$$

This implies that $x = y$. □

Lemma 2.3. *Let us consider the space* (X, S^*) *and suppose that* $\{x_p\}_{p\in\mathbb{N}}$ *of* X *is a convergent sequence with limit* x*. Then* $\{x_p\}_{p\in\mathbb{N}}$ *is of Cauchy.*

Proof. The relation $\lim_{p\to+\infty} x_p = x$ implies that for any $\epsilon > 0$,

$$\exists p_1 \in \mathbb{N}, \text{ for all } p \geq p_1 \Rightarrow S^*(x_p, ..., x_p, x) < \frac{\varepsilon}{2(n-1)} \tag{13}$$

and for the same ϵ,

$$\exists p_2 \in \mathbb{N}, \text{ for all } q \geq p_2 \Rightarrow S^*(x_q, ..., x_q, y) < \frac{\varepsilon}{2}. \tag{14}$$

Taking $p_0 = max\{p_1, p_2\}$, then for every $p, q \geq p_0$, we have

$$\begin{aligned} S^*(x_p, ..., x_p, x_q) &\leq (n-1) S^*(x_p, ..., x_p, x) + S^*(x_q, ..., x_q, x) \\ &< \frac{(n-1)\varepsilon}{2(n-1)} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{15}$$

Hence, $\{x_p\}_{p\in\mathbb{N}}$ is a Cauchy sequence. □

Lemma 2.4. *If there exist two sequences* $\{x_p\}_{p\in\mathbb{N}}$ *and* $\{y_p\}_{p\in\mathbb{N}}$ *of the space* (X, S^*) *that satisfy* $\lim_{p\to+\infty} x_p = x$ *and* $\lim_{p\to+\infty} y_p = y$*, then we have*

$$\lim_{p\to+\infty} S^*(x_p, ..., x_p, y_p) = S^*(x, ..., x, y). \tag{16}$$

Proof. Using (16), for any $\epsilon > 0$, we can state that

$$\exists p_1 \in \mathbb{N}, \text{ for each } p \geq p_1 \Rightarrow S^*(x_p, ..., x_p, x) < \frac{\varepsilon}{2(n-1)} \tag{17}$$

and

$$\exists p_2 \in \mathbb{N}, \text{ for each } p \geq p_2 \Rightarrow S^*(y_p, ..., y_p, y) < \frac{\varepsilon}{2(n-1)}. \tag{18}$$

If we take $p_0 = \max\{p_1, p_2\}$, then for every $p \geq p_0$, we get:

$$\begin{aligned} S^*(x_p, ..., x_p, y_p) &\leq (n-1) S^*(x_p, ..., x_p, x) + S^*(y_p, ..., y_p, x) \\ &\leq (n-1)[S^*(x_p, ..., x_p, x) + S^*(y_p, ..., y_p, y)] + S^*(x, ..., x, y) \\ &= \frac{(n-1)\varepsilon}{2(n-1)} + \frac{(n-1)\varepsilon}{2(n-1)} + S^*(x, ..., x, y). \end{aligned} \tag{19}$$

This is to say that

$$S^*(x_p, ..., x_p, y_p) - S^*(x, ..., x, y) < \varepsilon. \tag{20}$$

On the other hand, it follows that

$$\begin{aligned} S^*(x, ..., x, y) &\leq (n-1) S^*(x, ..., x, x_p) + S^*(y, ..., y, x_p) \\ &\leq (n-1)[S^*(x, ..., x, x_p) + S^*(y, ..., y, y_p)] + S^*(x_p, ..., x_p, y_p) \\ &< \frac{(n-1)\varepsilon}{2(n-1)} + \frac{(n-1)\varepsilon}{2(n-1)} + S^*(x_p, ..., x_p, y_p) \\ &< \varepsilon + S^*(x_p, ..., x_p, y_p). \end{aligned} \tag{21}$$

Consequently,

$$S^*(x, ..., x, y) - S^*(x_p, ..., x_p, y_p) < \varepsilon. \tag{22}$$

Then, thanks to (20) and (22), we can state that

$$|S^*(x,...,x,y) - S^*(x_n,...,x_n,y_n)| < \varepsilon. \tag{23}$$

Hence,

$$\lim_{p\to+\infty} S^*(x_p,...,x_p,y_p) = S^*(x,...,x,y). \tag{24}$$

□

We end this section by the following S^*-concept:

Definition 2.11. *Let us consider the space* (X, S^*). *A map* $F : X \to X$ *is contractive if there exists* $0 \leq L < 1$, *that satisfies*

$$S^*(F(x),...,F(x),F(y)) \leq LS^*(x,...,x,y), \quad \textit{for all } x,y \in X. \tag{25}$$

3. New Operators-Results on the Space (X, S^*)

We begin by the following auxiliary result:

Theorem 3.1. *Suppose that* (X, S^*) *is an* S^**-metric space and* $F : X \to X$, *is a contractive operator. Then* F *is continuous.*

Proof. Suppose that there exists a constant $0 \leq L < 1$, such that

$$S^*(F(x),...,F(x),F(y)) \leq LS^*(x,...,x,y), \quad \text{for all } x,y \in X. \tag{26}$$

Let us take $\{x_p\}_{p\in\mathbb{N}}$ in X which converges to x. Then, we can write $S^*(x_p,...,x_p,x) \to 0, p \to +\infty$. This implies that $S^*(F(x_p),...,F(x_p),F(x)) \to 0$. And then, $F(x_p) \to F(x)$, as $p \to +\infty$. Therefore, F is continuous. □

Before presenting our first main result, we need to define $\{x_p\}_{p\in\mathbb{N}} \subset X$ by the following recursive formula:

$$\begin{cases} x_{p+1} = F(x_p), \text{ for each } p \in \mathbb{N}, \\ x_0 \in X. \end{cases}$$

We have:

Theorem 3.2. *Suppose that* (X, S^*) *is a complete* S^**-metric space and* $F : X \to X$*, is a contraction operator. Then* F *has a unique fixed point* $x^* \in X$*:*

$$F(x^*) = x^*. \tag{27}$$

Proof. The proof of this theorem will be deviled into three steps:

Step1-Existence: By induction on $p \in \mathbb{N}$, we shall prove the property:

$$P(p) : S^*(x_{p+1}, ..., x_{p+1}, u_p) \leq L^p S^*(x_1, ..., x_1, x_0). \tag{28}$$

It is clear that $P(0)$ holds.

Let us show that for any $p \in \mathbb{N}$, $P(p) \Rightarrow P(p+1)$.

For each $p \in \mathbb{N}$, suppose that $P(p)$. Then, we have

$$\begin{aligned} S^*(x_{p+2}, ..., x_{p+2}, x_{p+1}) &= S^*(F(x_{p+1}), ..., F(x_{p+1}), F(x_p)) \\ &\leq LS^*(x_{p+1}, ..., x_{p+1}, x_p) \\ &\leq L^{p+1} S^*(x_1, ..., x_1, x_0). \end{aligned} \tag{29}$$

Therefore, $P(p+1)$ holds. Consequently, the property (28) is valid.

Step2-Cauchy Sequence: Let us take any $\varepsilon > 0, (p, q) \in \mathbb{N}^* \times \mathbb{N}^*$, with $q > p$, and consider $r = q - p$. So, we have

$$\begin{aligned} S^*(x_q, ..., x_q, x_p) &= S^*(x_{p+r}, ..., x_{p+r}, x_p) \\ &\leq (n-1) \sum_{i=p}^{p+r-1} S^*(x_{i+1}, ..., x_{i+1}, x_i) \\ &\leq (n-1) S^*(x_1, ..., x_1, x_0) \sum_{i=p}^{p+r-1} L^i \\ &\leq \frac{(n-1) L^p}{1-L} S^*(x_1, ..., x_1, x_0). \end{aligned} \tag{30}$$

Since $L \in [0, 1[$, then we have:

$$S^*(x_q, ..., x_q, x_p) \leq \frac{(n-1) L^p}{1-L} S^*(x_1, ..., x_1, x_0) \to 0, \text{ as } p \to +\infty. \tag{31}$$

Hence, $\{x_p\}_{p \in \mathbb{N}}$ is a Cauchy sequence.
Since X is a complete metric space, then there exists $x^* \in X$ with $\lim\limits_{p \to +\infty} x_p =$

x^*. Thanks to Theorem 3.1, we deduce that

$$x^* = \lim_{p \to +\infty} x_{p+1} = \lim_{p \to +\infty} F(x_p) = F(x^*). \tag{32}$$

Step3-Uniqueness: Suppose that we have two fixed points $x^*, y^* \in X$, with $x^* = F(x^*)$ and $y^* = F(y^*)$. Then, we can write:

$$S^*(F(x^*), ..., F(x^*), F(y^*)) \leq LS^*(x^*, ..., x^*, y^*). \tag{33}$$

This inequality is equivalent to

$$S^*(x^*, ..., x^*, y^*) \leq LS^*(x^*, ..., x^*, y^*) \Leftrightarrow (1-L) S^*(x^*, ..., x^*, y^*) \leq 0. \tag{34}$$

Therefore, $x^* = y^*$.
By the Steps $1, 2, 3$, the proof of Theorem 3.2 is achieved. □

To illustrate the above theorem, we give the following example:

Example 3.1. *Let us take $X = \mathbb{R}, n = 4$ and consider $S^*(x, y, z, t) = |x-y| + |x-z| + |x-t|$ is an S^*-metric on X. On X, we consider $F(x) := \frac{1}{4}\arctan x$. So, we have*

$$\begin{aligned} S^*(F(x), F(x), F(x), F(y)) &= \left|\frac{1}{4}\arctan x - \frac{1}{4}\arctan y\right| \\ &\leq \frac{1}{4} S^*(x, x, x, y). \end{aligned}$$

The conditions of Theorem 3.2 hold. Then, there exists a unique fixed point $x^* \in X$. (We remark that $0 = F(0)$.)

Using an "S^*-inequality", we prove the following theorem:

Theorem 3.3. *Let (X, S^*) be a compact S^*-metric space with $G : X \to X$ is an operator that satisfies the following inequality*

$$S^*(G(x), ..., G(x), G(y)) < S^*(x, ..., x, y), \text{ for all } x, y \in X, \ x \neq y. \tag{35}$$

Then F has exactly one fixed point $x_0 \in X$.

Proof. We proceed in two steps:

Step1-Existence: It is clear that x_0 is the minimum of the map $x \mapsto S^*(x, ..., x, G(x))$, and then we have $x_0 = G(x_0)$. In fact,

$$\begin{aligned} S^*(G(G(x_0)), ..., G(G(x_0)), G(x_0)) &< S^*(G(x_0), ..., G(x_0), x_0) \\ &= S^*(x_0, ..., x_0, G(x_0)). \end{aligned} \tag{36}$$

which is a contradiction.

Step2-Uniqueness: Suppose that we have $x_1, y_1 \in X$; with $x_1 = G(x_1)$ and $y_1 = G(y_1)$. Then the inequality

$$S^*(G(x_1), ..., G(x_1), G(y_1)) < S^*(x_1, ..., x_1, y_1) \tag{37}$$

is equivalent to

$$S^*(x_1, ..., x_1, y_1) < S^*(x_1, ..., x_1, y_1). \tag{38}$$

This is a contradiction. Hence, $x_1 = y_1$.
The proof of Theorem 3.3 is thus achieved. □

Based on Theorem 3.2 and using Definition 2.5, we finish this paper by presenting to the reader the following corollary:

Corollary 3.1. *For $x_0 \in X$, we suppose that (X, S^*) is a complete S^*-metric space and $F : B_{S^*}(x_0, r) \to X$ is contractive, for any $r > 0$.*
If $S^*(F(x_0), ..., F(x_0), x_0) < (1 - L)\frac{r}{n-1}$, then F has exactly one fixed point in $B_{S^*}(x_0, r)$.

4. Concluding Remarks

We have introduced new generalized metric spaces and proved some important properties. We have established some other properties of the associated operators in the case of certain S^*-metric inequalities. Also, we have generalized some fixed point results for contractive operators on the proposed S^*-metric spaces. For our results, some classical results have been obtained as some special cases.

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