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Entropy on Semi MV-Algebra

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ABSTRACT

In this paper, at first the notion of unity partition on semi MV-algebra with RDP is introduced. By using of state function, the concept of entropy and conditional entropy on partitions of semi MV-algebra with RDP are defined and, in addition some of basic properties related to this notions are investigated. Afterwards semi dynamical system on semi MV-algebra with RDP and lower and upper entropies on semi dynamical system are introduced, respectively. Then some theorem related to this concepts are proved. In final part of this article, entropy of some semi dynamical systems on semi MV-algebra with RDP are calculated.

Keywords: Dynamical system, Entropy, MV-Algebra, Semi MV-algebra.

1. Introduction

The notion of entropy was introduced by Clausius at his work in the context of thermodynamics in 1985. After that this concept extended to other fields. Shannon has defined a notion of entropy in information theory Shannon (1948). The Shannon's entropy was a measure of the average information content one loses when not knowing the value of the random variable.The concept of entropy in ergodic theory was introduced by Kolmogorov Kolmogorov (1958) and was improved by Sina Sinai (1959).The K-S entropy was a tool to measures the rate of the complexity of a dynamical system when the time passes.Also the K-S entropy has played an important role in ergodic theory. Actually the K-S entropy was an invariant and with the help of this notion the dynamical systems were classified. Adler, Konheim, and McAndrew Adler et al. (1965) have introduced the topological entropy as an invariant of topological conjugacy. The roles of topological entropy was like the entropy that was defined in the measure theory. The concept of fuzzy dynamical system and its entropy have been introduced by Markechova Markechova (1989).The main idea of fuzzy entropy is that the partitions are replaced by fuzzy partitions.Some researchers have defined fuzzy entropy considering algebraic structures like MV - algebra and effect algebra as a probability space Ebrahimi and Mosapour (2013) Petroviciova (2000) Petroviciova (2001).

Semi MV- algebra was introduced by Hasankhani and Borumand Saeid Hasankhani and Borumand Saeid (2013). Some properties of this algebraic structure were similar to properties of MV-algebra. In order to define entropy on algebraic structures, these structure must have some special conditions. One of the subclasses of Semi MV- algebra is Semi MV- algebra with the Rieze decomposition property. This subclass has the necessary condition to define entropy on it. In section 2, the notions partition, join partitions, entropy , conditional entropy and relative entropy on Semi MV-Algebra With RDP were defined and some properties of these entropies were investigated. The lower and upper entropies of a semi dynamical system on Semi MV-Algebra with RDP were introduced in section 3.In this section some theorems were proved and in one of these important theorems it was shown that two isomorphic semi dynamical system have equal entropy.In final section state and transformation functions of some semi dynamical systems on semi MV-algebra with RDP were computed and by using these functions, entropies of semi dynamical systems were calculated.

2. Entropy, conditional entropy and relative entropy on Semi MV-Algebra With RDP

Definition 2.1. *Cignoli et al. (2000) An MV- algebra* $M = (M, +, \cdot, \acute{}, 0)$ *is an algebra of type* $(2, 2, 1, 0)$ *where* $" + "$ *is associative and commutative with neutral element* 0*, and, in addition,* $\acute{0} = 1$*,* $\acute{1} = 0$*,* $x + 1 = x$*,* $x \cdot y = (\acute{x} + \acute{y})\acute{}$ *and* $y + (y + \acute{x})\acute{} = x + (x + \acute{y})\acute{}$ *for all* $x, y \in M$*.*

Definition 2.2. *A commutative l-group is an algebra* $(G, +, \leq)$*, where* $(G, +)$ *is a commutative group,* $(G, \leq)$ *is lattice ordered and if* $a \leq b$ *then* $a + c \leq b + c$*.*

Definition 2.3. *We say $u \in G, u > 0$ is a strong unit, if for any $g \in G$ there exists $n \in \mathbb{N}$ such that $nu \geq g$.*

Lemma 2.1. *For any $g \in MG$, $g \oplus 0 = g$ and $g \odot 0 = 0$.*

Theorem 2.1. *Mundici (1986) Let M be an MV- algebra. There exists a commutative l-group G with a strong unit such that $M = MG$.*

Remark 2.1. *By the above theorem, considering isomorphism, every MV-algebra could be identified with the unite interval of a unique l-group G with strong unite u.*

Definition 2.4. *We say an algebra M has Rieze decomposition property (RDP in short)if $x \leq y_1 + y_2$ implies that there exist two elements x_1, x_2 such that $x_1 \leq y_1, x_2 \leq y_2$ and $x = x_1 + x_2$.*

Definition 2.5. *Pu and Liu (1980) A fuzzy set of a set M is called fuzzy point if it takes value 0 for all $y \in M$ except one, say $x \in M$. If it's value at x is $t, 0 < t \leq 1$, then we denote the fuzzy point by x_t and the set of all fuzzy points of M by $FP(X)$.*

Definition 2.6. *Hasankhani and Borumand Saeid (2013) Let $(M, +, \cdot, ', 0, u)$ be an MV - algebra. We define the operations "$\oplus$", "$\odot$" and "$*$" by:*

$$\oplus : FP(M) \times FP(M) \to FP(M)$$
$$(x_\alpha, y_\beta) \mapsto (x + y)_{min\{\alpha,\beta\}},$$

$$\odot : FP(M) \times FP(M) \to FP(M)$$
$$(x_\alpha, y_\beta) \mapsto (x \cdot y)_{min\{\alpha,\beta\}},$$

and

$$* : FP(M) \to FP(M)$$
$$(x_\alpha)^* \mapsto (\grave{x})_\alpha,$$

for all $x_\alpha, y_\beta \in FP(M)$.

Proposition 2.1. *Let $x_\alpha, y_\beta, z_\gamma \in FP(M)$ then:*

i) $(x_\alpha \oplus y_\beta) \oplus z_\gamma = x_\alpha \oplus (y_\beta \oplus z_\gamma)$ *and* $x_\alpha \oplus y_\beta = y_\beta \oplus x_\alpha$ *for any* $x_\alpha, y_\beta, z_\gamma \in FP(M)$ *Hasankhani and Borumand Saeid (2013),*

ii) $(x_\alpha^*)^* = x_\alpha$ *for any* $x_\alpha \in FP(M)$*,Hasankhani and Borumand Saeid (2013),* *iii)* $x_\alpha \odot y_\beta = (x^*_{min\{\alpha,\beta\}} \oplus y^*_{min\{\alpha,\beta\}})^*$.

Proof. iii) $x_\alpha \odot y_\beta = (x \cdot y)_{min\{\alpha,\beta\}} = ((\acute{x} + \acute{y})\acute{})_{min\{\alpha,\beta\}} = ((\acute{x} + \acute{y})_{min\{\alpha,\beta\}})^* = (x^*_{min\{\alpha,\beta\}} \oplus y^*_{min\{\alpha,\beta\}})^*$. □

Definition 2.7. *We say $x_\alpha \leq y_\beta$ iff $x \leq y$ and $\alpha \leq \beta$.*

Proposition 2.2. *If M is a MV- algebra with RDP, then FP(M) has the Rieze decomposition property.*

Proof. $x_\alpha \leq y_\beta \oplus z_\gamma$ iff $x \leq y + z$ and $\alpha \leq \min\{\beta, \gamma\}$. Since M has RDP, $x \leq y + z$ implies there exist $e, f \in M$ such that $e \leq y, f \leq z, x = e + f$. $e_\alpha \oplus f_\alpha = x_\alpha$, $e_\alpha \leq y_\beta$ and $f_\alpha < z_\gamma$. □

In this paper, we say FP(M) with RDP if M is a MV- algebra with RDP.

Definition 2.8. *A subset $A = \{a^i_{\alpha_i} : i = 1, ..., n, a^i \in M\}$ of FP(M) is a partition of FP(M) if $\sum_{i=1}^{n} a^i_{\alpha_i} = u_\alpha$, and partition $B = \{b^j_{\beta_j}\}_{j=1}^m$ is refinement of partition $A = \{a^i_{\alpha_i}\}_{i=1}^n$ and it is written $A \prec B$ if for every $a^i_{\alpha_i}$ there exists a subset $\eta_i \subseteq \{1, ..., m\}$ such that $a^i_{\alpha_i} = \sum_{j\in\eta_i} b^j_{\beta_j}$, $\bigcup_{i=1}^{n} \eta_i = \{1, ..., m\}$, and $\eta_i \cap \eta_k = \emptyset$ for $k \neq j$.*

Definition 2.9. *Let $A = \{a^i_{\alpha_i}\}_{i=1}^n$ and $B = \{b^j_{\beta_j}\}_{j=1}^m$ be partitions of FP(M) with RDP. We say partition $C = \{c^{ij}_{\gamma_{ij}} : 1 \leq i \leq n, 1 \leq j \leq m\}$ is Rieze join refinement of A and B, showing with $C = A \vee B$, If $a^i_{\alpha_i} = \sum_{j=1}^{m} c^{ij}_{\gamma_{ij}}$ and $b^j_{\beta_j} = \sum_{i=1}^{n} c^{ij}_{\gamma_{ij}}$.*

Definition 2.10. *$m : FP(M) \to [0, 1]$ is said to be a state if satisfying the following conditions:*

i) $m(u_\alpha) = 1, \quad \forall\alpha \in [0, 1]$,

ii) If $x_\alpha = \sum_{i=1}^{m} x^i_{\beta_i}$ *then* $m(x_\alpha) = \sum_{i=1}^{m} m(x^i_{\beta_i})$.

Definition 2.11. *If* $A = \{a^i_{\alpha_i}\}_{i=1}^m$ *is a partition of FP(M), then its entropy* $H(A)$ *is defined by*

$$H(A) = \sum_{i=1}^{m} \varphi(m(a^i_{\alpha_i}))$$

where $\varphi(x) = -x \log x$, *if* $x > 0$, *and* $\varphi(0) = 0$.

Proposition 2.3.

i) $H(A)$ *is finite and,* $0 \leq H(A)$,

ii) $A = \{0_\beta, u_\alpha\}$ *is a partition of FP(M)and* $H(A) = 0$.

Definition 2.12. *If* $A = \{a^i_{\alpha_i}\}_{i=1}^n$ *and* $B = \{b^j_{\beta_j}\}_{j=1}^m$ *are partitions of FP(M) with RDP, and* $C = \{c^{ij}_{\gamma_{ij}} : 1 \leq i \leq n, 1 \leq j \leq m\}$ *is Rieze join refinement of A and B, then we define*

$$H_C(A \mid B) = \sum_{i=1}^{n} \sum_{j=1}^{m} m(b^j_{\beta_j}) \varphi(\frac{m(c^{ij}_{\gamma_{ij}})}{m(b^j_{\beta_j})}),$$

when ever $m(b^j_{\beta_j}) \neq 0$.

Proposition 2.4. *If* $A = \{a^i_{\alpha_i}\}_{i=1}^n$ *and* $B = \{b^j_{\beta_j}\}_{j=1}^m$ *are partitions of FP(M) with RDP, and* $C = \{c^{ij}_{\gamma_{ij}} : 1 \leq i \leq n, 1 \leq j \leq m\}$ *is a Rieze join refinement of A and B, then:*

i) $H_C(A \mid B) \leq H(A)$,

ii) $H(C) = H(A) + H_C(B \mid A)$,

Proof.

i) Let $A = \{a^i_{\alpha_i}\}_{i=1}^n$ and $B = \{b^j_{\beta_j}\}_{j=1}^m$ be partitions of FP(M) with RDP, and $C = \{c^{ij}_{\gamma_{ij}} : 1 \leq i \leq n, 1 \leq j \leq m\}$ be join refinement of A and B.For fix i, $\sum\limits_{j=1}^{m} m(b^j_{\beta_j})\frac{m(c^{ij}_{\gamma_{ij}})}{m(b^j_{\beta_j})} = \sum\limits_{j=1}^{m} m(c^{ij}_{\gamma_{ij}}) = m\sum\limits_{j=1}^{m} c^{ij}_{\gamma_{ij}} = m(a^i_{\alpha_i})$. since φ is convex, we have
$\sum\limits_{j=1}^{m} m(b^j_{\beta_j})\varphi(\frac{m(c^{ij}_{\gamma_{ij}})}{m(b^j_{\beta_j})}) \leq \varphi(\sum\limits_{j=1}^{m} m(b^j_{\beta_j})\frac{m(c^{ij}_{\gamma_{ij}})}{m(b^j_{\beta_j})})$.
$H_C(A \mid B) = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} m(b^j_{\beta_j})\varphi(\frac{m(c^{ij}_{\gamma_{ij}})}{m(b^j_{\beta_j})}) \leq \sum\limits_{i=1}^{n}\varphi(\sum\limits_{j=1}^{m} m(b^j_{\beta_j})\frac{m(c^{ij}_{\gamma_{ij}})}{m(b^j_{\beta_j})}) = \sum\limits_{i=1}^{n}\varphi(m(a^i_{\alpha_i})) = H(A)$.

ii) $H(c) = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}\varphi(m(c^{ij}_{\gamma_{ij}})) = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m}\varphi(m(a^i_{\alpha_i})\frac{m(c^{ij}_{\gamma_{ij}})}{m(a^i_{\alpha_i})}) = -\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} m(c^{ij}_{\gamma_{ij}})\log m(a^i_{\alpha_i}) - \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} m(c^{ij}_{\gamma_{ij}})\log\frac{m(c^{ij}_{\gamma_{ij}})}{m(a^i_{\alpha_i})} = -\sum\limits_{i=1}^{n}\log m(a^i_{\alpha_i})\sum\limits_{j=1}^{m} m(c^{ij}_{\gamma_{ij}}) + \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} m(a^i_{\alpha_i})\varphi(\frac{m(c^{ij}_{\gamma_{ij}})}{m(a^i_{\alpha_i})}) = -\sum\limits_{i=1}^{n} m(a^i_{\alpha_i})\log m(a^i_{\alpha_i}) + H_C(B \mid A) = H(A) + H_C(B \mid A)$. □

Corollary 2.1. *For any Rieze join refinement C of A and B there holds* $\max\{H(A), H(B)\} \leq H(C) \leq H(A) + H(B)$.

Proposition 2.5. *If $A \prec B$ then $H(A) \leq H(B)$.*

Proof. Assume that $A = \{a^i_{\alpha_i}\}_{i=1}^n$ and $B = \{b^j_{\beta_j}\}_{j=1}^m$ are partitions of FP(M) with RDP, and $A \prec B$. By definition there exists $\eta_i \subseteq \{1, ..., m\}$ such that $\bigcup\limits_{i=1}^{n}\eta_i = \{1, ..., m\}$, $\eta_j \cap \eta_k = \emptyset$ for $k \neq j$ and $a^i_{\alpha_i} = \sum\limits_{j\in\eta_i} b^j_{\beta_j}$. Without loss of generality, we can assume that $\eta_1 = \{1, ..., t_1\}, \eta_2 = \{t_1 + 1, ..., t_2\}, ..., \eta_n = \{t_{n-1} + 1, ..., t_n\}$, where $t_n = m$. Let

$$C = \{c^{ij}_{\gamma_{ij}} : 1 \leq i \leq n, 1 \leq j \leq m\} \textit{ and } c^{ij}_{\gamma_{ij}} = \begin{cases} b^j_{\beta_j} & \text{j} \in \eta_\text{i} \\ 0_1 & \text{o.w} \end{cases}$$

C is Rieze join refinement of A and B. By previous proposition $H(A) \leq H(C) = H(B)$. □

Definition 2.13. *Let $A = \{a^i_{\alpha_i}\}_{i=1}^n$ and $B = \{b^j_{\beta_j}\}_{j=1}^m$ be partitions of FP(M) with RDP, and $C = \{c^{ij}_{\gamma_{ij}} : 1 \leq i \leq n, 1 \leq j \leq m\}$ is Rieze join refinement of A and B. We say C is independent if $m(c^{ij}_{\gamma_{ij}}) = m(a^i_{\alpha_i})m(b^j_{\beta_j})$for any i, j.*

Proposition 2.6. *Let $A = \{a^i_{\alpha_i}\}_{i=1}^n$ and $B = \{b^j_{\beta_j}\}_{j=1}^m$ be partitions of FP(M) with RDP, and $C = \{c^{ij}_{\gamma_{ij}} : 1 \leq i \leq n, 1 \leq j \leq m\}$ be an independent Rieze join refinement of A and B, then:*

i) $H(C) = H(A) + H(B)$,

ii) $H_C(A \mid B) = H(A)$.

Proof.

i) $H(C) = -\sum_{i,j} m(a^i_{\alpha_i}) m(b^j_{\beta_j}) \log m(a^i_{\alpha_i}) m(b^j_{\beta_j}) = -\sum_i m(a^i_{\alpha_i}) \log m(a^i_{\alpha_i}) \sum_j m(b^j_{\beta_j}) - \sum_j m(b^j_{\beta_j}) \log m(b^j_{\beta_j}) \sum_i m(a^i_{\alpha_i}) = -\sum_i m(a^i_{\alpha_i}) \log m(a^i_{\alpha_i}) - \sum_j m(b^j_{\beta_j}) \log m(b^j_{\beta_j}) = H(A) + H(B)$.

ii) $H_C(A \mid B) = \sum_{i=1}^n \sum_{j=1}^m m(b^j_{\beta_j}) \varphi(\frac{m(c^{ij}_{\gamma_{ij}})}{m(b^j_{\beta_j})}) = \sum_{i=1}^n \sum_{j=1}^m m(b^j_{\beta_j}) \varphi(m(a^i_{\alpha_i})) = \sum_{i=1}^n \varphi(m(a^i_{\alpha_i})) \sum_{j=1}^m m(b^j_{\beta_j}) = H(A)$. □

Corollary 2.2. *Let A, B and C be partitions of $FP(M)$ with RDP. If $A \prec B$, $E = A \vee C$ and $F = B \vee C$ are independent, then:*

i) $H(E) \leq H(F)$,

ii) $H_E(A \mid C) \leq H_F(B \mid C)$.

Definition 2.14. *Let $A = \{a^i_{\alpha_i}\}_{i=1}^n$ and $B = \{b^j_{\beta_j}\}_{j=1}^m$ be two partitions of $FP(M)$ with RDP. The relative entropy of A with respect to B is defined as following:*

$$H(A||B) = \sum_{i=1}^n \sum_{j=1}^m m(a^i_{\alpha_i}) \log \frac{m(a^i_{\alpha_i})}{m(b^j_{\beta_j})},$$

when ever $m(b^j_{\beta_j}) \neq 0$.

Proposition 2.7. *Let* $A = \{a^i_{\alpha_i}\}_{i=1}^n$, $B = \{b^j_{\beta_j}\}_{j=1}^m$ *and* $C = \{c^k_{\gamma_k}\}_{k=1}^l$ *be partitions of* $FP(M)$ *with RDP. Then:*

i) $H(A||A^0) = H(A)$ *where* $A^0 = \{u_\alpha\}$,

ii) If D *is Rieze join refinement of* A *and* B *then* $H(D||B) = -H_D(A|B)$,

iii) If $m(b^j_{\beta_j}) \neq 0$ *for any* $j \in \{1, 2, ..., m\}$ *then* $H(A||B) \geq mH(A)$,

iv) $A \prec C$ *implies that* $H(C||B) \leq H(A||B)$,

v) $A \prec C$ *implies that* $H(B||C) \geq H(B||A)$.

Proof.

i) $m(u_\alpha) = 1$.

ii) Based on the definitions the proof is obvious.

iii) Since $0 < m(b^j_{\beta_j}) \leq 1$, For every $j \in \{1, ..., m\}$ thus $\frac{m(a^i_{\alpha_i})}{m(b^j_{\beta_j})} \geq m(a^i_{\alpha_i})$ and $\sum\limits_{j=1}^{m} \log \frac{m(a^i_{\alpha_i})}{m(b^j_{\beta_j})} \geq m \log m(a^i_{\alpha_i})$.

iv) $A \prec C$ thus for each $a^i_{\alpha_i} \in A$ there exists $\eta_i \subseteq \{1, ..., l\}$ such that $\bigcup\limits_{i=1}^{n} \eta_i = \{1, ..., l\}$, $\eta_j \cap \eta_k = \emptyset$ for $k \neq j$ and $a^i_{\alpha_i} = \sum\limits_{k \in \eta_i} c^k_{\gamma_k}$. Since m is a function of $FP(M)$ to $[0, 1]$ so we have $\log m(a^i_{\alpha_i}) = \log \sum\limits_{k \in \eta_i} m(c^k_{\gamma_k}) \geq \sum\limits_{k \in \eta_i} \log m(c^k_{\gamma_k})$ and this implies $m(a^i_{\alpha_i}) \log m(a^i_{\alpha_i}) \geq \sum\limits_{k \in \eta_i} m(c^k_{\gamma_k}) \log m(c^k_{\gamma_k})$. On the other hand $m(a^i_{\alpha_i}) \log m(b^j_{\beta_j}) = \sum\limits_{k \in \eta_i} m(c^k_{\gamma_k}) \log m(b^j_{\beta_j})$. Therefore, $H(C||B) \leq H(A||B)$.

v) $m(b^j_{\beta_j}) \log m(b^j_{\beta_j}) - m(b^j_{\beta_j}) \log m(a^i_{\alpha_i}) \leq m(b^j_{\beta_j}) \log m(b^j_{\beta_j}) - \sum\limits_{k \in \eta_i} m(b^j_{\beta_j}) \log m(c^k_{\gamma_k}) = \sum\limits_{k \in \eta_i} m(b^j_{\beta_j}) \log m(b^j_{\beta_j}) - m(b^j_{\beta_j}) \log m(c^k_{\gamma_k})$.

□

Definition 2.15. *Let $A, B, A_1, A_2, ..., A_n$ be partitions of FP(M) with RDP. We define*

$$H_*(A_1, ..., A_n) := \inf\{H(C) : C \in Ref(A_1, ..., A_n)\},$$
$$H^*(A_1, ..., A_n) := \sup\{H(C) : C \in Ref(A_1, ..., A_n)\},$$
$$H_*(A \mid B) := \inf\{H_C(A \mid B) : C \in Ref(A, B)\},$$
$$H^*(A \mid B) := \sup\{H_C(A \mid B) : C \in Ref(A, B)\}.$$

In view of Corollary 2.19, $H_*(A_1, ..., A_n)$ and $H^*(A_1, ..., A_n)$ are finite and $\max\{H(A_1), ..., H(A_n)\} \leq H_*(A_1, ..., A_n) \leq H^*(A_1, ..., A_n) \leq H(A_1) + ... + H(A_n)$.

Corollary 2.3. *Let A, B be partitions of FP(M) with RDP, then:*

i) $H^*(A \vee B) \geq H^*(A|B) + H(B)$,

ii) $H_*(A \vee B) \geq H_*(A|B) + H(B)$.

3. Entropy of semi dynamical system on semi MV-algebra with RDP

Definition 3.1. *A triple $(FP(M), m, \varphi)$ is said to be a semi dynamical system if $FP(M)$ is semi MV-algebra with RDP, m is a state on $FP(M)$,and $\varphi : FP(M) \to FP(M)$ is a mapping such that:*

i) $\varphi(u_\alpha) = u_\alpha$,

ii) $\varphi(x_\alpha \oplus y_\beta) = \varphi(x_\alpha) \oplus \varphi(y_\beta)$,

iii) $m(\varphi(x_\alpha)) = m(x_\alpha)$.
We say φ is a transformation.

Proposition 3.1. *If $A = \{a^i_{\alpha_i}\}_{i=1}^n$ is partition of FP(M) with RDP, then $\varphi(A) = \{\varphi(a^i_{\alpha_i})\}_{i=1}^n$is a partition of FP(M), and $H(\varphi(A)) = H(A)$.*

Proof. $\sum\limits_{i=1}^{n} a^i_{\alpha_i} = u_\beta$ and $\sum\limits_{i=1}^{n} \varphi(a^i_{\alpha_i}) = \varphi(\sum\limits_{i=1}^{n} a^i_{\alpha_i}) = u_\beta$. Also by definition we have $m(\varphi(x_\alpha)) = m(x_\alpha)$. □

Definition 3.2. *Let A be a partition and φ be a transformation of FP(M) with RDP. we define:*
$H_*^n(A,\varphi) := H_*(A \vee \varphi(A) \vee ... \vee \varphi^{n-1}(A))$,
$H_n^*(A,\varphi) := H^*(A \vee \varphi(A) \vee ... \vee \varphi^{n-1}(A))$.

Proposition 3.2. *If $C = A \vee B$, then $\varphi(C) = \varphi(A) \vee \varphi(B)$.*

Proof. Since $\varphi(x_\alpha \oplus y_\beta) = \varphi(x_\alpha) \oplus \varphi(y_\beta)$, the proof is trivial. □

Theorem 3.1. *Let $\{(a_i)\}_{i=1}^{\infty}$ be sequence of nonnegative numbers such that $a_{r+s} \leq a_r + a_s$ for each $r, s = 1, 2, \ldots$, then $\lim_{n\to\infty} \frac{1}{n} a_n$ exists.*

Proof. The proof can be found in Walters (1982). □

Proof of the next theorem is similar to the proof of the simillar theorem of other algebraic structures. Nevertheless we will mention the proof.

Theorem 3.2. *Let $(FP(M), m, \varphi)$ be a semi dynamical system. For any partition $A = \{a_{\alpha_i}^i\}_{i=1}^n$ of $FP(M)$, there exist limits*

$$h_*(A,\varphi) := \lim_{n\to\infty} \frac{1}{n} H_*^n(A,\varphi),$$
$$h^*(A,\varphi) := \lim_{n\to\infty} \frac{1}{n} H_n^*(A,\varphi).$$

Proof. By the previous theorem, it enough to prove that $H_*^{n+m}(A,\varphi) \leq H_*^n(A,\varphi) + H_*^m(A,\varphi)$. Let C be Rieze refinement of partitions $A, \varphi(A), ..., \varphi^{n-1}(A)$ and D be Rieze refinement of partitions $A, \varphi(A), ..., \varphi^{m-1}(A)$. $\varphi^n(D)$ is a Rieze refinement of $\varphi^n(A), \varphi^{n+1}(A), ..., \varphi^{n+m-1}(A)$. Let ε be any Rieze refinement of C and $\varphi^n(D)$. By Corollary 2.19 we have

$$H_*^{n+m}(A,\varphi) \leq H(\varepsilon) \leq H(C) + H(\varphi^n(D)) = H(C) + H(D).$$

C is arbitrary and $H_*^{n+m}(A,\varphi) - H(D) \leq H(c)$ thus $H_*^{n+m}(A,\varphi) - H(D) \leq H_*^n(A,\varphi)$, since D also is arbitrary, this implies $H_*^{n+m}(A,\varphi) \leq H_*^n(A,\varphi) + H_*^m(A,\varphi)$, therefore, $\lim_{n\to\infty} \frac{1}{n} H_*^n(A,\varphi)$ dose exist. With the similar argument we could conclude the existence of $\lim_{n\to\infty} \frac{1}{n} H_n^*(A,\varphi)$. □

Definition 3.3. *Let* $A = \{a^i_{\alpha_i}\}_{i=1}^n$ *and* $B = \{b^j_{\beta_j}\}_{j=1}^m$ *be partitions of* $FP(M)$ *with RDP. We say* $A \overset{\circ}{\subseteq} B$ *if for any* $a^i_{\alpha_i} \in A$ *there are* $b^j_{\beta_j} \in B$ *and* $c^{ij}_{\alpha_{ij}} \in FP(M)$ *such that* $b^j_{\beta_j} = a^i_{\alpha_i} \oplus c^{ij}_{\alpha_{ij}}$ *and* $m(c^{ij}_{\alpha_{ij}}) = 0_\alpha$.

Theorem 3.3. *Let* $A = \{a^i_{\alpha_i}\}_{i=1}^n$, $B = \{b^j_{\beta_j}\}_{j=1}^m$, $C = \{c^k_{\alpha_k}\}_{k=1}^r$, *and* $D = \{d^l_{\alpha_l}\}_{l=1}^s$ *be partitions of a semi dynamical system* $(FP(M), m, \varphi)$ *with RDP and* $A \overset{\circ}{\subseteq} C$, *then:*

a) $H(C) \leq H(A)$,

b) $\varphi(A) \overset{\circ}{\subseteq} \varphi(C)$,

c) If $P = \{p^{ij}_{\alpha_{ij}} : i = 1, ..., n, j = 1, ..., m\}$ *is independent Rieze join refinement of* A *and* B *and* $Q = \{q^{nj}_{\alpha_{nj}} : n = 1, ..., k, j = 1, ..., m\}$ *is Rieze join refinement of* C *and* B *then* $H(Q) \leq H(P)$,

d) If for every n *and* m, *Rieze join refinements of* $A, \varphi(A), \varphi^2(A), ..., \varphi^{n-1}(A)$ *and also Rieze join refinements of* $C, \varphi(C), \varphi^2(C), ..., \varphi^{m-1}(C)$ *are independent then* $h^*(\varphi, C) \leq h^*(\varphi, A)$ *and also* $h_*(\varphi, C) \leq h_*(\varphi, A)$.

Proof.

a) Since $A \overset{\circ}{\subseteq} C$ thus for any $a^i_{\alpha_i} \in A$ there is $c^k_{\beta_k} \in C$ such that $m(a^i_{\alpha_i}) = m(c^k_{\beta_k})$ therefore $H(C) \leq H(A)$.

b) For any $a^i_{\alpha_i} \in A$ there are $c^k_{\beta_k} \in C$ and $e^{ik}_{\alpha_{ik}} \in FP(M)$ such that $c^k_{\beta_k} = a^i_{\alpha_i} \oplus e^{ik}_{\alpha_{ik}}$ and $m(e^{ik}_{\alpha_{ik}}) = 0_\alpha$. We have $\varphi(c^k_{\beta_k}) = \varphi(a^i_{\alpha_i}) \oplus \varphi(e^{ik}_{\alpha_{ik}})$, and $m(\varphi(p^{ik}_{\alpha_{ik}})) = m(e^{ik}_{\alpha_{ik}}) = 0_\alpha$.

c) Since for every $a^i_{\alpha_i} \in A$ there is $c^k_{\beta_k} \in C$ such that $m(a^i_{\alpha_i}) = m(c^k_{\beta_k})$ and also $m(p^{ij}_{\alpha_{ij}}) = m(a^i_{\alpha_i})m(b^j_{\beta_j})$, and $m(q^{nj}_{\alpha_{nj}}) = m(c^k_{\beta_k})m(b^j_{\beta_j})$ thus, the proof is trivial.

d) If P is a Rieze join refinement of $A, \varphi(A), \varphi^2(A), ..., \varphi^{n-1}(A)$ and Q is a Rieze join refinement of $C, \varphi(C), \varphi^2(C), ..., \varphi^{n-1}(C)$ then part three implies

$$H(Q) = H(C \vee \varphi(C) \vee ... \vee \varphi^{n-1}(C)) \leq H(P) = H(A \vee \varphi(A) \vee ... \vee \varphi^{n-1}(A)).$$

□

Definition 3.4. *Let $(FP(M), m, \varphi)$ be semi dynamical system with RDP. The lower and upper entropy, $h_*(\varphi)$ and $h^*(\varphi)$ of semi dynamical system $(FP(M), m, \varphi)$ are defined as follow:*

$$h_*(\varphi) := sup\{h_*(A, \varphi) : A \text{ is a partition of } FP(M)\},$$

$$h^*(\varphi) := sup\{h^*(A, \varphi) : A \text{ is a partition of } FP(M)\}.$$

Proposition 3.3. *Let A be a partition of $FP(M)$, then $h^*(A, \varphi) \leq H(A)$ and also $h_*(A, \varphi) \leq H(A)$.*

Proof. Let C be a Rieze join refinement of $A, \varphi(A), \varphi^2(A), ..., \varphi^{n-1}(A)$ then $H(C) \leq \sum\limits_{i=0}^{n-1} H(\varphi^i(A)) = nH(A)$ thus $sup\ H(C) \leq n\, H(A)$ and also $inf\ H(C) \leq n\, H(A)$. □

Definition 3.5. *Two semi dynamical systems $(FP(M_1), m_1, \varphi_1)$ and $(FP(M_2), m_2, \varphi_2)$ with RDP are said to be isomorphic if there exists a bijective mapping $\psi : FP(M_1) \to FP(M_2)$ such that:*

i) $\psi(u^1_\alpha) = u^2_\alpha$,

ii) $\psi(\overset{n}{\underset{i=1}{\oplus_1}} x^i_{\alpha_i}) = \overset{n}{\underset{i=1}{\oplus_2}} \psi(x^i_{\alpha_i})$,

iii) $m_2(\psi(x_\alpha)) = m_1(x_\alpha)$,

iv) $\varphi_2(\psi(x_\alpha)) = \psi(\varphi_1(x_\alpha))$ *for all* $x_\alpha \epsilon FP(M_1)$.

Proposition 3.4. *If two semi dynamical systems $(FP(M_1), m_1, \varphi_1)$ and $(FP(M_2), m_2, \varphi_2)$ with RDP are isomorphic with isomorphism ψ, then:*

i) $A = \{a^i_{\alpha_i}\}^n_{i=1}$ *is a partition of* $FP(M_1)$ *iff* $\psi(A) = \{\psi(a^i_{\alpha_i})\}^n_{i=1}$ *is a partition of* $FP(M_2)$,

ii) $H(A) = H(\psi(A))$,

iii) If $C = A \vee B$, then $\psi(C) = \psi(A) \vee \psi(B)$

Proof.

i) Let $A = \{a^i_{\alpha_i}\}_{i=1}^n$ be a partition of $FP(M_1)$, $\overset{n}{\underset{i=1}{\oplus_2}} \varphi_2(\psi(a^i_{\alpha_i})) = \overset{n}{\underset{i=1}{\oplus_2}} \psi(\varphi_1(a^i_{\alpha_i})) = \psi(\varphi_1(\overset{n}{\underset{i=1}{\oplus_1}} a^i_{\alpha_i})) = \psi(u^1_\alpha) = u^2_\alpha$. The proof of inverse is similar.

ii) $m_2(\psi(a_{\alpha_i})) = m_1(a_{\alpha_i})$ implies $H(A) = H(\psi(A))$.

iii) Let $A = \{a^i_{\alpha_i}\}_{i=1}^n$, $B = \{b^j_{\beta_j}\}_{j=1}^m$ and $\{c^{ij}_{\gamma_{ij}} : 1 \leq i \leq n, 1 \leq j \leq m\}$. $\psi(a^i_{\alpha_i}) = \sum_{j=1}^{m} \psi(c^{ij}_{\gamma_{ij}})$ and $\psi(b^j_{\beta_j}) = \sum_{i=1}^{n} \psi(c^{ij}_{\gamma_{ij}})$.

□

Theorem 3.4. *If semi dynamical systems $(FP(M_1), m_1, \varphi_1)$ and $(FP(M_2), m_2, \varphi_2)$ with RDP are isomorphic with isomorphism ψ, then $h_*(\varphi_1) = h_*(\varphi_2)$ and $h^*(\varphi_1) = h^*(\varphi_2)$.*

Proof. $H^n_*(A, \varphi_1) = H_*(A \vee \varphi_1(A) \vee ... \vee \varphi_1^{n-1}(A)) = \inf\{H(C) : C \in Ref(A, \varphi_1(A), ..., \varphi_1^{n-1}(A))\} = \inf\{H(\psi(C)) : \psi(C) \in Ref(\psi(A), \varphi_2(\psi(A)), ..., \varphi_2^{n-1}(\psi(A)))\} = H^n_*(\psi(A), \varphi_2)$ and this proved $h^n_*(A, \varphi_1) = h^n_*(\psi(A), \varphi_2)$. If B is a partition of $FP(M_2)$ then it could be proved $h^n_*(\psi^{-1}(B), \varphi_1) = h^n_*(B, \varphi_2)$ thus $\sup_A h^n_*(A, \varphi_1) = \sup_B h^n_*(B, \varphi_2)$ and this implies $h^n_*(\varphi_1) = h^n_*(\varphi_2)$. In a similar way we can prove the second equality. □

4. Examples

In this section we want to compute the lower and upper entropies of some examples of semi MV-algebra. At first we introduce some definitions and prove some properties.

Remark 4.1. *Interval $[0, 1]$ is MV-algebra that we define its continuous t-conorm "$\oplus$", continuous t-norm "$\odot$", and negation "$*$" by $x \oplus y = \min\{1, x+y\}$, $x \odot y = \max\{0, x + y - 1\}$, and $x^* = 1 - x$.*
Let $FP([0, 1]) = \{x_\alpha : x, \alpha \in [0, 1]\}$. Since $[0, 1]$ has Rieze decomposition property thus $FP([0, 1])$ is a semi MV-algebra with RDP.

Definition 4.1. *An ideal of a semi MV-algebra $FP(M)$ is a nonempty set I of $FP(M)$ satisfying the two following conditions:*

i) If $x_\alpha, y_\beta \in I$, then $x_\alpha \oplus y_\beta \in I$,

ii) If $y_\beta \in I$, $x_\alpha \in FP(M)$, and $x_\alpha \leq y_\beta$, then $x_\alpha \in I$.

Remark 4.2. *Fix α and define $[0,1]_\alpha = \{x_\alpha : x \in [0,1]\}$ and suppose that for every $p \in [1,\infty)$, $\frac{x_\alpha}{p} = (\frac{x}{p})_\alpha$ and $k(x_\alpha) = (kx)_\alpha$ when $kx \leq 1$.*

Lemma 4.1. *If I is an ideal of $FP(M)$, then $I = \bigcup_{\beta \in K} [0,1]_\beta$, $K = \{\beta : \exists\ x_\beta \neq 0_\beta, s.t\ x_\beta \in I\}$.*

Proof. Suppose that $x_\beta \neq 0_\beta$ and $x_\beta \in I$. There exists $n \in N$ such that $\underbrace{\{x_\beta \oplus x_\beta \oplus ... \oplus x_\beta\}}_{n} = \min\{1, nx\}_\beta = 1_\beta$. Since I is ideal so $1_\beta \in I$ and this implies $[0,1]_\beta \subset I$, and $I = \bigcup_{\beta \in K} [0,1]_\beta$. □

Lemma 4.2. *If $\psi : FP([0,1]) \to FP([0,1])$ is an increasing transformation, $m : FP([0,1]) \to [0,1]$ is an increasing state, then $\ker\psi$ and $\ker m$ are ideals and $\ker\psi = \ker m = \{0_\alpha : \alpha \in [0,1]\}$.*

Proof. $\psi(0_\alpha) = 0_\alpha$ so $0_\alpha \in \ker\psi$. If $x_\alpha \leq y_\beta$ and $y_\beta \in \ker\psi$ then $\psi(x_\alpha) \leq \psi(y_\beta) = 0_\gamma$, and $\psi(x_\alpha) = 0_\delta, \delta \leq \gamma$ thus $x_\alpha \in \ker\psi$. If $x_\alpha, y_\beta \in \ker\psi$ then $x_\alpha \oplus y_\beta \in \ker\psi$. On the other hand we have $\psi(1_\alpha) = 1_\alpha$ thus $1_\alpha \notin \ker\psi$ and this implies $\ker\psi = \{0_\alpha : \alpha \in [0,1]\}$.
A similar proof could be found for the state m. □

Lemma 4.3. *If $\psi_\alpha : [0,1]_\alpha \to [0,1]_\alpha$ is a transformation, and $m_\alpha : [0,1]_\alpha \to [0,1]$ is a state, then for any $x_\alpha \in [0,1]_\alpha$:*

i) $\psi_\alpha(x_\alpha) = x_\alpha$,

ii) $m_\alpha(x_\alpha) = x$.

Proof. The proof "i" will be done step by step.
Step 1. ψ_α is an increasing transformation.
Let $x_\alpha \leq y_\alpha$ then $x \leq y$ thus there exists $z \in [0,1]$ such that $x + z = y$.

$\psi_\alpha(x_\alpha \oplus z_\alpha) = \psi_\alpha(y_\alpha)$ hence $\psi_\alpha(x_\alpha) \leq \psi_\alpha(y_\alpha)$.

Step 2. $\ker \psi_\alpha = \{0_\alpha\}$ iff ψ_α is injective.
$\psi_\alpha(x_\alpha) = \psi_\alpha(y_\alpha)$ iff $\psi_\alpha(x_\alpha \ominus y_\alpha) = 0_\alpha$ iff $x_\alpha \ominus y_\alpha = 0_\alpha$.

Step 3. $\psi_\alpha((\frac{1}{2})_\alpha) = (\frac{1}{2})_\alpha$.
$\psi_\alpha((1 - \frac{1}{2})_\alpha) = \psi_\alpha(1_\alpha) \ominus \psi_\alpha((\frac{1}{2})_\alpha)$ thus $\psi_\alpha((\frac{1}{2})_\alpha) = \frac{1_\alpha}{2} = (\frac{1}{2})_\alpha$.
Since ψ_α is increasing transformation thus $\psi_\alpha[0, \frac{1}{2}]_\alpha \subseteq [0, \frac{1}{2}]_\alpha$ therefore in order to determine ψ_α is sufficient to specify $\psi_\alpha \mid_{[0, \frac{1}{2}]_\alpha}$.

Step 4. $\psi_\alpha((\frac{k}{2^n})_\alpha) = (\frac{k}{2^n})_\alpha$, for $k \in N$ and $\frac{k}{2^n} \leq \frac{1}{2}$.
$\psi_\alpha((\frac{1}{2})_\alpha) = \psi_\alpha((\frac{1}{4})_\alpha) \oplus \psi_\alpha((\frac{1}{4})_\alpha) = (\frac{1}{2})_\alpha$ thus $\psi_\alpha((\frac{1}{4})_\alpha) = (\frac{1}{4})_\alpha$. By induction we have $\psi_\alpha((\frac{1}{2^n})_\alpha) = (\frac{1}{2^n})_\alpha$ for any $n \geq 1$ and also $\psi_\alpha((\frac{k}{2^n})_\alpha) = k\psi_\alpha((\frac{1}{2^n})_\alpha) = k(\frac{1}{2^n})_\alpha = (\frac{k}{2^n})_\alpha$.
Step 5. $\psi_\alpha(x_\alpha) = x_\alpha$ for any $x_\alpha \in [0, \frac{1}{2}]_\alpha$.
There are two sequences $\{a^n\}_{n=1}^{\infty}, \{b^n\}_{n=1}^{\infty}$ of $[0, \frac{1}{2}]$ by the form $\frac{k}{2^n}$ such that $a^n \leq x \leq b^n$ with $a^n \leq a^{n+1} \leq x \leq b^{n+1} \leq b^n$ for every $n \in N$. Since ψ_α is increasing transformation, $\psi_\alpha(a^n_\alpha) = a^n_\alpha$ and $\psi_\alpha(b^n_\alpha) = b^n_\alpha$ thus $a^n_\alpha \leq a^{n+1}_\alpha \leq \psi(x_\alpha) = y_\alpha \leq b^{n+1}_\alpha \leq b^n_\alpha$ therefore $x_\alpha = (\lim\limits_{n\to\infty} a^n)_\alpha = (\lim\limits_{n\to\infty} b^n)_\alpha = y_\alpha = \psi_\alpha(x_\alpha)$.
By 5 steps we prove $\psi_\alpha(x_\alpha) = x_\alpha$ for every $x_\alpha \in [0, 1]_\alpha$.
The proof "ii" is similar to the proof of part "i". □

Lemma 4.4. *If $\psi : FP([0, 1]) \to FP([0, 1])$ is a transformation, and $m : FP([0, 1]) \to [0, 1]$ is a state, then for every $x_\alpha \in FP([0, 1])$:*

i) $\psi(x_\alpha) = (x_\alpha)$,

ii) $m(x_\alpha) = x$.

Proof. $\psi(x_\alpha) = \psi \mid_{[0,1]_\alpha} (x_\alpha) = \psi_\alpha(x_\alpha) = (x_\alpha)$ and also $m(x_\alpha) = m \mid_{[0,1]_\alpha} (x_\alpha) = m_\alpha(x_\alpha) = x$. □

Remark: A similar conclusion should be obtained if we consider the MV-algebra $\mathbb{Q} \cap [0, 1]$.

Example 4.1. *Let* $M_k = \{0, \frac{1}{k}, \frac{2}{k}, ..., \frac{k}{k}\}$ *be a finite MV-algebra.* M_k *has the Rieze decomposition property thus* $FP(M_k)$ *is a semi MV-algebra with RDP.* $FP(M_k)$ *possesses a unique translation that is identity. Let* $M_k^\alpha = \{0_\alpha, (\frac{1}{k})_\alpha, (\frac{2}{k})_\alpha, ..., (\frac{k}{k})_\alpha\}$ *for* $k \geq 2$ *such that* $p(\frac{m}{k})_\alpha = (\frac{pm}{k})_\alpha$ *if* $pm \leq k$, *and* $\psi_\alpha : M_k^\alpha \to M_k^\alpha$ *be a transformation.* $\underbrace{(\frac{1}{k})_\alpha \oplus ... \oplus (\frac{1}{k})_\alpha}_{k\ ori} = 1_\alpha$ *thus* $\underbrace{\psi_\alpha((\frac{1}{k})_\alpha) \oplus ... \oplus \psi_\alpha((\frac{1}{k})_\alpha)}_{k\ ori} = 1_\alpha$ *and we deduce* $\psi_\alpha((\frac{1}{k})_\alpha) \neq 0_\alpha$. *Suppose that* $\psi_\alpha((\frac{1}{k})_\alpha) = (\frac{m}{k})_\alpha$, $m > 1$. *Then* $\psi_\alpha((\frac{k-1}{k})_\alpha) = \underbrace{\psi_\alpha((\frac{1}{k})_\alpha) \oplus ... \oplus \psi_\alpha((\frac{1}{k})_\alpha)}_{k-1\ ori} \geq \underbrace{(\frac{2}{k})_\alpha \oplus ... \oplus (\frac{2}{k})_\alpha}_{k-1\ ori} = 1_\alpha$ *so we get* $\psi_\alpha((\frac{k-1}{k})_\alpha) = 1_\alpha$. *On the other hand we have* $\psi_\alpha((\frac{k-1}{k})_\alpha) + \psi_\alpha((\frac{1}{k})_\alpha) = 1_\alpha$ *that implies* $\psi_\alpha((\frac{k-1}{k})_\alpha) < 1_\alpha$ *and a contradiction. In conclusion* ψ_α *must be identity.Now let* $\psi : FP(M_k) \to FP(M_k)$ *be a transformation,* $\psi((\frac{1}{k})_\alpha) = \psi_\alpha((\frac{1}{k})_\alpha) = (\frac{1}{k})_\alpha$.
If $m_\alpha : M_k^\alpha \to [0, 1]$ *is a state then it's unique by the form* $m_\alpha((\frac{1}{k})_\alpha) = \frac{1}{k}$ *for every* $(\frac{1}{k})_\alpha \in M_k^\alpha$. $\underbrace{m_\alpha((\frac{1}{k})_\alpha) \oplus ... \oplus m_\alpha((\frac{1}{k})_\alpha)}_{k\ ori} = 1$ *thus* $m_\alpha((\frac{1}{k})_\alpha)) \neq 0$. *Let* $m_\alpha((\frac{1}{k})_\alpha)) = t$. *If* $\frac{1}{k} < t \leq 1$ *then* $1 = m_\alpha((\frac{k-1}{k})_\alpha) \oplus m_\alpha((\frac{1}{k})_\alpha) > \frac{k-1}{k} + \frac{1}{k}$ *and a contradiction. If* $0 \leq t < \frac{1}{k}$ *then* $1 = m_\alpha((\frac{k-1}{k})_\alpha) \oplus m_\alpha((\frac{1}{k})_\alpha) < \frac{k-1}{k} + \frac{1}{k}$ *and this is also a contradiction. Thus we proved* $m_\alpha((\frac{1}{k})_\alpha) = \frac{1}{k}$. *If* $m : FP(M_k) \to [0, 1]$ *is a state then* $m((\frac{1}{k})_\alpha) = m_\alpha((\frac{1}{k})_\alpha) = \frac{1}{k}$.
In the next step we will calculate the entropies of $FP(M_k)$.

Partitions of the form $L_k = \{(\frac{1}{k})_{\alpha_1}, ..., (\frac{1}{k})_{\alpha_k}\}$ *are the finest refinement of* $FP(M_k)$, *and* $H(L_k) = \log k$. *Thus* $0 \leq H_*^n(A, \psi) \leq H_n^*(A, \psi) \leq \log k$ *which implies* $0 \leq h_*^n(A, \psi) \leq h_n^*(A, \psi) \leq \lim_{n\to\infty} \frac{1}{n} \log k = 0$ *so* $h_*(\psi) = h^*(\psi) = 0$.

Example 4.2. *Let* $M_Q = Q \cap [0, 1]$ *be an MV-algebra.*$FP(M_Q)$ *possesses a unique state* m *and a unique transformation* ψ. *Let* $L_k = \{(\frac{1}{k})_{\alpha_1}, ..., (\frac{1}{k})_{\alpha_k}\}$ *be a partition for every integer* $k \geq 1$.*Then* $H_n^*(L_k, \psi) = \sup\{H(C) : C \in Ref(L_k, \psi(L_k), ..., \psi^{n-1}(L_k)\} = n \log k$ *and* $h^*(L_k, \psi) = \log k$.
Let $L = \{p_{\alpha_1}^1, ..., p_{\alpha_k}^k\}$ *be an arbitrary partition for* $FP(M_Q)$. *We define a*

partition $C = \{C^{i_1...i_n}_{\alpha_{i_1...i_n}} : 1 \leq i_j \leq k, j = 1, ..., n\}$ *where* $C^{i_1...i_n}_{\alpha_{i_1...i_n}} = p^i_{\alpha_i}$ *if* $i_1 = i_2 = ... = i_n = i$, *and* $C^{i_1...i_n}_{\alpha_{i_1...i_n}} = 0$ *otherwise.* C *is a Rieze refinement of* $L, \psi(L), ..., \psi^{n-1}(L)$ *and* $H(L) \leq H^n_*(L, \psi) \leq H(C) = H(L)$. *This argument implies* $h_*(L, \psi) = 0$ *and* $h_*(\psi) = 0$. *Now we consider partition* $C = \{p^{i_1}_{\alpha_{i_1}} p^{i_2}_{\alpha_{i_2}} ... p^{i_m}_{\alpha_{i_m}} : p^{i_j}_{\alpha_{i_j}} \in L, \alpha_{i_j} \in \{\alpha_1, ..., \alpha_k\}, j = 1, ..., n\}$*.Then* C *is a Rieze refinement of* $L, \psi(L), ..., \psi^{n-1}(L)$. *Hence,* $nH(L) \geq H^*_n(L, \psi) = \sup\{H(C) : C \in Ref(L, \psi)\} \geq H(C) = nH(L)$. *Consequently,* $H^*_n(L, \psi) = nH(L)$, $h^*(L, \psi) = H(L)$, *and* $h^*(L) = \infty$.

Example 4.3. *Let* $FP([0, 1])$ *be semi MV-algebra. We have proved that state* m *and transformation* ψ *are unique by the form* $m(x_\alpha) = x$ *and* $\psi(x_\alpha) = x_\alpha$ *for any* $x_\alpha \in FP([0, 1])$. *With a similar argument in* 4.7 *we can conclude* $h^*(\psi) = \infty$ *and* $h_*(\psi) = h_*(C, \psi)$ *for any partition* C *in* $FP([0, 1])$.

5. Concluding Remarks

In this work some properties of semi dynamical system on semi MV-algebra were investigated. Most results were similar to those obtained in classical theory. Entropies of some examples were calculated. These newly introduced entropies of the fuzzy dynamical system were also isomorphism invariant, which is an important property. Maybe one of the most useful results of the theory of invariant measures for practical purposes is the Kolmogorov-Sina theorem stating that $h(T) = h(T, A)$, whenever A is a generating partition of the given σ-algebra.An interesting open problem could be trying to find the properties of generators. Another notable research would be calculating entropies of semi dynamical systems that in this paper were not computed.

References

Adler, R. L., Konheim, A. G., and McAndrew, M. H. (1965). Topological entropy. *Trans Amer Math Soc*, 114:309 – 319.

Cignoli, R. L. O., D'Ottaviano, I. M. L., and Mundici, D. (2000). *Algebraic Foundations of Many-Valued Reasoning.* Springer, Kluwer Academic, Dordrecht, Boston.

Ebrahimi, M. and Mosapour, B. (2013). The concept of entropy on d-posets. *Cankaya University Journal of Science and Engineering*, 10:137–151.

Hasankhani, M. and Borumand Saeid, A. (2013). Semi mv -algebras. *Cankaya University Journal of Science and Engineering*, 10:51–64.

Kolmogorov, A. N. (1958). New metric invariant of transitive dynamical systems and endomorphisms of lebesgue spaces. *Doklady of Russian Academy of Sciences*, 119(5):861 – 864.

Markechova, D. (1989). The entropy of fuzzy dynamical systems. *Busefal*, 38:38–41.

Mundici, D. (1986). Interpretation of afc algebras in lukasiewicz sentential calculus. *J. Functional Analysis*, 65:15–63.

Petroviciova, J. (2000). On the entropy of partitions in product mv algebras. *Soft Computing*, 4:41– 44.

Petroviciova, J. (2001). On the entropy of dynamical systems in product mv algebras. *Fuzzy Sets and Systems*, 121:347–351.

Pu, P. M. and Liu, Y. M. (1980). Fuzzy topology. i. neighborhood structure of a fuzzy point and moore-smith convergence. *Journal of Mathematical Analysis and Applications*, 76:571–599.

Shannon, C. (1948). A mathematical theory of communicationn. *Bell Syst, Tech Journal*, 27:379 – 423, 623 – 653.

Sinai, Y. G. (1959). On the notion of entropy of a dynamical system. *Doklady of Russian Academy of Sciences*, 124(3):768 – 771.

Walters, P. (1982). *An Introduction to Ergodic Theory*. Springer-Verlag, New York.