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Confidence Intervals for the Process Capability Index C_p Based on Confidence Intervals for Variance under Non-Normality

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ABSTRACT

One of the indicators for evaluating the capability of a process is the process capability index C_p. The confidence interval of C_p is important for statistical inference on the process. Usually, the calculation of the confidence interval for a process capability index requires an assumption about the underlying distribution. Therefore, three confidence intervals for based on the confidence intervals for the variance under non-normality were proposed in this paper. The confidence intervals considered were the adjusted degrees of freedom (ADJ) confidence interval, large-sample (LS) confidence interval, and augmented-large-sample (ALS) confidence interval. The estimated coverage probability and expected length of 95% confidence intervals for C_p were studied by means of a Monte Carlo simulation under different settings. Simulation results showed that the ALS confidence interval performed well in terms of coverage probability for all conditions. The LS and ADJ confidence intervals had much lower coverage probability than the nominal level for skewed distributions.

Keywords: confidence interval, process capability index, simulation study, non-normality.

1. Introduction

Statistical process control (SPC) has been widely applied in order to monitor and control processes in many industries. In recent years, process capability indices (PCIs) have generated much interest and there is a growing body of statistical process control literature. PCIs are one of the quality measurement tools that provide a numerical measure of whether a production process is capable of producing items within specification limits (Maiti and Saha (2012)). PCI is convenient because it reduces complex information about a process to a single number. If the value of the PCI exceeds one, it implies that the process is satisfactory or capable.

Although there are several PCIs, the most commonly applied index is C_p (Kane, 1986, Zhang, 2010). In this paper, we focus only on the process capability index C_p, defined by Kane (1986) as:

$$C_p = \frac{USL - LSL}{6\sigma}, \tag{1}$$

where USL and LSL are the upper and lower specification limits, respectively, and σ is the process standard deviation, assuming the process output is approximately normally distributed. The numerator of C_p gives the size of the range over which the process measurements can vary. The denominator gives the size of the range over which the process actually varies (Kotz and Lovelace (1998)). Due to the fact that the process standard deviation is unknown, it must be estimated from the sample data $\{X_1, X_2, \ldots, X_n\}$. The sample standard deviation S; $S = \sqrt{(n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$ is used to estimate the unknown parameter σ in Equation (1). The estimator of the process capability index C_p is therefore

$$\hat{C}_p = \frac{USL - LSL}{6S}, \tag{2}$$

Although the point estimator of the capability index C_p shown in Equation (2) can be a useful measure, the confidence interval is more useful. A confidence interval provides much more information about the population characteristic of interest than does a point estimate (e.g. Smithson (2001); Thompson (2002); Steiger (2004)). The confidence interval for the capability index C_p is constructed by using a pivotal quantity $Q = (n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$. Therefore, the $(1-\alpha)100\%$ confidence interval for the capability index C_p is

$$\left(\hat{C}_p\sqrt{\frac{\chi^2_{\alpha/2,n-1}}{n-1}}, \hat{C}_p\sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{n-1}}\right) \tag{3}$$

where $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ are the $(\alpha/2)100^{th}$ and $(1-\alpha/2)100^{th}$ percentiles of the central chi-squared distribution with $(n-1)$ degrees of freedom.

The confidence interval for the process capability index C_p shown in Equation (3) is to be used for data that are normal. When the data are normally distributed, the coverage probability of this confidence interval is close to a nominal value of $1-\alpha$. However, the underlying process distributions are non-normal in many industrial processes (e.g., Chen and Pearn (1997); Bittanti and Moiraghi (1998); Wu and Messimer (1999); Chang and D.S. (2002); Ding (2004)). In these situations, the coverage probability of the confidence interval can be considerably below $1-\alpha$. Hummel and Hettmansperger (2004) presented a confidence interval for population variance by adjusting the degrees of freedom of chi-square distribution. In order to develop approximate confidence intervals for variance under non-normality, Burch (2014) considered a number of kurtosis estimators combined with large-sample. Thus, the researcher deemed that it would be interesting to construct three confidence intervals for the process capability index C_p based on the confidence intervals for the variance proposed by Hummel and Hettmansperger (2004) and Burch (2014).

The structure of the paper is as follows. In Section 2, we review the confidence intervals for the variance under non-normality. Confidence intervals for the process capability index are presented in Section 3. In Section 4, simulations are undertaken to see how the confidence intervals perform under different conditions. Conclusions are presented in the final section.

2. Confidence Intervals for the Variance Under Non-normality

In this section, we review the confidence intervals for the variance under non-normality proposed by Hummel and Hettmansperger (2004) and Burch (2014).

2.1 Adjusted degrees of freedom confidence interval

Suppose $X_i \sim N(\mu, \sigma^2), i = 1, 2, ..., n$, this method depends on the fact that the sample variance is a sum of squares, and, for samples sufficiently large, can be approximated as a chi-square with an appropriate estimate for the degrees of freedom. Hummel and Hettmansperger (2004) found an estimate for the degrees of freedom using the method of matching moments (e.g., Shoemaker (1999); Mood and Boes (1974); Searls and Intarapanich (1990)). They matched the first two moments of the distribution of S^2 with that of a random variable X distributed as $c\chi^2_r$. The solution for r and c is solved using the following system of equations: (1) $\sigma^2 = cr$, and (2) $\frac{\sigma^4}{n}\left(\kappa - \frac{n-3}{n-1}\right) = 2rc^2$, where κ is the kurtosis of the distribution defined by $\kappa = \frac{E[(X-\mu)^4]}{(E[(X-\mu)^2])^2}$. Hummel and Hettmansperger (2004) proposed the confidence interval for σ^2 by adjusting the degrees of freedom of chi-squared distribution is given by

$$\left(\frac{\hat{r}S^2}{\chi^2_{1-\alpha/2,\hat{r}}}, \frac{\hat{r}S^2}{\chi^2_{\alpha/2,\hat{r}}}\right), \tag{4}$$

where $\chi^2_{\alpha/2,\hat{r}}$ and $\chi^2_{1-\alpha/2,\hat{r}}$ are the $\alpha/2$ and $1-\alpha/2$ quantiles of the central chi-squared distribution with $\hat{r}$ degrees of freedom, respectively with

$$\hat{r} = \frac{2n}{\hat{\gamma} + 2n/(n-1)}$$

and

$$\hat{\gamma} = \frac{n(n+1)}{(n-1)(n-2)(n-3)} \frac{\sum_{i=1}^{n}(X_i - \bar{X})^4}{S^4} - \frac{3(n-1)^2}{(n-2)(n-3)}.$$

2.2 Large-sample confidence interval for the variance

Suppose $X_i \sim N(\mu, \sigma^2), i = 1, 2, ..., n$, an equal-tailed $(1-\alpha)100\%$ confidence interval for σ^2, using a pivotal quantity $Q = (n-1)S^2/\sigma^2$, is Cojbasic and Loncar (2011)

$$\left(\frac{(n-1)S^2}{\chi^2_{1-\alpha/2,n-1}}, \frac{(n-1)S^2}{\chi^2_{\alpha/2,n-1}}\right),$$

where $S^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$, and $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ are the $\alpha/2$ and $1-\alpha/2$ quantiles of the central chi-squared distribution with $n-1$ degrees of freedom, respectively. If the normality assumption is not true, then one can depend on large-sample theory, which indicates that S^2 is asymptotically normally distributed. Namely,

$$S^2 \overset{asymp}{\sim} N\left(\sigma^2, \frac{\sigma^4}{n}\left(\kappa_e + \frac{2n}{n-1}\right)\right),$$

where κ_e is excess kurtosis of the distribution defined by

$$\kappa_e = \frac{E[(X-\mu)^4]}{(E[(X-\mu)^2])^2} - 3.$$

While the asymptotic variance of S^2 is unbiasedly estimated, the confidence interval defined as $S^2 \pm z_{1-\alpha/2}\sqrt{\widehat{\text{var}}(S^2)}$, where $z_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the standard distribution, is infrequently used since the distribution of S^2 is positively skewed when sample sizes are small. In practice, a natural logarithm transformation of S^2 is aplied in order to achieve approximate normality for the distribution of $\log(S^2)$ in finite sample-size applications. The mean and variance of $\log(S^2)$ are estimated using the first two terms of a Taylor's series expansion. It follows that

$$\log(S^2) \overset{approx}{\sim} N\left(\log(\sigma^2), \frac{1}{n}\left(\kappa_e + \frac{2n}{n-1}\right)\right),$$

and a large-sample confidence interval for σ^2 given by

$$\left(S^2 \exp\left(-z_{1-\alpha/2}\sqrt{A}\right), S^2 \exp\left(z_{1-\alpha/2}\sqrt{A}\right)\right), \tag{5}$$

where $A = \dfrac{G_2 + 2n/(n-1)}{n}$, in this case κ_e has been replaced with the commonly used estimator G_2 defined by

$$G_2 = \frac{n-1}{(n-2)(n-3)}[(n-1)g_2 + 6],$$

with $g_2 = \dfrac{m_4}{m_2^2} - 3$, $m_4 = n^{-1}\sum_{i=1}^n (X_i - \bar{X})^4$ and $m_2 = n^{-1}\sum_{i=1}^n (X_i - \bar{X})^2$.

2.3 Augmented-large-sample confidence interval for the variance

Burch (2014) considered a modification to the approximate distribution of $\log(S^2)$ by using a three-term Taylor's series expansion. Employing the large-sample properties of S^2, the mean and variance of $\log(S^2)$ are given by

$$E(\log(S^2)) \approx \log(\sigma^2) - \frac{1}{2n}\left(\kappa_e + \frac{2n}{n-1}\right),$$

$$\text{var}(\log(S^2)) \approx \frac{1}{n}\left(\kappa_e + \frac{2n}{n-1}\right)\left(1 + \frac{1}{2n}\left(\kappa_e + \frac{2n}{n-1}\right)\right).$$

Both the mean and the variance of $\log(S^2)$ are dependent on the kurtosis of the underlying distribution. The augmented-large-sample confidence interval for σ^2 is

$$\left(S^2 \exp\left(-z_{1-\alpha/2}\sqrt{B} + C\right), S^2 \exp\left(z_{1-\alpha/2}\sqrt{B} + C\right)\right), \tag{6}$$

where $B = \widehat{\text{var}}(\log(S^2))$, $C = \dfrac{\hat{\kappa}_{e,5} + 2n/(n-1)}{2n}$, in this case κ_e has been replaced with the modified estimator $\hat{\kappa}_{e,5}$ defined by

$$\hat{\kappa}_{e,5} = \left(\frac{n+1}{n-1}\right) G_2 \left(1 + \frac{5G_2}{n}\right).$$

3. Proposed Confidence Intervals for the Process Capability Index

From Equation (4), we construct the confidence interval for the C_p based on the confidence interval for σ^2 by adjusting the degrees of freedom of chi-square distribution, which is

$$P\left(\frac{\hat{r}S^2}{\chi^2_{1-\alpha/2,\hat{r}}} < \sigma^2 < \frac{\hat{r}S^2}{\chi^2_{\alpha/2,\hat{r}}}\right) = 1 - \alpha$$

$$P\left(\sqrt{\frac{\chi^2_{\alpha/2,\hat{r}}}{\hat{r}S^2}} < \frac{1}{\sigma} < \sqrt{\frac{\chi^2_{1-\alpha/2,\hat{r}}}{\hat{r}S^2}}\right) = 1 - \alpha$$

$$P\left(\frac{USL-LSL}{6S}\sqrt{\frac{\chi^2_{\alpha/2,\hat{r}}}{\hat{r}}} < \frac{USL-LSL}{6\sigma} < \frac{USL-LSL}{6S}\sqrt{\frac{\chi^2_{1-\alpha/2,\hat{r}}}{\hat{r}}}\right)$$
$$= 1-\alpha.$$

Therefore, the $(1-\alpha)100\%$ confidence interval for the C_p based on the confidence interval for σ^2 by adjusting the degrees of freedom of chi-square distribution is given by

$$CI_{ADJ} = \left(\frac{USL-LSL}{6S}\sqrt{\frac{\chi^2_{\alpha/2,\hat{r}}}{\hat{r}}}, \frac{USL-LSL}{6S}\sqrt{\frac{\chi^2_{1-\alpha/2,\hat{r}}}{\hat{r}}}\right). \tag{7}$$

Similarly, from Equation (5) the confidence interval for the C_p based on the large-sample confidence interval for σ^2 can be derived as follows

$$P\left(S^2\exp\left(-z_{1-\alpha/2}\sqrt{A}\right) < \sigma^2 < S^2\exp\left(z_{1-\alpha/2}\sqrt{A}\right)\right) = 1-\alpha$$
$$P\left(\frac{1}{S\sqrt{\exp\left(z_{1-\alpha/2}\sqrt{A}\right)}} < \frac{1}{\sigma} < \frac{1}{S\sqrt{\exp\left(-z_{1-\alpha/2}\sqrt{A}\right)}}\right) = 1-\alpha$$
$$P\left(\frac{USL-LSL}{6S\sqrt{\exp\left(z_{1-\alpha/2}\sqrt{A}\right)}} < \frac{USL-LSL}{6\sigma} < \frac{USL-LSL}{6S\sqrt{\exp\left(-z_{1-\alpha/2}\sqrt{A}\right)}}\right)$$
$$= 1-\alpha.$$

Thus, $(1-\alpha)100\%$ confidence interval for the C_p based on the large-sample confidence interval for σ^2 is

$$CI_{LS} = \left(\frac{USL-LSL}{6S\sqrt{\exp\left(z_{1-\alpha/2}\sqrt{A}\right)}}, \frac{USL-LSL}{6S\sqrt{\exp\left(-z_{1-\alpha/2}\sqrt{A}\right)}}\right). \tag{8}$$

Using Equation (6), we find the confidence interval for the C_p based on the augmented-large-sample confidence interval for σ^2, which is

$$P\left(S^2\exp\left(-z_{1-\alpha/2}\sqrt{B}+C\right)<\sigma^2<S^2\exp\left(z_{1-\alpha/2}\sqrt{B}+C\right)\right)=1-\alpha$$

$$P\left(\frac{1}{S\sqrt{\exp\left(z_{1-\alpha/2}\sqrt{B}+C\right)}}<\frac{1}{\sigma}<\frac{1}{S\sqrt{\exp\left(-z_{1-\alpha/2}\sqrt{B}+C\right)}}\right)=1-\alpha$$

$$P\left(\frac{USL-LSL}{6S\sqrt{\exp\left(z_{1-\alpha/2}\sqrt{B}+C\right)}}<\frac{USL-LSL}{6\sigma}<\frac{USL-LSL}{6S\sqrt{\exp\left(-z_{1-\alpha/2}\sqrt{B}+C\right)}}\right)$$
$$=1-\alpha.$$

Therefore, the $(1-\alpha)100\%$ confidence interval for the C_p based on the augmented-large-sample confidence interval for σ^2 is given by

$$CI_{ALS}=\left(\frac{USL-LSL}{6S\sqrt{\exp\left(z_{1-\alpha/2}\sqrt{B}+C\right)}},\frac{USL-LSL}{6S\sqrt{\exp\left(-z_{1-\alpha/2}\sqrt{B}+C\right)}}\right). \quad (9)$$

4. Simulation Results

This section provides the simulation studies for the estimated coverage probabilities and expected lengths of the three confidence intervals for the capability index C_p proposed in the previous section. The estimated coverage probability and the expected length (based on M replicates) are given by $\widehat{1-\alpha}=\#(L\leq C_p\leq U)/M$, and $\widehat{Length}=\sum_{j=1}^{M}(U_j-L_j)/M$, where $\#(L\leq C_p\leq U)$ denotes the number of simulation runs for which the true process capability index C_p lies within the confidence interval. The right-skewed data were generated with the population mean $\mu=50$ and the population standard deviation $\sigma=1$ (Kotz and Lovelace (1998)) given in the Table 1.

Table 1: Probability distributions generated and the coefficient of skewness for Monte Carlo simulation.

Probability Distributions	Coefficient of Skewness
N(50,1)	0.000
Gamma(4,2)+48	1.000
Gamma(0.75,0.867)+49.1340	2.309
Gamma(0.25,0.5)+49.5	4.000

The true values of the process capability index C_p, LSL and USL are set in the Table 2.

Table 2: True values of C_p, LSL and USL.

True Values of C_p	LSL	USL
1.00	47.00	53.00
1.33	46.01	53.99
1.50	45.50	54.50
1.67	44.99	55.01
2.00	44.00	56.00

The sample sizes were set at $n = 30$, 50, 75 and 100 and the number of simulation trials was 50,000. The nominal level was fixed at 0.95. All simulations were carried out using programs written in the open source statistical package R (Ihaka and Gentleman, 1996). The simulation results are shown in case of N(50,1) and Gamma(4,2)+48. Figures 1 and 2 provide simulation results for the estimated coverage probability of the confidence intervals in case of N(50,1) and while the expected length is presented in Figures 3 and 4. Clearly, the augmented-large-sample (ALS) confidence interval has estimated coverage probabilities close to the nominal level when the data were generated from a normal distribution for all the sample sizes. On the other hand, for more skewed distributions, all the confidence intervals provided estimated coverage probabilities that were much different from the nominal level, especially the large-sample (LS) confidence interval and the adjusted degrees of freedom (ADJ) confidence interval. The ALS confidence interval performed well when compared with other confidence intervals in terms of coverage probability. Additionally, when the underlying distributions were much skewed, i.e., Gamma(0.25,0.5)+49.5, the LS and ADJ confidence intervals had coverage below 90% for all sample sizes. The expected lengths of LS and ADJ confidence intervals were shortest for all situations. This is not surprising because the coverage probabilities of LS and ADJ confidence intervals were below the nominal level. Hence, using the LS and ADJ confidence intervals is not recommended. When sample sizes increase, the expected lengths become shorter. On the other hand, when the underlying distributions are much skewed, the expected lengths become wider. Other simulations not shown in this paper yield the results similar to the simulations of N(50,1) and Gamma(4,2)+48.

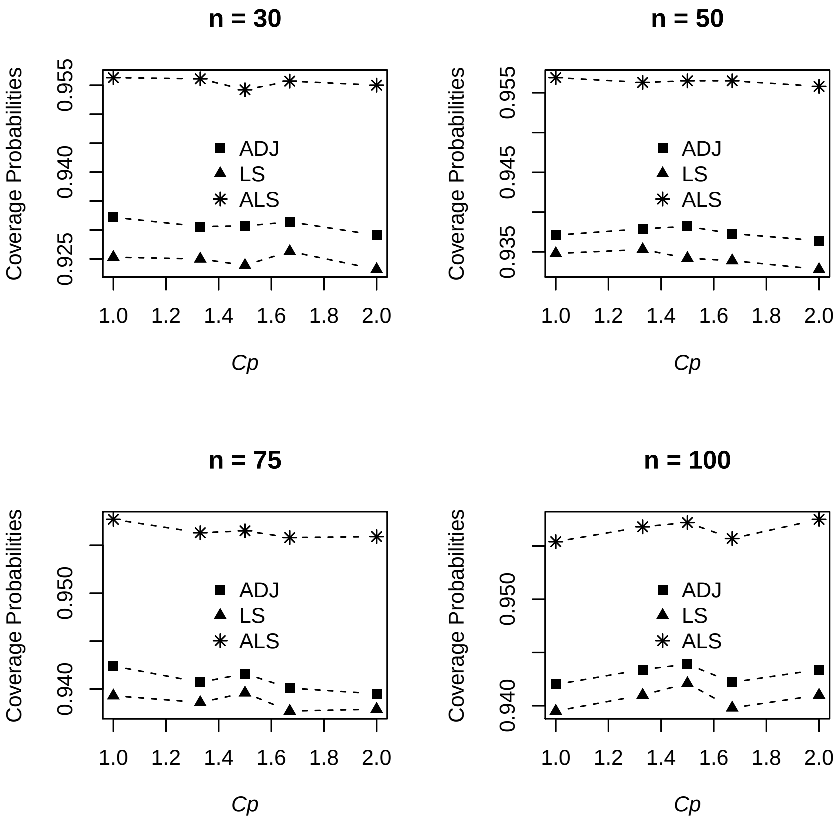

Figure 1: Estimated coverage probabilities of 95% confidence intervals for C_p in case of N(50,1).

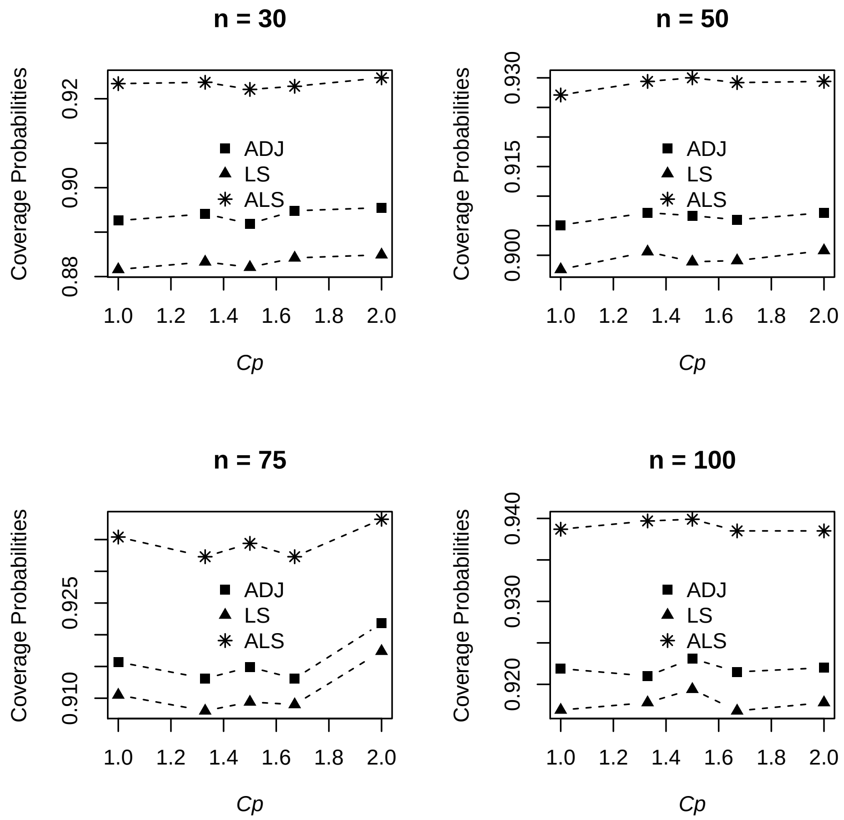

Figure 2: Estimated coverage probabilities of 95% confidence intervals for C_p in case of Gamma(4,2)+48.

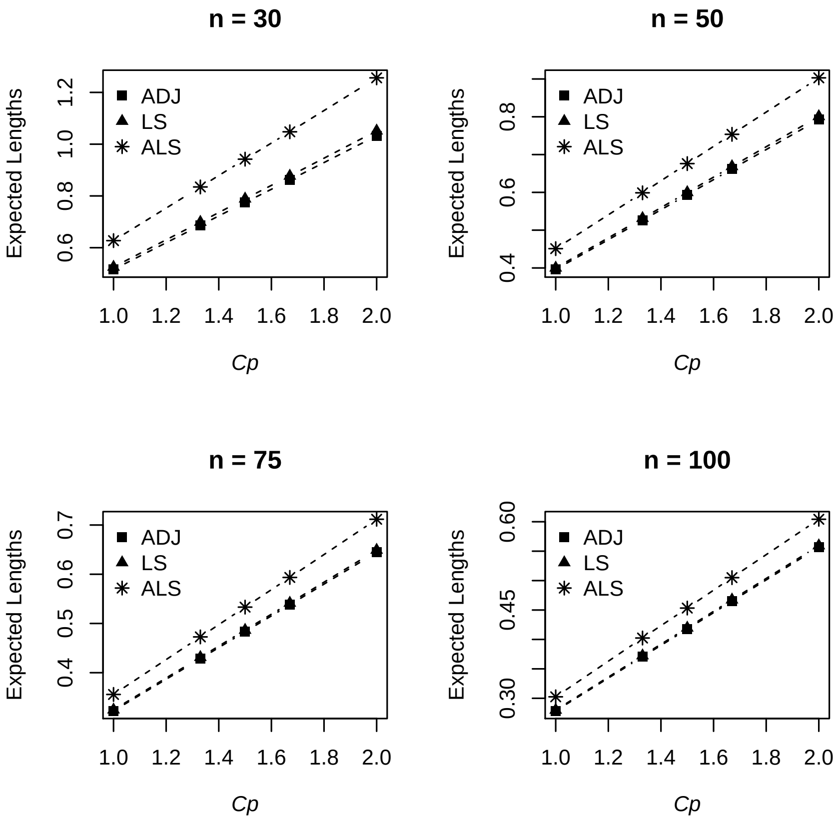

Figure 3: Expected lengths of 95% confidence intervals for C_p in case of N(50,1).

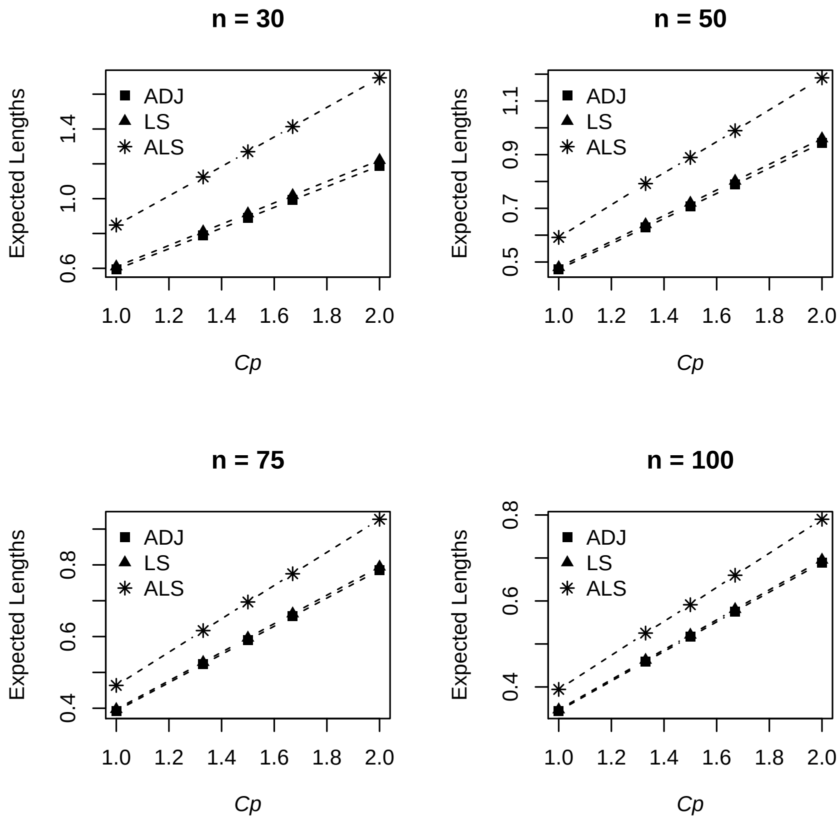

Figure 4: Expected lengths of 95% confidence intervals for C_p in case of Gamma(4,2)+48.

5. Conclusions

This paper studied three confidence intervals for the process capability index C_p. The proposed confidence intervals were based on the confidence intervals for the variance under non-normality, which was constructed by Hummel and Hettmansperger (2004) and Burch (2014). They consisted of the adjusted degrees of freedom (ADJ) confidence interval, large-sample (LS) confidence interval, and augmented-large-sample (ALS) confidence interval. All proposed confidence intervals were compared through a Monte Carlo simulation study. The ALS confidence interval proved to be better than the other confidence intervals in terms of coverage probability for all situations. Therefore, the use of the ALS confidence interval is recommended.

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