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On Characterizations of Real Hypersurface related to the Shape Operator and Structure Tensor in Nonflat Complex Space Form

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ABSTRACT

In this paper, I investigate the real hypersurfaces in complex space form $M_n(c)$, $c \neq 0$ under the condition that $L_\xi(\phi A + A\phi) = 0$, where L_ξ and $\phi A + A\phi$ denote the structure Lie operator of M and the operator is composed by the shape operator and the structure tensor ϕ.

Keywords: real hypersurface, Structure Lie operator, Shape operator, Hopf hypersurfaces, structure tensor, model space of type A.

1. Introduction

A complex n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. As is well-known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbf{C}$, a complex Euclidean space $\mathbf{C}^n$ or a complex hyperbolic space $H_n\mathbf{C}$, according to $c > 0$, $c = 0$ or $c < 0$.

In this paper we consider a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kaehler metric and complex structure J on $M_n(c)$. The Reeb vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant Ki and Suh (1990) and that M is called a *Hopf hypersurface*.

Takagi (1973) completely classified homogeneous real hypersurfaces in such hypersurfaces as six model spaces A_1, A_2, B, C, D and E. Berndt (1989) classified all homogeneous Hopf hyersurfaces in $H_n\mathbf{C}$ as four model spaces which are said to be A_0, A_1, A_2 and B. A real hypersurface of A_1 or A_2 in $P_n\mathbf{C}$ or A_0, A_1, A_2 in $H_n\mathbf{C}$, then M is said to be a type A for simplicity.

The induced operator L_ξ on real hypersurface M from the 2-form $\mathcal{L}_\xi g$ is defined by $(\mathcal{L}_\xi g)(X, Y) = g(L_\xi X, Y)$ for any vector field X and Y on M, where $\mathcal{L}_\xi$ denotes the operator of the Lie derivative with respect to the structure vector field ξ. This operator L_ξ is given

$$L_\xi = \phi A - A\phi$$

on M, and call it structure Lie operator of M. Some works have studied several conditions on the structure Lie operator L_ξ and given some results on the classification of real hypersurfaces of type A in $M_n(c)$ (see Perez et al. (2005),Kim and Lim (2014) and Kim et al. (2014))

As for structure Lie operator, Okumura (1975) for $c > 0$ and Montiel and Romero (1986) for $c < 0$ showed the following:

Theorem 1 (Montiel and Romero (1986), Okumura (1975)). *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $L_\xi = 0$ on M if and only if M is locally congruent to one of the type A.*

With respect to the structure Lie operator,Ki et al. (2010) gave a characterization of real hypersurface in complex space form $M_n(c)$.

Theorem 2 (Ki et al. (2010)). *Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. Then it is satisfies $R_\xi L_\xi g = 0$ on M if and only if M is locally congruent to one of the model space of type A.*

Now we consider the operator defined by

$$(\phi A + A\phi)X = 0 \tag{1}$$

for any vector field X in M. If M satisfying $\phi A + A\phi = 0$, then it is well known that M is cylinderical (seeKi and Suh (1990)). About the existence of real hypersurface, many researchers have been studying real hypersurfaces in nonflat complex space forms under certain geometric conditions such as $\phi A + A\phi = 0$ or integrable (see Kimura and Maeda (1989),Niebergall and Ryan (1998) and Aiyama et al. (1990) etc). Nevertheless, the classification of real hypersurfaces under the above conditions remains open problem up to this point (see Niebergall and Ryan (1998)).

As for,Yano and Kon (1973) investigated the conditions skew-symmetric of shape operator and structure tensor.

Theorem 2 (Yano and Kon (1973)). *Let M be a connected complete real hypersurface in $P_n(c)$, $n \geq 2$. If $\phi A + A\phi = k\phi$ for some constant $k \neq 0$, then M is of type A_1 or B*

Also, about the holomorphic distribution D is integrable, Kimura and Maeda (1989) showed the following:

Theorem 3 (Kimura and Maeda (1989)). *Let M be a real hypersurface of $P_n(c)$. Then the second fundamental form of M is η-parallel and the holomorphic distribution D is integrable if and only if M is locally congruent to a ruled real hypersurface.*

In this paper we shall study a real hypersurface in a non-flat complex space form $M_n(c)$ which satisfies $L_\xi(\phi A + A\phi) = 0$ on M. We give another characterization of real hypersurface of type A in $M_n(c)$ by above condition.

All manifolds in the present paper are assumed to be connected and of class C^∞ and the real hypersurfaces supposed to be oriented.

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c)$, and N be a unit normal vector field of M. By $\widetilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\widetilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \widetilde{\nabla}_X N = -AX$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric tensor of M induced from $\widetilde{g}$, and A is the shape operator of M in $M_n(c)$. For any vector field X on M we put

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where J is the almost complex structure of $M_n(c)$. Then we see that M induces an almost contact metric structure (ϕ, g, ξ, η), that is,

$$\begin{aligned} \phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \end{aligned} \tag{2}$$

for any vector fields X and Y on M. Since the almost complex structure J is parallel, we can verify from the Gauss and Weingarten formulas the followings:

$$\nabla_X \xi = \phi A X, \tag{3}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{4}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the following Gauss and Codazzi equations respectively :

$$\begin{aligned} R(X,Y)Z = \;& \tfrac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & -2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{5}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{6}$$

for any vector field X, Y and Z on M, where R denotes the Riemannian curvature tensor of M.

By the virtue of (3), $(\mathcal{L}_\xi g)(X,Y) = g((\phi A - A\phi)X, Y)$ for any vector fields X and Y on M, and hence the induced operator L_ξ from $\mathcal{L}_\xi g$ is given by

$$L_\xi X = (\phi A - A\phi)X.$$

On the other hand, since $\phi A + A\phi$ is skew-symmetric, we define an operator on M in $M_n(c)$ by

$$g((\phi A + A\phi)X, Y) = 0. \tag{7}$$

for any vector field X and Y on M. Also, if the holomorphic distribution D satisfies the condition (7), then we call the holomorphic distribution D is integrable.

Let Ω be the open subset of M defined by

$$\Omega = \{p \in M | A\xi - \alpha\xi \neq 0\}. \tag{8}$$

where $\alpha = \eta(A\xi)$. We Put

$$A\xi = \alpha\xi + \beta U, \tag{9}$$

where U is a unit vector field orthogonal to ξ, and β does not vanish on Ω.

3. Proof of Theorems

In this section, we assume that Ω is not empty, and we shall prove Theorem 3.1 and Theorem 3.2.

Theorem 3.1. *Let M be a real hypersurface satisfying $L_\xi(\phi A + A\phi) = 0$ in a complex space form $M_n(c)$, $c \neq 0$. then M is a Hopf hypersurface in $M_n(c)$.*

Theorem 3.2. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. Then M satisfies $L_\xi(\phi A + A\phi) = 0$ on M if and only if M is the following hypersurface:*

(1) Cylindrical, $(\lambda + \mu = 0)$

(2) locally congruent to one of the model spaces of type A (otherwise).

Proof of Theorem 3.1. Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$, satisfying $L_\xi(\phi A + A\phi) = 0$. We assume that the open set Ω given in (9) is not empty, then the above condition together with (2),(3) and (8) implies that

$$(\phi A\phi A - A\phi A\phi + \phi A^2\phi + A^2)X = \alpha\eta(AX)\xi + \beta\eta(AX)U \tag{10}$$

for any vector field X on M. If we put $X = \xi$ into (11) and using (9), then we have

$$\beta\{\phi A\phi U + AU\} = 0.$$

Since $\beta \neq 0$ in Ω, we obtain

$$\phi A\phi U + AU = 0. \tag{11}$$

If we take inner product of ξ and make use of (2) and (9), then we have $\beta = 0$ and hence and it is a contradiction. Thus the set Ω is empty, and hence M is a Hopf hypersurface.

We shall prove Theorem 3.2, that is, as the characterization of Hopf hypersurface we can state:

Proof of Theorem 3.2. By Theorem 3.1, M is a Hopf hypersurface in $M_n(c)$. Since ξ is a Reeb vector field, the assumption $L_\xi(\phi A + A\phi)X = 0$ is equivalent to

$$\{\phi A\phi A - A\phi A\phi + \phi A^2\phi + A^2\}X = \alpha^2\eta(X)\xi \tag{12}$$

On the other hand, if we differentiate $A\xi = \alpha\xi$ covariantly and make use of the (6) of Codazzi, then we have

$$A\phi A - \frac{\alpha}{2}(\phi A + A\phi) - \frac{c}{4}\phi = 0. \tag{13}$$

For any vector field X on M such that $AX = \lambda X$, it follows from (13) that

$$\lambda\phi A\phi X - A\phi A\phi X + \phi A^2\phi X + \lambda^2 X = 0 \tag{14}$$

By virtue of the equation (13), we also obtain

$$(\lambda - \frac{\alpha}{2})A\phi X = \frac{1}{2}(\alpha\lambda + \frac{c}{2})\phi X. \tag{15}$$

If $\lambda \neq \frac{\alpha}{2}$, then we see from (15) that ϕX is also a principal direction, say $A\phi X = \mu\phi X$. From the equation (14), we obtain

$$(\lambda - \mu)(\lambda + \mu) = 0 \tag{16}$$

If $\lambda + \mu = 0$, then we have $\phi AX + A\phi X = 0$. Therefore M is cylinderical (see[5]) If $\lambda = \mu$, M has at most 3 distinct principal curvatures α and the two roots of the quadratic equation, and hence $\phi AX = A\phi X$. If $\lambda = \mu$ or $\lambda + \mu = 0$, We see from that the above equation that M exist in the case of $\lambda = \mu$ and hence $\phi AX = A\phi X$ for any X on M (see [2]). If $\lambda = \frac{\alpha}{2}$, then it is easily seen that $A\phi X = \frac{\alpha}{2}\phi X = \phi AX$. Therefore from this result we obtain

$$L_\xi = \phi A - A\phi = 0 \tag{17}$$

on the whole M. Conversely if it satisfies (17), then it is easily seen that (15) holds, that is, $L_\xi(\phi A + A\phi) = 0$ is satisfies on M. Thus Theorem B follows from Theorem 1.

4. Conclusion

Takaki and Berndt classified real hypersurface into 6 model spaces in complex projective space and into 4 model spaces in complex hyperbolic space. Afterward, many geometricians researched operators inherent in real hypersurface to find the features of it. This research shows the features of real hypersurface with operators composed of the combination of shape operators and structure tensors.

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