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Notes on Certain Arithmetic Inequalities Involving Two Consecutive Primes

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ABSTRACT

Let r, k be positive integers (parameters) with $r \geq 2$, and let p_r be the r-th prime number. Let W_k denote the set of positive integers n for which the number of distinct prime factors of n is greater or equal to k. By using the prime number theorem and Bertrand's theorem, we will determine arithmetic functions $f, g : \mathbb{N} \longrightarrow \mathbb{N}$ for which $f(n) - \alpha_r g(n)$ has infinitely many sign changes on the set W_k, where $\alpha_r = \frac{p_{r-1}}{p_r}$. In the framework of internal set theory (for more details, see Nelson (1977)), some notions concerning nonstandard analysis as well as unlimited positive integers have been used.

Keywords: Arithmetic functions, prime number theorem, Bertrand's theorem, sign changes, internal set theory.

1. Introduction

Let $\omega(n)$ denote the number of distinct prime factors of n. For every $k \geq 1$, we put

$$W_k = \{n \in \mathbb{N} \; ; \; \omega(n) \geq k\}. \tag{1}$$

Recall that a Diophantine inequality or equation is an inequality (resp. equation) whose solution required to be integers. In arithmetic functions, one of important topics is to establish Diophantine inequalities (resp. equations) for infinitely many $n \in \mathbb{N}$ (see Sándor (2008, 2014), and De Koninck and Mercier (2004)). Many researchers have obtained their results on the set $\mathbb{N} = W_1 \cup \{1\}$. The purpose of this work is to study (on the set W_k, with $k \geq 1$) some Diophantine inequalities involving $\varphi_s(n)$, $\pi(n)$ and d_n. One can refer to, Nathanson (2000), and Yan (2002).

Let p_r denote the r-th prime number, with $r \geq 2$. In this paper, we will determine a couple of arithmetic functions (f, g) such that $f(n) - \alpha_r g(n)$ has infinitely many sign changes on the set W_k, where $\alpha_r = \dfrac{p_{r-1}}{p_r}$. That is, we prove that there is an infinite sequence of positive integers $(n_i)_{i=1,2,\ldots} \subset W_k$ and there is a couple of arithmetic functions (f, g) such that

$$\alpha_r < \frac{f(n_i)}{g(n_i)}, \text{ for } i = 1, 2, \ldots$$

and also, there is an infinite sequence of positive integers $(m_i)_{i=1,2,\ldots} \subset W_k$ such that

$$\frac{f(m_i)}{g(m_i)} < \alpha_r, \text{ for } i = 1, 2, \ldots$$

or, equivalently

$$\ldots < \frac{f(m_i)}{g(m_i)} < \ldots < \alpha_r < \ldots < \frac{f(n_i)}{g(n_i)} < \ldots, \text{ for } i = 1, 2, \ldots$$

Because, it is very difficult to determine the value of p_r whenever r is sufficiently large. Then we can say that there is an approximation of α_r by rationals, where α_r is the rapport of two consecutive primes p_{r-1} and p_r. Thus, we can surround α_r from the right and from the left by infinitely many rational numbers.

2. Materials

In this work, we use the following results. One can refer to, De Koninck and Mercier (2004), and Wells (2005).

Theorem 2.1 (Euclid's Theorem)**.** *There are infinitely many primes.*

Theorem 2.2 (Twin Prime Conjecture)**.** *There are infinitely many twin primes.*

Theorem 2.3 (Prime Number Theorem)**.** *Let $\pi(x)$ denote the number of prime numbers not exceeding x, that is,*

$$\pi(x) = \sum_{p \leq x} 1.$$

Then

$$\lim_{x \longrightarrow +\infty} \frac{\pi(x) \log x}{x} = 1.$$

Theorem 2.4 (Bertrand's theorem)**.** *If n is an integer greater than 2, then there is at least one prime between n and $2n-1$.*

Definition 2.1. *A positive integer is called square-free if it is the product of distinct prime numbers.*

Definition 2.2. *Let $\gamma(n)$ denote the Kernel of n given by*

$$\gamma(1) = 1 \quad \text{and} \ \gamma(n) = \prod_{p|n} p, \ \text{for } n \geq 2.$$

Definition 2.3. *Let n be a positive integer. We have*

- *$\tau(n)$ is the number of the positive divisors of n, i.e.,*

$$\tau(n) = \sum_{d|n} 1.$$

- *$\sigma(n)$ is the sum of the positive divisors of n, i.e.,*

$$\sigma(n) = \sum_{d|n} d.$$

- *$\sigma_2(n)$ is the sum of the square of the positive divisors of n, i.e.,*

$$\sigma_2(n) = \sum_{d|n} d^2.$$

- *$\varphi(n) = \varphi_1(n)$ or Euler's function: is defined to be the numbers of non-negative integers m less than n which are prime to n, i.e.,*

$$\varphi_1(n) = \sum_{\substack{0 \leq m < n \\ \gcd(m,n)=1}} 1 = n \prod_{p|n} \left(1 - \frac{1}{p}\right), \ \varphi_1(1) = 1.$$

- *For every $s \geq 1$, $\varphi_s(n)$ is given by*

$$\varphi_s(n) = n^s \prod_{p|n} \left(1 - \frac{1}{p^s}\right), \quad \varphi_s(1) = 1.$$

Theorem 2.5 (see De Koninck and Mercier (2004) in p. 254)**.** *If $n = p_1^{r_1} p_2^{r_2} \dots p_s^{r_s}$ is the standard factorization of n as a product of powers of distinct primes, then*

$$\prod_{i=1}^{s} \frac{p_i}{p_i - 1} > \frac{\sigma(n)}{n}.$$

In addition, we need the following notions (see Bellaouar and Boudaoud (2015), Van den Berg (1992), and Nelson (1977)).

1. A real number x is called unlimited if its absolute value $|x|$ is larger than any standard integer n. So a nonstandard integer ω is also an unlimited real number.
2. A real number ε is called infinitesimal if its absolute value $|\varepsilon|$ is smaller than $\frac{1}{n}$ for any standard n.
3. A real number r is called limited if is not unlimited.
4. Two real numbers x and y are equivalent or infinitely close (i.e., we can write $x \simeq y$) if their difference $x - y$ is infinitesimal.

3. On the sign changes of $f(n) - \alpha_r g(n)$

Let p_n be the n-th prime number. From the Prime Number Theorem, we deduce that

$$\lim_{n \longrightarrow +\infty} \frac{p_{n-1}}{d_{n-1}} = +\infty,$$

where d_{n-1} is the gap between p_n and p_{n-1}.

For every $k \geq 1$, we can choose the r-th prime number p_r and the integer s as the following way: p_r is sufficiently large and $s \geq 1$, such that

$$p_k < \left(1 + \frac{p_{r-1} - 1}{d_{r-1} + 1}\right)^{\frac{1}{s}}. \tag{2}$$

For example, in the case when $k = 50000$, then for

$$p_{r-1} = 1161979280141205742956 29.$$

That is

$$p_r = 1161979280141205742957 21,$$

and also $d_{r-1} = p_r - p_{r-1} = 92$. Thus, for $s = 3$, it follows that

$$\begin{aligned} p_{50000} &= 611953 < \left(1 + \frac{p_{r-1} - 1}{d_{r-1} + 1}\right)^{\frac{1}{s}} \\ &= \left(1 + \frac{116197928014120574295629 - 1}{93}\right)^{\frac{1}{3}} \\ &= 10000000.07. \end{aligned}$$

Theorem 3.1. *Let $k \geq 1$ be the integer of (1). Under the same assumption as in (2), $p_r\varphi_s(n) - p_{r-1}n^s$ has infinitely many sign changes on the set W_k.*

Proof. We show that (i) and (ii) are each true for infinitely many $n \in W_k$, where

(i) $p_r\varphi_s(n) > p_{r-1}n^s$,

(ii) $p_r\varphi_s(n) < p_{r-1}n^s$.

First, we prove that there exists a positive integer n_0 such that

$$p_r\varphi_s(n_0) > p_{r-1}n_0^s.$$

Let p_N be a prime number satisfying

$$p_N > \frac{1}{\left(\frac{p_r}{p_{r-1} + 1}\right)^{\frac{1}{k}} - 1} + 1. \tag{3}$$

Suppose the opposite, this means that for all $n \in W_k$ we have

$$p_r\varphi_s(n) \leq p_{r-1}n^s.$$

In particular, for $n = p_N p_{N+1} \dots p_{N+k-1} \in W_k$, it follows from (3) that

$$\frac{p_r}{p_{r-1}} \leq \frac{n^s}{\varphi_s(n)} = \prod_{p|n} \frac{p^s}{p^s - 1} \leq \prod_{p|n} \frac{p}{p-1} < \left(\frac{p_N}{p_N - 1}\right)^k < \frac{p_r}{p_{r-1} + 1},$$

which implies that $p_{r-1} > p_{r-1} + 1$. A contradiction.

Second, from the hypothesis of (2), we prove that there exists a positive integer n_1 such that

$$p_r \varphi_s(n_1) < p_{r-1} n_1^s.$$

Assume, by way of contradiction, that for all $n \in W_k$ we get

$$p_r \varphi_s(n) \geq p_{r-1} n^s.$$

In particular, for $n = 2.3 \ldots p_k$ (which is an element of W_k), it follows from (2) that

$$\frac{p_r}{p_{r-1}} \geq \frac{n^s}{\varphi_s(n)} = \prod_{p|n} \frac{p^s}{p^s - 1} \geq \frac{p_k^s}{p_k^s - 1} > \frac{p_r}{p_{r-1} - 1}.$$

A contradiction.

Finally, we return to prove the inequalities (i) and (ii) for infinitely many $n \in W_k$. In fact, from the definition of φ_s we see that $p_r \varphi_s(n_0^i) > p_{r-1}\left(n_0^i\right)^s$ and $p_r \varphi_s(n_1^i) < p_{r-1}\left(n_1^i\right)^s$ both hold for every $i \geq 1$. This completes the proof of Theorem 3.1. □

Example 3.1. From the proof of Theorem 3.1, it is clear that if

$$p_r = 1161979280141205742957 21$$

and

$$p_{r-1} = 116197928014120574295629,$$

then $p_r \varphi_3(n) - p_{r-1} n^3$ has infinitely many sign changes on the set W_{50000}. That is, there are infinitely many $n \in W_{50000}$ such that $p_r \varphi_3(n) > p_{r-1} n^3$ and there are infinitely many $m \in W_{50000}$ such that $p_r \varphi_3(m) < p_{r-1} m^3$.

Theorem 3.2. *The inequality* $p_r d_n - p_{r-1} d_{n+1}$ *has infinitely many sign changes on the set* W_1.

Proof. For all positive integers n, write $d_n = p_{n+1} - p_n$ so that $d_1 = 1$ and all other d_n are even. Since $(3, 5, 7)$ is the only prime triplet of the form $p, p+2, p+4$ (see Santos (2004) in p. 76), by using twin prime conjecture, there are infinitely many primes (p_n, p_{n+1}, p_{n+2}) such that

$$\begin{cases} d_n = p_{n+1} - p_n = 2, \\ d_{n+1} = p_{n+2} - p_{n+1} \geq 4. \end{cases} \tag{4}$$

From Bertrand's theorem and (4), we have

$$p_r d_n - p_{r-1} d_{n+1} \leq 2\left(p_r - 2p_{r-1}\right) < 0.$$

Thus, the inequality holds infinitely often. On the other hand, from Guy (1994) in p. 26, Erdös and Turán have shown that $d_n > d_{n+1}$ infinitely often. Then $p_r d_n - p_{r-1} d_{n+1} > 0$ for infinitely many n. This completes the proof. □

Theorem 3.3. *Let $\pi(n)$ be the number of primes which satisfy $2 \leq p \leq n$, and let ℓ be a positive integer. Then $p_r\pi(n) - p_{r-1}\pi(n+\ell)$ has finitely many sign changes on the set W_1.*

Proof. We suppose that $p_r\pi(n)-p_{r-1}\pi(n+\ell)$ has infinitely many sign changes, that is, $p_r\pi(n) < p_{r-1}\pi(n+\ell)$ holds for infinitely many n. For each such integer n, we must have

$$\pi(n) < \pi(n+\ell) \leq \pi(n) + \ell. \tag{5}$$

Noticing that the right hand side of (5) can be deduced by induction on ℓ. Thus,

$$\pi(n) < \frac{p_{r-1}\ell}{p_r - p_{r-1}}.$$

A contradiction, because $\pi(n)$ is asymptotic to $\dfrac{n}{\log n}$ which tends to the infinity. The proof is finished. □

Corollary 3.1. *The inequality $p_r\pi(n) > p_{r-1}\pi(n+\ell)$ holds for infinitely many $n \in W_1$.*

Proof. In 1849, Polignac conjectured the following statement: For every even natural number $2k$ there are infinitely many pairs of consecutive primes p_n, p_{n+1} such that $d_n = p_{n+1} - p_n = 2k$. For more details, see Rebenboim (1996) in p. 250. If ℓ is either even or odd, there are infinitely many pairs of consecutive primes p_n, p_{n+1} such that $p_{n+1} - p_n > \ell$. For each such prime p_n, let $n = p_n$. It follows that $\pi(n) = \pi(n+\ell)$, and therefore $p_r\pi(n) > p_{r-1}\pi(n+\ell)$. □

Proposition 3.1. *The inequality $\sigma(n) < \sigma(n-1)$ holds for infinitely many $n \in W_2$.*

Proof. Since there are infinitely many distinct primes p, q such that

$$N = \frac{pq-1}{2} > p + q.$$

Then for $n = pq$, we have

$$\sigma(n-1) \geq 1 + 2 + N + n - 1 > 1 + p + q + n = \sigma(n).$$

This completes the proof. □

Remark 3.1. Using the same idea of the proof of Proposition 3.1, we can prove the inequality $\sigma(n) < \sigma(n-1)$ for infinitely many $n \in W_k$, with $k = 3, 4, \ldots$

Lemma 3.1. *Let p, q and r be distinct primes. If $n = pqr$, then*

$$8n < \sum_{d|n,\ d<n} d^2.$$

Proof. Suppose that $n = pqr$, where $p < q < r$ and $8 < r$. Since $qr|n$ and $qr < n$, it follows for $d_t = qr$ that $d_t^2 = q^2r^2 > r(pqr) > 8n$. Then, evidently

$$8n < \sum_{d|n,\ d<n} d^2.$$

It therefore remains to verify the triplets $(2,3,5)$, $(2,3,7)$, $(2,5,7)$ and $(3,5,7)$ which are all true. □

Theorem 3.4. *If $k \geq 3$, then the inequality $\sigma_2(n) > n\tau(n) + n^2$ holds for infinitely many $n \in W_k$.*

Proof. Let $n \in W_k$ be a square-free integer, with $k \geq 3$. We have

$$\frac{\sigma_2(n)}{n\tau(n)} = \frac{n^2 + \sum\limits_{d|n,\ d<n} d^2}{n\tau(n)} = \frac{n}{\tau(n)} + \frac{\sum\limits_{d|n,\ d<n} d^2}{n\tau(n)},$$

with $\tau(n) = 2^k$. By induction on k, it suffices to prove that $\sum\limits_{d|n,\ d<n} d^2 > n.2^k$. In fact, let $k = 3$ and let $n = q_1q_2q_3$ with q_1, q_2, q_3 are distinct primes. From Lemma 3.1, we have

$$\begin{aligned} n.2^k &= q_1q_2q_3.2^3 \\ &< \sum_{d|n,\ d<n} d^2 \\ &= 1 + q_1^2 + q_2^2 + q_3^2 + (q_1q_2)^2 + (q_1q_3)^2 + (q_2q_3)^2. \end{aligned}$$

Let $k \geq 4$, and assume that the result holds for $k-1$. Let $n = q_1q_2 \ldots q_k$ with the primes $q_1, q_2, \ldots, q_k$ are distinct. Since $2q_1 \leq q_1^2$, then

$$\begin{aligned} n.2^k &= 2q_1 \left(q_2q_3 \ldots q_k\right).2^{k-1} \\ &< 2q_1 \sum_{\substack{d|q_2q_3\ldots q_k, \\ d<q_2q_3\ldots q_k}} d^2 \\ &\leq q_1^2 \sum_{\substack{d|q_2q_3\ldots q_k, \\ d<q_2q_3\ldots q_k}} d^2 \\ &< \sum_{\substack{d|q_1q_2\ldots q_k, \\ d<q_1q_2\ldots q_k}} d^2. \end{aligned}$$

This completes the proof of Theorem 3.4. □

Proposition 3.2. *The inequality*

$$\tau\left(\gamma\left(n\right)\right) \geq \gamma\left(\tau\left(n\right)\right) > \frac{p_r}{p_{r-1}}$$

holds for infinitely many $n \in W_k$.

Proof. Let $(q_1, q_2, \ldots, q_k)$ be an k-tuple of distinct primes and let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be positive integers of the form $2^a - 1$ with $a \geq 1$. For $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k}$, we have

$$\tau\left(\gamma\left(n\right)\right) = \tau\left(q_1q_2\ldots q_k\right) = 2^k,$$

and

$$\gamma\left(\tau\left(n\right)\right) = \gamma\left(\prod_{i=1}^{k}\left(1+\alpha_i\right)\right) = 2.$$

Finally, the right hand side of the inequality of Proposition 3.2 follows from Bertrand's theorem. □

Proposition 3.3. *If $k \geq 2$, then $\tau\left(\gamma\left(n\right)\right) - \gamma\left(\tau\left(n\right)\right)$ has infinitely many sign changes on the set W_k.*

Proof. From Proposition 3.2, it suffices to prove that $\tau\left(\gamma\left(n\right)\right) < \gamma\left(\tau\left(n\right)\right)$ holds for infinitely many $n \in W_k$. In fact, let $(q_1, q_2, \ldots, q_k)$ be an k-tuple of distinct primes and let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be positive integers of the form $l_i^a - 1$, for $i = 1, 2, \ldots k$ respectively, where $l_1, l_2, \ldots, l_k$ are distinct odd primes and $a \geq 1$. For $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k}$, we obtain

$$\gamma\left(\tau\left(n\right)\right) = \gamma\left(\prod_{i=1}^{k}\left(1+\alpha_i\right)\right) = l_1l_2\ldots l_k.$$

and

$$\tau(\gamma(n)) = \tau(q_1 q_2 \dots q_k) = 2^k.$$

This completes the proof of Proposition 3.3. □

Theorem 3.5. *There exist two arithmetic functions $f, g : \mathbb{N} \longrightarrow \mathbb{R}$ for which $p_r f(n) - p_{r-1} g(n)$ changes sign infinitely often on the set W_k. Moreover, if $f(1)$ and $g(1)$ are two equal positive integers, then $(p_r, p_{r-1}) = (3, 2)$ and $f(1) = g(1) = 1$.*

Proof. We shall use the n-th convergent of an irrational number (because, any irrational number can be written uniquely as an infinite simple continued fraction, see Yan (2002) in p. 44). Let $[a_0, a_1, \dots, a_n, \dots]$ be an infinite simple continued fraction. We denote the n-th convergent by $\frac{P_n}{Q_n}$, where for every $n \geq 2$

$$\begin{cases} \dfrac{P_0}{Q_0} = \dfrac{a_0}{1}, \\ \dfrac{P_1}{Q_1} = \dfrac{a_0 a_1 + 1}{a_1} \\ \quad\vdots \\ \dfrac{P_n}{Q_n} = \dfrac{a_n P_{n-1} + P_{n-2}}{a_n Q_{n-1} + Q_{n-2}}. \end{cases} \tag{6}$$

Then from Yan (2002), for every $n \geq 1$ we have

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1}.$$

Now, let p_r be the r-th prime number with $r \geq 2$. We put for all positive integer n,

$$f(n) = \frac{P_n Q_{n-1}}{p_r}, \;\; g(n) = \frac{P_{n-1} Q_n}{p_{r-1}} \;;\; n \geq 1$$

It is clear that $p_r f(n) - p_{r-1} g(n)$ changes sign infinitely often on the set W_k. Which leads to the following result: If $f(1) = g(1) \in \mathbb{N}$, then $p_r = 3$ and $f(1) = g(1) = 1$. Indeed, if $f(1) = g(1) = m$ for a certain positive integer m, it follows that

$$\frac{P_1 Q_0}{p_r} = \frac{P_0 Q_1}{p_{r-1}} = m.$$

According to equations (6), we find

$$\frac{a_0 a_1 + 1}{p_r} = \frac{a_0 a_1}{p_{r-1}} = m.$$

Therefore, $m\left(p_r - p_{r-1}\right) = 1$. Thus, we must have $m = 1$ and $p_r = 3$. That completes the proof of Theorem 3.5. □

4. Notes on the set $A_{r,s}$

In this section, we present some properties of the set $A_{r,s}$ which is defined by the following notation.

Notation 4.1. Let r, s be positive integers (parameters) with $r \geq 2$. We put

$$A_{r,s} = \left\{n \in \mathbb{N} \ ; \ p_r \varphi_s\left(n\right) > p_{r-1} n^s\right\}. \tag{7}$$

Proposition 4.1. *If $r \geq 3$, then there are no positive integers $n \in W_1$ satisfying*

$$\frac{\varphi_1\left(n\right)}{n} = \frac{p_{r-1}}{p_r}. \tag{8}$$

Proof. Let n be an odd positive integer. Because $\varphi_1\left(n\right)$ is always an even, then $p_r \varphi_1\left(n\right) \neq p_{r-1} n$. Let n be an even positive integer that satisfies (8). Since $\left(p_r, p_{r-1}\right) = 1$, then p_r divides n. Therefore, there exist two positive integers r_1 and β_1 such that

$$n = r_1 p_r^{\beta_1} \ ; \ \left(r_1, p_r\right) = 1,$$

where r_1 is even. It follows from equation (8) that

$$\frac{\varphi_1\left(r_1\right)}{r_1} = \frac{p_{r-1}}{p_r - 1}.$$

Now, for $r_1 = 2^a N$ with $a \geq 1$ and N is odd, we have

$$\frac{\varphi_1\left(r_1\right)}{r_1} = \frac{\varphi_1\left(2^a N\right)}{2^a N} = \frac{\varphi_1\left(N\right)}{2N} = \frac{p_{r-1}}{p_r - 1}.$$

Finally, using Bertrand's theorem we obtain

$$\frac{\varphi_1\left(N\right)}{N} = \frac{2p_{r-1}}{p_r - 1} > \frac{p_r}{p_r - 1} > 1.$$

Which is impossible, since $\varphi_1\left(t\right) \leq t$ for every $t \geq 1$. □

Remark 4.1. If n is an even positive integer, then $n \notin A_{r,1}$. Indeed, for $n = 2^a N$ with $\left(2, N\right) = 1$ and $a \geq 1$. It follows from Bertrand's theorem that

$$p_r \varphi_1\left(n\right) = p_r 2^{a-1} \varphi_1\left(N\right) < p_{r-1} 2^a \varphi_1\left(N\right) \leq p_{r-1} n, \tag{9}$$

which we may assure the right hand side of (9) because $\varphi_1\left(N\right) \leq N$.

Proposition 4.2. *For every $r \geq 3$, $A_{r,s}$ has an infinity of prime numbers.*

Proof. Using Bertrand's theorem, for every prime number $p \geq (p_{r-1})^{\frac{1}{s}}$, we have

$$(p_r - p_{r-1})\, p^s - p_r > 0,$$

where $p_r > 3$. It follows that

$$p_r\,(p^s - 1) > p_{r-1} p^s,$$

and therefore $p \in A_{r,s}$. As required. □

Theorem 4.1. *Suppose that*

$$\left(\frac{p_r}{d_{r-1}}\right)^{\frac{1}{s}} \geq 3, \tag{10}$$

and let $n \notin A_{r,s}$ be an odd prime power (for example, by (10), $3^m \notin A_{r,s}$ for every $m \geq 1$). Then there exists a positive integer r_0 such that $n \in A_{r+r_0,s}$ or $n \in A_{r-r_0,s}$.

Proof. For every $r' \geq 1$, assume that $n = p^m \notin A_{r \pm r',s}$ with $m \geq 1$. Then $p \in A_{i,s}$ for all $i \geq 2$. Which implies that the following inequalities

$$\frac{p_i}{p_{i-1}} \leq \frac{p^s}{p^s - 1} \leq \frac{p}{p-1} \leq \frac{3}{2}$$

hold for every $i \geq 2$. This is a contradiction, since $\frac{p_3}{p_2} = \frac{5}{3} > \frac{3}{2}$. □

Proposition 4.3. *Let $a \geq 2$ be an almost perfect number (that is, a number such that $\sigma(n) = 2n - 1$. For more details, see Guy (1994) in p .45). For all $n \in \{am\ ;\ 2 \leq m \leq a$ and $(a, m) = 1\}$, we have $n \notin A_{r,1}$.*

Proof. It is clear that for all primes $p \leq a$,

$$\left(2 - \frac{1}{a}\right)\left(\frac{p}{p-1}\right) > 2. \tag{11}$$

Let $n = am$, with $2 \leq m \leq a$. Suppose the contrary, that is, $n \in A_{r,1}$. It follows that

$$p_r > p_{r-1} \prod_{p|n} \frac{p}{p-1}.$$

Moreover, if $m = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_s^{\alpha_s}$, where $(q_i)_{i=1,2,\ldots,s}$ are distinct primes and $(a, m) = 1$, then

$$p_r > p_{r-1} \prod_{p|a} \frac{p}{p-1} \left(\frac{q_s}{q_s - 1} \right)^s .$$

On the other hand, from Theorem 2.5 and Bertrand's theorem, we get

$$2 > \left(2 - \frac{1}{a}\right) \left(\frac{q_s}{q_s - 1} \right)^s ,$$

because a is an almost perfect number. Which contradicts (11). □

Theorem 4.2. *Let $\ell \geq 2$ be a limited positive integer and let q be an unlimited prime number. If r is limited, then $\ell \in A_{r,s}$ if and only if $\ell q \in A_{r,s}$.*

Proof. From the definition of $A_{r,s}$ in (7), we see that $\ell \in A_{r,s}$ if and only if

$$p_r \prod_{p|\ell} \left(1 - \frac{1}{p^s}\right) - p_{r-1} > 0. \tag{12}$$

or, equivalently, for every positive infinitesimal ε we get

$$p_r \prod_{p|\ell} \left(1 - \frac{1}{p^s}\right) - p_{r-1} - \varepsilon > 0,$$

because ℓ and p_r are limited. In particular, for

$$\varepsilon = \frac{p_r \prod_{p|\ell} \left(1 - \frac{1}{p^s}\right)}{q^s} \simeq 0,$$

we get

$$p_r \prod_{p|\ell} \left(1 - \frac{1}{p^s}\right) - p_{r-1} - \frac{p_r \prod_{p|\ell} \left(1 - \frac{1}{p^s}\right)}{q^s} > 0. \tag{13}$$

Since φ_s is multiplicative, we have $\ell q \in A_{r,s}$.

Conversely, if $\ell q \in A_{r,s}$, it follows from (12) and (13) that $\ell \in A_{r,s}$. □

Proposition 4.4. *If n is an unlimited almost perfect number, then $n \notin A_{r,1}$.*

Proof. Let n be an unlimited almost perfect number (for example, $n = 2^t$ with t is unlimited). Suppose the contrary, that is, $n \in A_{r,1}$. It follows that

$$\frac{\sigma(n)}{n} < \frac{p_r}{p_{r-1}}.$$

There are two cases:

- In the case when p_r is unlimited, then from the Prime Number Theorem we have
$$2 \simeq 2 - \frac{1}{n} = \frac{\sigma(n)}{n} < \frac{p_r}{p_{r-1}} \simeq 1.$$
It is an impossible case.
- In the case when p_r is limited, then from Bertrand's theorem we get
$$2 - \frac{1}{n} = \frac{\sigma(n)}{n} < \frac{p_r}{p_{r-1}} \leq 2 - \frac{1}{p_{r-1}}.$$
Thus, $n \leq p_{r-1}$. Since n is unlimited, then it is also an impossible case as well.

This completes the proof. □

Proposition 4.5. *Let $2 = p_1 < p_2 < \ldots$ be the sequence of primes in increasing order, and let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$ with $s \simeq +\infty$ and $\alpha_i \geq 1$ for $i = 1, 2, \ldots, s$. Let $N \geq 2$ be a limited divisor of n, then we have $\frac{n}{N} \notin A_{r,1}$.*

Proof. Let $[x]$ denote the integer part of x. Since $\left[\prod_{p|n} \frac{p}{p-1}\right]$ is unlimited, then for every limited divisor N of n with $N \geq 2$, we get

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) \leq \frac{n}{\left[\prod_{p|n} \frac{p}{p-1}\right]} < \frac{n}{N}.$$

It follows from Bertrand's theorem that

$$p_r \varphi\left(\frac{n}{N}\right) < \frac{2p_{r-1}\frac{n}{N}}{\left[\prod_{p|\frac{n}{N}} \frac{p}{p-1}\right]} < \frac{2p_{r-1}n}{N^2} \leq \frac{p_{r-1}n}{N}.$$

This completes the proof. □

Finally, we prove the following proposition.

Proposition 4.6. *If p_r is unlimited and $s \geq 1$ is limited. Then for every limited $n \geq 2$, we have $n \notin A_{r,s}$. Furthermore, for each such integer n, the number $p_{r-1}n^s - p_r\varphi_s(n)$ is always unlimited.*

Proof. Let $n \geq 2$ be a limited positive integer. Since r is unlimited, then there exist an infinitesimal positive real number ϕ_r and an appreciable rational number $a_n(s)$ such that

$$1 + \phi_r = \frac{p_r}{p_{r-1}} < \frac{n^s}{\varphi_s(n)} = 1 + a_n(s).$$

From (7), it follows that $n \notin A_{r,s}$.

Now, for each such integer n, we assume that there exists a limited integer N_0 satisfying

$$N_0 \geq p_{r-1}n^s - p_r\varphi_s(n).$$

Therefore, we can deduce that $N_0 \geq \frac{p_{r-1}}{2}$, because

$$n^s - \frac{p_r}{p_{r-1}}\varphi_s(n) = n^s - (1 + \phi_r)\varphi_s(n) > \frac{n^s - \varphi_s(n)}{2} \geq \frac{1}{2}.$$

Which contradicts the fact that p_{r-1} is unlimited and $n^s - \varphi_s(n) \geq 1$. This completes the proof of Proposition 4.6. □

5. Conclusion

The results presented in this paper give us the solubility of certain Diophantine inequalities and equations, where our working set is a subset of positive integers which is denoted as W_k with $k \geq 1$. For further research, by the help of internal set theory we ask if $f(n) - \alpha\, g(n)$ changes sign infinitely often on a proper external subset of W_k, where f, g are two arithmetic functions and α is a fixed parameter. Similarly as in Section 4, it would be interesting to know some other properties of the set $A_{r,s}$. Namely, it is necessary to know whether $A_{r,s}$ contains Niven numbers, Smidth numbers, Woodall numbers, Cullen numbers and unlimited Fermat numbers.

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References

Bellaouar, D. and Boudaoud, A. (2015). Nonclassical study on the simultaneous rational approximation. *Malaysian J. Math. Sci*, 9(2):209–225.

De Koninck, J. M. and Mercier, A. (2004). 1001 problèmes en théorie classique des nombres. In *Number Theory*, Paris. Ellipses.

Guy, R. (1994). Unsolved problems in number theory. In *Number Theory*, New York. Springer-Verlag.

Nathanson, M. (2000). Elementary Methods in Number Theory. In *Number Theory*, New York. Springer-Verlag.

Nelson, E. (1977). Internal set theory: A new approach to non standard analysis. *Bull. Amer. Math.soc*, 83:1165–1198.

Rebenboim, P. (1996). The new book of prime number records. In *Number Theory*, New York. Springer-Verlag.

Sándor, J. (2008). On inequalities $\sigma(n) > n + \sqrt{n}$ and $\sigma(n) > n + \sqrt{n} + \sqrt[3]{n}$. *Octogon Math. Mag*, 16(1):276–278.

Sándor, J. (2014). On certain inequalities for σ, φ, ψ and related function. *Notes Numb. Theor*, 20(2):52–60.

Santos, D. (2004). Elementary number theory notes. In *Number Theory*, New York. School of Mary Washington.

Van den Berg, I. P. (1992). Extended use of IST. *Annals of Pure and Applied Logic*, 58:73–92.

Wells, D. (2005). Prime numbers, the most mysterious figures in math. In *Number Theory*, Canada. Wiley & Sons, Inc.

Yan, S. (2002). Number theory for computing. In *Number Theory*, New York. Springer Science & Business Media.