Malaysian Journal of Mathematical Sciences 10(3): 297–306 (2016)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

Counting the Number of Weakly Connected Dominating Sets of Graphs

Saeid Alikhani *, Somayeh Jahari and Mohammad Mehryar

Department of Mathematics, Yazd University, Iran

E-mail:alikhani@yazd.ac.ir
∗Corresponding author

ABSTRACT

Let $G = (V(G), E(G))$ be a simple graph. A non-empty set $S \subseteq V(G)$ is a weakly connected dominating set in G, if the subgraph obtained from G by removing all edges each joining any two vertices in $V(G) \setminus S$ is connected. In this paper, we consider some graphs and study the number of their weakly connected dominating sets.

Keywords: Dominating sets; Weakly connected; Path; Cycle.

1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph. A set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V(G)\backslash S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. There are various domination numbers in the literature (Domke et al. (2005)). For a detailed treatment of domination theory, the reader is referred to Haynes et al. (1998). A non-empty $S \subseteq V(G)$, S is called a weakly connected dominating set (w.c.d.s.) of G, if the subgraph obtained from G by removing all edges each joining any two vertices in $V(G) \setminus S$ is connected. The weakly connected domination number $\gamma_w(G)$, is defined to be the minimum integer k

with $|S| = k$ for some weakly connected dominating set S of G (see Domke et al. (2005), Koh et al. (2010)).

A dominating set with cardinality $\gamma_w(G)$ is called a γ_w*-set*. Let $\mathcal{D}_w(G,i)$ be the family of weakly connected dominating sets of a graph G with cardinality i and let $d_w(G,i) = |\mathcal{D}_w(G,i)|$. The number of dominating sets of a graph has been actively studied in recent years (Akbari et al. (2010), Alikhani et al. (2013), Alikhani and Peng (2014), Kotek et al. (2012)). In this paper, we shall count the number of weakly connected dominating sets of a graph G.

For two graphs $G = (V,E)$ and $H = (W,F)$, the corona $G \circ H$ is the graph arising from the disjoint union of G with $|V|$ copies of H, by adding edges between the ith vertex of G and all vertices of ith copy of H (Frucht and Harary (1970)). The join $G + H$ of two graph G and H with disjoint vertex sets V and W and edge sets E and F is the graph union $G \cup H$ together with all the edges joining V and W.

In the next section, we consider specific graphs and study the number of their weakly connected dominating sets. In Section 3, we consider graphs with specific construction, denoted by $G(m)$ and construct all their weakly connected dominating sets. As an example of these graphs, we study the structure of weakly connected dominating sets and the number of weakly connected dominating sets of paths. Finally, we study the number of weakly connected dominating sets of cycles in the last section.

As usual, we use $\lceil x \rceil$, $\lfloor x \rfloor$ for the smallest integer greater than or equal to x and the largest integer less than or equal to x, respectively. Also we denote the complete graph, path and cycle of order n by K_n, P_n and C_n, respectively. Also $K_{1,n}$ is the star graph with $n+1$ vertices. In this article, we denote $\{1, 2, \ldots, n\}$ simply by $[n]$.

2. Weakly connected dominating sets of specific graphs

In this section we consider specific graphs and study their weakly connected dominating sets with cardinality i, for $\gamma_w(G) \leq i \leq |V(G)|$. It is well-known and generally accepted that the problem of determining the domination number and dominating sets (and so weakly connected domination number and weakly connected dominating sets) of an arbitrary graph is difficult. Since this problem has been shown to be NP-complete (see Garey and Johnson (1979)), we shall consider in this section, specific graphs.

First we consider the complete graph K_n and the star graphs $K_{1,n}$. The number of weakly connected dominating sets of K_n and $K_{1,n}$ are easy to compute.

Theorem 2.1. *(i) For every $n \in \mathbb{N}$, and $1 \leq i \leq n$, $d_w(K_n, i) = \binom{n}{i}$.*

(ii) For every $1 \leq i \leq n-1$, $d_w(K_{1,n}, i) = \binom{n}{i-1}$.

(iii) $d_w(K_{1,n}, n) = n+1$ and $d_w(K_{1,n}, n+1) = 1$.

The following theorems give the weakly connected domination number of corona and join of two graphs:

Theorem 2.2. Sandueta and Canoy Jr. (2011) Let G be a connected graph with $|V(G)| \geq 2$ and H an arbitrary graph. Then $\gamma_w(G \circ H) = |V(G)|$.

Theorem 2.3. Sandueta and Canoy Jr. (2011) For two graphs G and H,

$$\gamma_w(G+H) = \begin{cases} 1 & \text{if } \gamma(G) = 1 \text{ or } \gamma(H) = 1; \\ 2 & \text{otherwise.} \end{cases}$$

The following theorem gives the number of w.c.d.s. of $G_1 + G_2$.

Theorem 2.4. *Let G_1 and G_2 be connected graphs of order n_1 and n_2, respectively. Then, for two natural numbers i_1, i_2, and $i \geq \gamma_w(G_1 + G_2)$,*

$$d_w(G_1 + G_2, i) = d_w(G_1, i) + d_w(G_2, i) + \sum_{i_1 + i_2 = i} \binom{n_1}{i_1}\binom{n_2}{i_2}.$$

Proof. . Let i be a natural number $1 \leq i \leq n_1 + n_2$. We want to determine $d_w(G_1 + G_2, i)$. If i_1 and i_2 are two natural numbers such that $i_1 + i_2 = i$, then clearly, for every $D_1 \subseteq V(G_1)$ and $D_2 \subseteq V(G_2)$, such that $|D_j| = i_j$, $j = 1, 2$, $D_1 \cup D_2$ is a weakly connected dominating set of $G_1 + G_2$. Moreover, if $D \in \mathcal{D}_w(G_1, i)$, then D is a weakly connected dominating set for $G_1 + G_2$ of size i. The same is true for every $D \in \mathcal{D}_w(G_2, i)$. Therefore we have the result. □

The following corollary gives the relationship between the number of w.c.d.s. of wheels W_n and cycles C_n:

Corollary 2.1. *For every* $n \geq 4$, $d_w(W_n, i) = \begin{cases} 1 & \text{if } i = 1; \\ d_w(C_{n-1}, i) + \binom{n-1}{i-1} & \text{if } i \geq 2. \end{cases}$

Proof. Since $W_n = C_{n-1} + K_1$, by Theorem 2.4 we have,

$$d_w(W_n, i) = \begin{cases} d_w(C_{n-1}, i) + 1 & \text{if } i = 1; \\ d_w(C_{n-1}, i) + \binom{n-1}{i-1} & \text{otherwise.} \end{cases}$$

Since for every $n \geq 4$, $d_w(C_{n-1}, 1) = 0$, we have the result. □

3. Weakly connected dominating sets of $G(m)$

In this section, we shall study the weakly connected dominating sets (w.c.d.s.) of specific graphs denoted by $G(m)$. As an example of graphs $G(m)$, we construct w.c.d.s. of paths and count the number of w.c.d.s. of paths.

A path is a connected graph in which two vertices have degree one and the remaining vertices have degree two. Let P_n be the path with $V(P_n) = [n]$ and $E(P_n) = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}\}$, see Figure 1.

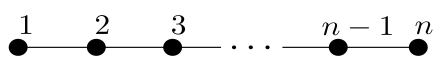

Figure 1: The path P_n with vertices labeled $[n]$.

Let P_{m+1} be a path with vertices labeled by $y_0, y_1, \dots, y_m$, for $m \geq 0$ and let v_0 be a specific vertex of a graph G. Denote by $G_{v_0}(m)$ a graph obtained from G by identifying the vertex v_0 of G with an end vertex y_0 of P_{m+1}. It is clear that if the path is glued to a different vertex v_1 of G, then the two graphs $G_{v_1}(m)$ and $G_{v_0}(m)$ may not be isomorphic. It depends on the vertex to which we glue the path. If throughout our discussion, this vertex is fixed, then we shall simply use the notation $G(m)$ (if there is no likelihood of confusion).

We need the following lemma to obtain our main results in this section:

Lemma 3.1. (i) $\mathcal{D}_w(G(m), i) = \emptyset$ *if and only if* $i > |V(G(m))|$ *or* $i < \gamma_w(G(m))$,

(ii) *If* $e \in E(G)$, *then* $\gamma_w(G - e) - 1 \leq \gamma_w(G) \leq \gamma_w(G - e)$, (see Lemańska and Raczek)

(iii) *For any* $m \in \mathbb{N}$, $\gamma_w(G(m-1)) \le \gamma_w(G(m)) \le \gamma_w(G(m-1))+1$. (by *(ii)* above).

We need the following easy lemma and theorem:

Lemma 3.2. *For every* $n \in \mathbb{N}$, $\gamma_w(P_n) = \lfloor \frac{n}{2} \rfloor$.

Theorem 3.1. *Suppose that* $G(m)$ *is the graph defined in this section. Then for every* $m \ge 0$,

$$\gamma_w(G(m)) = \begin{cases} \gamma_w(G) + \lfloor \frac{m-1}{2} \rfloor & \text{if } G \text{ has a } \gamma\text{-set containing } y_0; \\ \gamma_w(G) + \lfloor \frac{m}{2} \rfloor & \text{otherwise.} \end{cases}$$

Proof. If y_0 is in the γ_w-set of G, then to obtain the γ_w-set of $G(m)$ it suffices to dominate the path with vertices $\{y_2, \dots, y_m\}$, otherwise we dominate the path with vertices $\{y_1, y_2, \dots, y_m\}$. Therefore; by Lemma 3.2, the proof is complete. □

To enumerate the weakly connected dominating set of $G(m)$ with cardinality i, no need to consider w.c.d.s. of $G(m-3)$ with cardinality $i-1$. Therefore, we only need to consider w.c.d.s. in $G(m-1)$ and $G(m-2)$ with cardinality $i-1$. The families of these weakly connected dominating sets (w.c.d.s.) can be empty or otherwise. Thus, we have four cases of whether these two families are empty or not. We do not need to consider the case that $\mathcal{D}_w(G(m-1), i-1) = \mathcal{D}_w(G(m-2), i-1) = \emptyset$, because it implies $\mathcal{D}_w(G(m), i) = \emptyset$. Also the case $\mathcal{D}_w(G(m-1), i-1) \neq \emptyset$, $\mathcal{D}_w(G(m-2), i-1) = \emptyset$ does not exists. Thus, we only need to consider two cases. We consider these cases in Theorem 3.2 which construct the w.c.d.s. of $G(m)$.

Theorem 3.2. (i) *If* $\mathcal{D}_w(G(m-1), i-1) = \emptyset$ *and* $\mathcal{D}_w(G(m-2), i-1) \neq \emptyset$, *then* $\mathcal{D}_w(G(m), i) = \Big\{ \{y_{m-1}\} \cup X \mid X \in \mathcal{D}_w(G(m-2), i-1) \Big\}$,

(ii) *If* $\mathcal{D}_w(G(m-2), i-1) \neq \emptyset$, $\mathcal{D}_w(G(m-1), i-1) \neq \emptyset$, *then* $\mathcal{D}_w(G(m), i) = \Big\{ \{y_m\} \cup X_1, \{y_{m-1}\} \cup X_2 \mid X_1 \in \mathcal{D}_w(G(m-1), i-1), X_2 \in \mathcal{D}_w(G(m-2), i-1) \Big\}$

Proof. (i) Obviously $\Big\{ \{y_{m-1}\} \cup X \mid X \in \mathcal{D}_w(G(m-2), i-1) \Big\} \subseteq \mathcal{D}_w(G(m), i)$. Now suppose that $Y \in \mathcal{D}_w(G(m), i)$. Then at least one of the vertices y_m or y_{m-1} is in Y. If $y_m \in Y$ then at least one of the vertices y_{m-1}

or y_{m-2} is in Y. If $y_{m-1} \in Y$, then $Y - \{y_m\} \in \mathcal{D}_w(G(m-1), i-1)$ a contradiction. So $y_{m-2} \in Y$ and $Y - \{y_{m-1}\} \in \mathcal{D}_w(G(m-2), i-1)$. Therefore $\mathcal{D}_w(G(m), i) \subseteq \Big\{\{y_{m-1}\} \cup X \,|\, X \in \mathcal{D}_w(G(m-2), i-1)\Big\}$.

(ii) Obviously $\Big\{ \{y_m\} \cup X_1, \{y_{m-1}\} \cup X_2 \,|\, X_1 \in \mathcal{D}_w(G(m-1), i-1), X_2 \in \mathcal{D}_w(G(m-2), i-1)\Big\} \subseteq \mathcal{D}_w(G(m), i)$.

Now, let $Y \in \mathcal{D}_w(G(m), i)$, then $y_m \in Y$ or $y_{m-1} \in Y$. If $y_m \in Y$, then at least one vertex labeled y_{m-1} or y_{m-2} is in Y. If $y_{m-1} \in Y$, then $Y = X \cup \{y_m\}$ for some $X \in \mathcal{D}(G(m-1), i-1)$. If $y_{m-2} \in Y$, then $Y = X \cup \{y_{m-1}\}$ for some $X \in \mathcal{D}(G(m-2), i-1)$. So we have the result. □

Theorem 3.3. *For every $m \geq 2$,*

$$d_w(G(m), i) = d_w(G(m-1), i-1) + d_w(G(m-2), i-1).$$

Proof. It follows from Theorem 3.2. □

Since $P_n = P_1(n-1)$, we can apply the results for the graph $G(m)$ to obtain some properties of w.c.d.s. and their numbers for paths. We denote $\mathcal{D}_w(P_n, i)$ simply by $\mathcal{P}_n^i$.

For the construction of $\mathcal{P}_n^i$, by Theorem 3.2, we only need to consider two families $\mathcal{P}_{n-1}^{i-1}$ and $\mathcal{P}_{n-2}^{i-1}$.

Theorem 3.4. *For every $n \geq 3$ and $i \geq \lfloor \frac{n}{2} \rfloor$,*

(i) If $\mathcal{P}_{n-1}^{i-1} = \emptyset$ and $\mathcal{P}_{n-2}^{i-1} \neq \emptyset$, then $\mathcal{P}_n^i = \Big\{X \cup \{n-1\} |\, X \in \mathcal{P}_{n-2}^{i-1}\Big\}$.

(ii) If $\mathcal{P}_{n-1}^{i-1} \neq \emptyset$ and $\mathcal{P}_{n-2}^{i-1} \neq \emptyset$, then

$$\mathcal{P}_n^i = \Big\{\{n\} \cup X_1, \{n-1\} \cup X_2 \,|\, X_1 \in \mathcal{P}_{n-1}^{i-1}, X_2 \in \mathcal{P}_{n-2}^{i-1}\Big\}.$$

Proof. It follows from Theorem 3.2. □

The following theorem gives a recurrence for the number of w.c.d.s. of P_n.

Theorem 3.5. *For every $n \geq 3$ and $\lfloor \frac{n}{2} \rfloor \leq i \leq n$, $d_w(P_n, i) = d_w(P_{n-1}, i-1) + d_w(P_{n-2}, i-1)$, with initial values $d_w(P_1, 1) = 1$, $d_w(P_2, 1) = 2$ and $d_w(P_2, 2) = 1$.*

Table 1: $d_w(P_n, j)$, the number of w.c.d.s. of P_n with cardinality j.

j	1	2	3	4	5	6	7	8	9	10
$d_w(P_1, j)$	1									
$d_w(P_2, j)$	2	1								
$d_w(P_3, j)$	1	3	1							
$d_w(P_4, j)$		3	4	1						
$d_w(P_5, j)$		1	6	5	1					
$d_w(P_6, j)$			4	10	6	1				
$d_w(P_7, j)$			1	10	15	7	1			
$d_w(P_8, j)$				5	20	21	8	1		
$d_w(P_9, j)$				1	15	35	28	9	1	
$d_w(P_{10}, j)$					6	35	56	36	10	1

Proof. It follows from Theorem 3.3. □

Using Theorem 3.5 , we obtain $d_w(P_n, j) = |\mathcal{P}_n^j|$ for $1 \leq n \leq 10$ in Table 1.

Here, we shall solve the recurrence relation with two variables for $d_w(P_n, j)$ in Theorem 3.5. Corresponding to this recurrence relation, we state an elementary combinatorial problem.

Suppose that we have n boxes in the row and j objects. We want to count the number of permutations of these items in boxes such that there is at most one object in each box and no two adjacent boxes can be empty. It is easy to see that the answer of this problem is $\binom{j+1}{n-j}$. We can see that if $a_{n,j}$ is the solution of this problem, then we have the following recurrence relation with these initial values $a_{1,1} = 1$, $a_{2,1} = 2$ and $a_{2,2} = 1$:

$$a_{n,j} = a_{n-1,j-1} + a_{n-2,j-1}.$$

So we have the following result:

Theorem 3.6. *For every $n \in \mathbb{N}$ and $\lfloor \frac{n}{2} \rfloor \leq j \leq n$, $d_w(P_n, j) = \binom{j+1}{n-j}$.*

4. Weakly connected dominating sets of C_n

Let $C_n, n \geq 3$, be the cycle with n vertices $V(C_n) = [n]$ and $E(C_n) = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}, \{n, 1\}\}$, see Figure 2. In this section, we consider the number of weakly dominating sets of cycle C_n. Using Maple programme, we obtain $d_w(C_n, j) = |\mathcal{C}_n^j|$ for $1 \leq n \leq 14$ in Table 2.

Lemma 4.1. *For every $n \in \mathbb{N}$, $\gamma_w(C_n) = \lfloor \frac{n}{2} \rfloor$.*

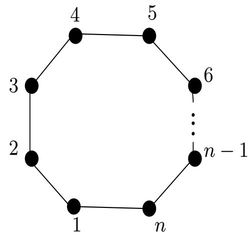

Figure 2: The cycle C_n with vertices labelled $[n]$.

Table 2: $d_w(C_n, j)$, the number of weakly connected dominating sets of C_n with cardinality j.

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$d_w(C_1, j)$	1													
$d_w(C_2, j)$	2	1												
$d_w(C_3, j)$	3	3	1											
$d_w(C_4, j)$		6	4	1										
$d_w(C_5, j)$		5	10	5	1									
$d_w(C_6, j)$			14	15	6	1								
$d_w(C_7, j)$			7	28	21	7	1							
$d_w(C_8, j)$				26	48	28	8	1						
$d_w(C_9, j)$				9	63	75	36	9	1					
$d_w(C_{10}, j)$					42	125	110	45	10	1				
$d_w(C_{11}, j)$					11	121	220	154	55	11	1			
$d_w(C_{12}, j)$						62	276	357	208	66	12	1		
$d_w(C_{13}, j)$						13	208	546	546	273	78	13	1	
$d_w(C_{14}, j)$							86	539	980	798	350	91	14	1

Proof. We consider graph C_n with the vertex set $V(C_n) = \{v_1, v_2, \dots, v_n\}$ and the edge set $E(C_n) = \{\{v_{i-1}, v_i\} \mid 1 \le i \le n \; and \; v_0 = v_n\}$. we have $\gamma_w(C_n) = 1$ for $n \le 3$. Now assume $n \ge 4$ and use induction on n. Suppose that S is a minimum weakly dominating set of C_n and consider the vertex $v_i \in S$. Since $N(v_i) = \{v_{i-1}, v_{i+1}\}$ and $C_n - \{v_i, v_{i-1}\}$ is a path with $n-2$ vertices, by induction, $\gamma_w(C_n) = 1 + \gamma_w(P_{n-2}) = 1 + \lfloor \frac{n-2}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$. □

The following theorem gives the number of w.c.d.s. of C_n with cardinality $n - 3 \le i \le n$.

Lemma 4.2. *(i) For every $n \ge 4$, and $n-2 \le i \le n$, $d_w(C_n, i) = \binom{n}{i}$.*

(ii) For every $n \ge 6$, $d_w(C_n, n-3) = \frac{(n+1)n(n-4)}{6}$.

Proof. (i) Since $d_w(C_n, n) = 1$ and $d_w(C_n, n-1) = n$, so the result is true for $i \in \{n-1, n\}$. We prove $d_w(C_n, n-2) = \binom{n}{n-2}$. Clearly for every $S \in \mathcal{D}_w(C_{n-1}, n-2)$ we have $S \in \mathcal{D}_w(C_n, n-2)$. Also if $S_1 \in \mathcal{D}_w(C_{n-1}, n-3)$, then $S_1 \cup \{n\} \in \mathcal{D}_w(C_n, n-2)$. On the other hand, if $S \in \mathcal{D}_w(C_n, n-2)$, and $n \in S$, then $S \setminus \{n\} \in \mathcal{D}_w(C_{n-1}, n-3)$. If $n \notin S$, then $S \in \mathcal{D}_w(C_{n-1}, n-2)$. Therefore

$$d_w(C_n, n-2) = d_w(C_{n-1}, n-3) + d_w(C_{n-1}, n-2).$$

Now using induction, we have,

$$d_w(C_n, n-2) = \binom{n}{n-2}.$$

(ii) First we prove that,

$$d_w(C_n, n-3) = d_w(C_{n-1}, n-4) + d_w(C_{n-1}, n-3) - 1. \quad (1)$$

Clearly every w.c.d.s. S of C_n with cardinality $n-3$ is a w.c.d.s. of C_{n-1} with cardinality $n-3$, except for the following cases:

- the set S contains vertices 1 and 2,
- the set S contains vertices 1 and $n-1$,
- the set S contains vertices $n-1$ and $n-2$.

Also it is easy to see that if $S \in \mathcal{D}_w(C_{n-1}, n-4)$ or S is any of the sets $\{2, \cdots, n-3\}$ and $\{3, \cdots, n-2\}$, then $S \cup \{n\}$ is a w.c.d.s. of size $n-3$ in cycle C_n. Consequently

$$\begin{aligned} d_w(C_n, n-3) &= d_w(C_{n-1}, n-4) + 2 + d_w(C_{n-1}, n-3) - 3 \\ &= d_w(C_{n-1}, n-4) + d_w(C_{n-1}, n-3) - 1. \end{aligned}$$

Using equation (1), we have,

$$d_w(C_n, n-3) = \sum_{i=5}^{n} \frac{(i-1)(i-2)}{2} - (n-4),$$

and by easy computation we have the result. □

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