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On Primitive 11-Centralizer Groups of Odd Order

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ABSTRACT

Let G be a finite group and let $|Cent(G)|$ be the number of distinct centralizers of its elements. G is called n-centralizer if $|Cent(G)| = n$ and is called primitive n-centralizer if $|Cent(G)| = |Cent(\frac{G}{Z(G)})| = n$. In this paper, we characterize all primitive 11-centralizer groups of odd order.

Keywords: Covering, n-centralizer group, Primitive n-centralizer group, Odd.

1. Introduction

In this paper, all groups are finite and all notations are standard. For example $\mathbb{Z}_n$ denotes the cyclic group of order n, $Z(G)$ denotes the center of a group G, D_{2n} denotes the dihedral group of order $2n$ and $K \rtimes H$ denotes the semidirect product of K and H with normal subgroup K. A finite group G is said to be an AC-group if $C_G(x)$ is abelian for all $x \in G \setminus Z(G)$. It was shown in Baishya (2013) that if $|\frac{G}{Z(G)}| = pq$ or pqr where p, q, r are primes not necessarily distinct, then G is an AC-group. We recall that a group G is called capable if there exists a group H such that $G \cong \frac{H}{Z(H)}$. Given a group G, let $Cent(G)$ be the set of centralizers of elements of G, i.e., $Cent(G) := \{C_G(x) | x \in G\}$, where $C_G(x)$ is the centralizer of the element x in G. A finite group G is called *n-centralizer* if $|Cent(G)| = n$ and *primitive n-centralizer* if $|Cent(G)| = |Cent(\frac{G}{Z(G)})| = n$. Obviously a finite group G is 1-centralizer if and only if it is abelian. It was shown in Belcastro and Sherman (1994) that there is no finite n-centralizer group for $n \in \{2, 3\}$. Also all finite n-centralizer groups for $n \in \{4, 5\}$ were classified. As a simple result, we can see that there is no finite primitive 4-centralizer group. Moreover a finite group G is primitive 5-centralizer if and only if $\frac{G}{Z(G)} \cong S_3$. In Ashrafi (2000a), all finite 6-centralizer groups were studied. Also it was shown in Ashrafi (2000b) that if G is primitive 6-centralizer, then $\frac{G}{Z(G)} \cong A_4$. In Abdollahi et al. (2007), all n-centralizer groups were characterized for $n \in \{7, 8\}$ and it was shown that there is no finite primitive 8-centralizer group. In Ashrafi and Taeri (2006), the structure of finite primitive 7-centralizer groups were verified. Also in Foruzanfar and Mostaghim (2015), finite primitive 9-centralizer groups were classified and it was shown that if G is primitive 9-centralizer, then $\frac{G}{Z(G)} \cong D_{14}$, $T_{21} = < a, b \,|\, a^3 = b^7 = 1, ba = ab^2 >$ or $< a, b \,|\, a^6 = b^7 = 1, a^{-1}ba = b^3 >$. Finally in Foruzanfar and Mostaghim (2014), it was proved that there is no finite 10-centralizer group of odd order. The purpose of this paper is to classify all primitive 11-centralizer groups of odd order.

Theorem 1.1. *If G is a primitive 11-centralizer group of odd order, then $\frac{G}{Z(G)}$ is isomorphic to $(\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$.*

2. Preliminary Results

A *cover* for a group G is a collection $\mathcal{H}$ of proper subgroups of G such that $G = \cup_{H \in \mathcal{H}} H$. A cover with n members is called *n-cover*, for a natural number n. Also it is called *irredundant* if no proper sub-collection is also a cover and is called a *partition* with *kernel* K if the intersection of pairwise members of the cover is K. Neumann (1954) proved that for a group G with an irredundant n-cover, the index of the intersection of the cover in G is bounded by a function of n and Tomkinson (1987) improved that bound. Let $f(n)$ be the largest index $|G : D|$, where G is a group with an irredundant n-cover whose intersection of its members is D. Scorza (1926) was the first who posed a question on finite covers. He settled the question which groups are the union of three proper subgroups and proved $f(3) = 4$. Furthermore, Greco (1957), Bryce et al. (1997), Abdollahi et al. (2005) and Abdollahi and Jafarian Amiri (2007) obtained the values of $f(n)$, for $n = 4, 5, 6, 7$, respectively and they proved that $f(4) = 9$, $f(5) = 16$, $f(6) = 36$ and $f(7) = 81$. In order to prove Theorem 1.1, we first present some lemmas and propositions that will be used in proof of it.

Lemma 2.1. *(Theorem 6 of Belcastro and Sherman (1994)) Let p be a prime. If* $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, *then* $|Cent(G)| = p + 2$.

Corollary 2.1. *(Corollary 2.5 of Baishya (2013)) Let* $p \geq q$ *be primes. If* $|\frac{G}{Z(G)}| = pq$, *then* $|Cent(G)| = p + 2$.

Lemma 2.2. *(Lemma 2.6 of Foruzanfar and Mostaghim (2015)) Let G be a finite non-abelian group and* $|G| = p^2q$ *where p and q are distinct prime numbers.*
a) If $p < q$, *then* $G = A_4$ *or* $|Cent(G)| = q + 2$.
b) If $p > q$, *then* $|Cent(G)| = p + 2$ *or* $p^2 + 2$.

Lemma 2.3. *(Lemma 2.1 of Baishya (2013)) Let* $|\frac{G}{Z(G)}| = pqr$, *where p, q, r are primes not necessarily distinct. Then G is an AC-group.*

Lemma 2.4. *(Lemma 3.3 of Tomkinson (1987)) Let M be a proper subgroup of the finite group G and let* H_1,H_2,...,H_k *be subgroups of G with* $|G : H_i| = \beta_i$ *and* $\beta_1 \leq ... \leq \beta_k$. *If* $G = M \cup H_1 \cup ... \cup H_k$, *then* $\beta_1 \leq k$. *Furthermore, if* $\beta_1 = k$, *then* $\beta_1 = \beta_2 = ... = \beta_k = k$ *and* $H_i \cap H_j \leq M$ *for any two distinct i and j.*

Definition 2.1. *A non-empty subset* $X = \{x_1, ..., x_r\}$ *of a finite group G is called a set of pairwise non − commuting elements if* $x_ix_j \neq x_jx_i$ *for all distinct* $i, j \in \{1, ..., r\}$. *A set of pairwise non-commuting elements of G is said to have maximal size if its cardinality is the largest one among all such sets.*

Remark 2.1. *Let G be a finite group and* $\{x_1, ..., x_r\}$ *be a set of pairwise non-commuting elements of G having maximal size. Then*
(1) $\{C_G(x_i)|i = 1, ..., r\}$ *is an irredundant r-cover with the intersection* $Z(G) = \cap_{i=1}^{r} C_G(x_i)$ *(see Theorem 5.1 of Tomkinson (1987)).*
(2) $|\frac{G}{Z(G)}| \leq f(r)$ *(see Corollary 5.2 of Tomkinson (1987)).*
(3) $f(3) = 4$, $f(4) = 9$, $f(5) = 16$, $f(6) = 36$ *and* $f(7) = 81$ *(see Scorza (1926),*

Greco (1957), Bryce et al. (1997), Abdollahi et al. (2005) and Abdollahi and Jafarian Amiri (2007), respectively).
(4) *Let G be a group such that every proper centralizer in G is abelian. Then for all $a, b \in G \setminus Z(G)$ either $C_G(a) = C_G(b)$ or $C_G(a) \cap C_G(b) = Z(G)$.*
If $z \in (C_G(a) \cap C_G(b)) \setminus Z(G)$, then $C_G(z)$ contains both $C_G(a)$ and $C_G(b)$, since $C_G(a)$ and $C_G(b)$ are abelian. Since z is not in $Z(G)$, $C_G(z) \leq C_G(a)$ and $C_G(z) \leq C_G(b)$. Thus $C_G(z) = C_G(a) = C_G(b)$. Hence, in such a group G, $\{C_G(x) \,|\, x \in G \setminus Z(G)\}$ forms a partition with kernel $Z(G)$. It follows that $\{\frac{C_G(x)}{Z(G)} \,|\, x \in G \setminus Z(G)\}$ forms a partition whose kernel is the trivial subgroup (see also Proposition 1.2 of Ito (1953)).

Lemma 2.5. *(Lemma 2.4 of Abdollahi et al. (2007)) Let G be a finite non-abelian group and $\{x_1, ..., x_r\}$ be a set of pairwise non-commuting elements of G with maximal size. Then*
(1) $r \geq 3$.
(2) $r + 1 \leq \ |Cent(G)|$.
(3) $r = 3$ *if and only if* $|Cent(G)| = 4$.
(4) $r = 4$ *if and only if* $|Cent(G)| = 5$.

Proposition 2.1. *(Proposition 2.5 of Abdollahi et al. (2007)) Let G be a finite group and let $X = \{x_1, ..., x_r\}$ be a set of pairwise non-commuting elements of G having maximal size.*
(a) If $|Cent(G)| < r + 4$, then
(1) *For each element $x \in G$, $C_G(x)$ is abelian if and only if $C_G(x) = C_G(x_i)$ for some $i \in \{1, ..., r\}$.*
(2) *If $C_G(x_i)$ is a maximal subgroup of G for some $i \in \{1, ..., r\}$, then $Z(G) = C_G(x_i) \cap C_G(x_j)$ for all $j \in \{1, ..., r\} \setminus \{i\}$. In particular, if $|G : C_G(x_1)| \leq |G : C_G(x_2)| \leq 2$, then $|Cent(G)| = 4$, and if $|G : C_G(x_1)| \leq |G : C_G(x_2)| = 3$, then $|Cent(G)| = 5$.*
(b) If $|Cent(G)| = r+2$, then there exists a proper non-abelian centralizer $C_G(x)$ which contains $C_G(x_{i_1})$, $C_G(x_{i_2})$ and $C_G(x_{i_3})$ for three distinct $i_1, i_2, i_3 \in \{1, ..., r\}$.
(c) If $|Cent(G)| = r + 3$, then there exists a proper non-abelian centralizer $C_G(x)$ which contains $C_G(x_{i_1})$ and $C_G(x_{i_2})$ for two distinct $i_1, i_2 \in \{1, ..., r\}$.

Lemma 2.6. *(Lemma 2.6 of Abdollahi et al. (2007)) Let G be a finite non-abelian group. Then every proper centralizer of G is abelian if and only if $|Cent(G)| = r + 1$, where r is the maximal size of a set of pairwise non-commuting elements of G.*

Theorem 2.1. *(Theorem 4.2 of Tomkinson (1987)) Suppose that G is covered by n abelian subgroups $A_1, A_2, \cdots, A_n$, then:*
(i) If $G = \langle A_1, A_2 \rangle$, then $|\frac{G}{Z(G)}| \leq (n-1)^2$.
(ii) If $\langle A_1, A_2 \rangle < G$, then $|\frac{G}{Z(G)}| \leq 2(n-2)^{\log_2(n-2)}$.

Theorem 2.2. *(Theorem 1 of Cohn (1994)) Suppose that* $H_n \leq H_{n-1} \leq \cdots \leq H_1$ *are proper subgroups of a group* G. *If* $G = \cup_{r=1}^{n} H_r$ *assumed to be maximal, then* $|G| \leq \sum_{r=2}^{n} |H_r|$, *with equality if and only if* (a) $H_1 H_r = G$; $r \neq 1$ *and* (b) $H_r \cap H_s \subset H_1$; $r \neq s$.

3. The Proof of Theorem 1.1

Let G be a finite primitive 11-centralizer group of odd order. Let $\{x_1, \cdots x_r\}$ be a set of pairwise non-commuting elements of G having maximal size. Then $X_i = C_G(x_i), 1 \leq i \leq r$ is an irredundant r-cover with intersection $Z(G)$. Assume that $|G : X_i| = \alpha_i$, where $\alpha_1 \leq \cdots \leq \alpha_r$. Since G is a primitive 11-centralizer group, by Lemma 2.5, we have $5 \leq r \leq 10$. Then we first have the following lemma.

Lemma 3.1. *With the above notations we have* $r \neq 5, 6$.

Proof. First, we show that $r \neq 5$. Suppose, for a contradiction, that $r = 5$. By Remark 2.1, $|\frac{G}{Z(G)}| \leq 16$. Now Lemma 2.4 implies that $\alpha_2 \leq 4$. Since $|G|$ is odd, we have $\alpha_2 = 3$ and $|\frac{G}{Z(G)}| \in \{9, 15\}$. Therefore $\frac{G}{Z(G)}$ is abelian, which is not possible. It implies that $r \neq 5$.
Now we show that $r \neq 6$. Suppose, for a contradiction, that $r = 6$. So by Remark 2.1, $|\frac{G}{Z(G)}| \leq 36$. Now Lemma 2.4 implies that $\alpha_2 \leq 5$. Suppose that $\alpha_2 = 3$. Since 3 is a divisor of $|\frac{G}{Z(G)}|$, we have $|\frac{G}{Z(G)}| \in \{9, 15, 21, 27, 33\}$. Therefore G is an AC-group and by Lemma 2.6, $|Cent(G)| = 7$, a contradiction. Now suppose that $\alpha_2 = 5$. Since 5 is a divisor of $|\frac{G}{Z(G)}|$, we have $|\frac{G}{Z(G)}| \in \{15, 25, 35\}$. Therefore $\frac{G}{Z(G)}$ is abelian, which is not possible. It implies that $r \neq 6$. □

Now by Lemma 3.1, it is enough to investigate four cases that will be verified separately.

Case 1: $r = 7$. By Remark 2.1, $|\frac{G}{Z(G)}| \leq 81$. Also Lemma 2.4 implies that $\alpha_2 \leq 6$. Since $|G|$ is odd, then $\alpha_2 = 3$ or 5. If $\alpha_2 = 3$, then $|\frac{G}{Z(G)}| \in \{9, 15, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75, 81\}$. If $|\frac{G}{Z(G)}| \in \{9, 15, 21, 27, 33, 39, 45, 51, 57, 63, 69, 75\}$, then G is an AC-group and by Lemma 2.6, $|Cent(G)| = 8$, a contradiction. Now suppose that $|\frac{G}{Z(G)}| = 81$. By The GAP Group (2013), we conclude that there are four 11-centralizer groups of order 81 and the only capable group between them is $(\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$. Therefore we have $\frac{G}{Z(G)} \cong$

$(\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$. Now suppose that $\alpha_2 = 5$. So $|\frac{G}{Z(G)}| \in \{15, 25, 35, 45, 55, 65, 75\}$. It implies that G is an AC-group and by Lemma 2.6, $|Cent(G)| = 8$, a contradiction.

Case 2: $r = 8$. In this case $\alpha_2 \leq 7$. If $\alpha_2 = 3$, then by Proposition 2.1, $|Cent(G)| = 5$, a contradiction. If $\alpha_2 = 5$, then $\alpha_1 = 3$ or 5. If $\alpha_1 = 3$, then $G = X_1X_2$ and by Proposition 2.1, $X_1 \cap X_2 = Z(G)$. Hence, $|\frac{G}{Z(G)}| = 15$ which is not possible. If $\alpha_1 = 5$, then $|\frac{G}{Z(G)}| \leq 25$. Now since 5 is a divisor of $|\frac{G}{Z(G)}|$, $|\frac{G}{Z(G)}| \in \{15, 25\}$, which is again a contradiction. Finally, suppose that $\alpha_2 = 7$. By Lemma 2.4, $\alpha_2 = \alpha_3 = \cdots = \alpha_8 = 7$ and so $|G| = \sum_{i=2}^{8} |X_i|$. Also $G = X_1X_2$ by Theorem 2.2 and $X_1 \cap X_2 = Z(G)$ by Proposition 2.1. Since $|G|$ is odd, $\alpha_1 = 3, 5$ or 7. If $\alpha_1 = 3$, then $|\frac{G}{Z(G)}| = 21$, a contradiction. If $\alpha_1 = 5$, then $|\frac{G}{Z(G)}| = 35$, which is not possible. If $\alpha_1 = 7$, then $|\frac{G}{Z(G)}| = 49$ and $\frac{G}{Z(G)}$ is abelian, which is a contradiction.

Case 3: $r = 9$. In this case by Lemma 2.4, $\alpha_2 \leq 8$. If $\alpha_2 = 3$, by Proposition 2.1, $|Cent(G)| = 5$, a contradiction. If $\alpha_2 = 5$, then $\alpha_1 = 3$ or 5. If $\alpha_1 = 3$, then $G = X_1X_2$ and by Proposition 2.1, $X_1 \cap X_2 = Z(G)$. Hence, $|\frac{G}{Z(G)}| = 15$, which is not possible. If $\alpha_1 = 5$, then $|\frac{G}{Z(G)}| \leq 25$. Now since 5 is a divisor of $|\frac{G}{Z(G)}|$, $|\frac{G}{Z(G)}| \in \{15, 25\}$, which is again a contradiction. Finally, suppose that $\alpha_2 = 7$. Therefore $\alpha_1 = 3, 5$ or 7. If $\alpha_1 = 3$, then $G = X_1X_2$ and since $X_1 \cap X_2 = Z(G)$, therefore $|\frac{G}{Z(G)}| = 21$, which is a contradiction. Similarly, If $\alpha_1 = 5$, then $|\frac{G}{Z(G)}| = 35$, a contradiction. Finally, if $\alpha_1 = 7$, then $G = \langle X_1, X_2 \rangle$. Therefore by Theorem 2.1, $|\frac{G}{Z(G)}| \leq 64$. Since $|G|$ is odd and G is non-abelian, $|\frac{G}{Z(G)}| \in \{21, 35, 49, 63\}$. It implies that G is an AC-group and by Lemma 2.6, $|Cent(G)| = 10$, a contradiction.

Case 4: $r = 10$. In this case by Lemma 2.6, every proper centralizer of G is abelian and by Lemma 2.4, we obtain that $\alpha_2 \leq 9$.
If $\alpha_2 = 3$, then $|Cent(G)| = 5$ by Proposition 2.1, a contradiction. Now suppose that $\alpha_2 = 5$, then $|\frac{G}{Z(G)}| \leq 25$ and since 5 is a divisor of $|\frac{G}{Z(G)}|$, then $|\frac{G}{Z(G)}| \in \{15, 25\}$. Therefore $\frac{G}{Z(G)}$ is abelian, which is not possible. If $\alpha_2 = 7$, then $|\frac{G}{Z(G)}| \leq 49$ and since 7 is a divisor of $|\frac{G}{Z(G)}|$, then $|\frac{G}{Z(G)}| \in \{21, 35, 49\}$. If $|\frac{G}{Z(G)}| = 21$, then $\frac{G}{Z(G)} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$ and $|Cent(G)| = 9$ by Corollary 2.1, a contradiction. Now if $|\frac{G}{Z(G)}| \in \{35, 49\}$, then $\frac{G}{Z(G)}$ is abelian, which is not possible. Finally suppose that $\alpha_2 = 9$, then $|\frac{G}{Z(G)}| \leq 81$ and since 9 is a divisor of $|\frac{G}{Z(G)}|$, then $|\frac{G}{Z(G)}| \in \{9, 27, 45, 63, 81\}$. Suppose that $|\frac{G}{Z(G)}| = 27$. Since center of every group of order 27 is of order 3, Lemma 2.1 implies that

$|Cent(\frac{G}{Z(G)})| = 5$, a contradiction. If $|\frac{G}{Z(G)}| = 45$, then by Lemma 2.2, $\frac{G}{Z(G)} \cong A_4$ or $|Cent(\frac{G}{Z(G)})| = 7$, a contradiction. Now suppose that $|\frac{G}{Z(G)}| = 63$. Then by Lemma 2.2, $\frac{G}{Z(G)} \cong A_4$ or $|Cent(\frac{G}{Z(G)})| = 9$, which is not possible. Finally, if $|\frac{G}{Z(G)}| = 81$, then $\frac{G}{Z(G)} \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$.
Now the proof of Theorem 1.1 is complete.

4. Conclusion

In this paper, we have studied all finite primitive 11-centralizer groups G of odd order. We have considered centralizers of a set of pairwise non-commuting elements of G of maximal size as an irredundant r-cover and have shown that the quotient group $\frac{G}{Z(G)}$ is isomorphic to $(\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$. We propose as two open problems to verify the cases when G is a 11-centralizer group or G is a primitive 11-centralizer group of even order.

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