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Dirichlet's Theorem Related Prime Gap Statistics.

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ABSTRACT

In this paper, we investigate a special prime gap problem which results from Dirichlet's theorem and relate it to the twin prime conjecture. Then we numerically study the density of several classes of the prime gap that not exceeding a natural number N. A few open problems are given.

1. Introduction

Dirichlet's theorem (Rosen, 2011) stated that there are infinite many prime in any arithmetic progression $an + b$, for any fixed a, b, while Green-Tao theorem (Green and Tao, 2008) stated that there are arbitrary long arithmetic progression in primes. The common problems for prime gaps are the first occurrence of large gaps Brent (1973), Nicely (1999), Nyman and Nicely (2003) and the frequency distribution of prime gaps (Brent, 1974). If the frequency of gap with distance 2 has a nonzero measure, then this simply implies the twin primes conjecture is true. In this paper, we suggest an investigation on a special prime gap problem which result from Dirichlet's theorem and we relate it to twin prime conjecture. It is obvious that each pair of twin prime will take the form of $(6k + 5, 6k + 1)$. According to Dirichlet's theorem, there are infinitely many primes are in the form of either 6k+5 or 6k+1. We are going to prove that there are also infinitely many pairs of consecutive prime in the form of $(6k+5, 6k+1)$. Therefore we would like to ask the converse: given the set of consecutive primes $(6k + 5, 6k + 1)$, how many of them are twin primes?

2. Basic Setting

Let P denotes the sequences of prime numbers $\{P_1, P_2, P_3, ...\}$. Let $CP = \{(P_i, P_i + 1)\}$ be the set of all consecutive prime pairs and the subset of CP, $D = \{(P_i, P_i + 1 \in CP | P_i = 6k_1 + 5 \text{and} P_{i+1} = 6k_2 + 1)\}$. Obviously, any P_i cannot be in two distinct pairs in D.

Theorem 2.1. *There are infinitely many consecutive primes* $(P_i, P_i + 1) \in D$.

We prove this by contradiction. Assume there are finite consecutive primes$(P_i, P_i + 1) \in D$. Let (P_n, P_{n+1}) be the largest one and we consider following primes after P_n.

Case 1: Assume $P_{k>n} = 6m + 5$ exists. Since (P_n, P_{n+1}) is the largest pair, therefore any $P_{i>n}$ must be in the form of $6k + 5$, due to the existence of the new $6k + 1$ would lead to another larger prime pairs. This contradicts with (P_n, P_{n+1}) is the largest one .

Case 2. $P_{k>n} = 6m + 5$ doesn't exist. It contradicts with Dirichlet's theorem.

From the set of D we could define subsets as follows:

$$D_i = \{(P_k, P_{k+1} \in D | P_{k+1} - P_k = i.)\}$$

Note that i could only take certain positive integers in the form of $6k + 2$ i.e. $2, 8, 14, 20, ...$ Clearly $D_2 \subset D$ is the set of all twin primes.

In this short paper, we would like to study the density of each class over D numerically for given primes less than natural number N , as well as D itself. In the spirit of Prime Number Theorem, let

$$D(N) = \{(x, y) \in D | x \leq N\}$$

as well as

$$D_i(N) = \{(x, y) \in D_i | x \leq N\}.$$

We define the density of each class as:

$$\alpha_i = \frac{|D_i(N)|}{|D_i(N)|}$$

where $|x|$ is the number of elements in x.

3. Numerical Results

A MAPLE program is written where only class 2, 8, 14, and 20 are counted. The distribution is different from the study of Brent (1973), Nicely (1999), Nyman and Nicely (2003) since not all the prime pairs are included.

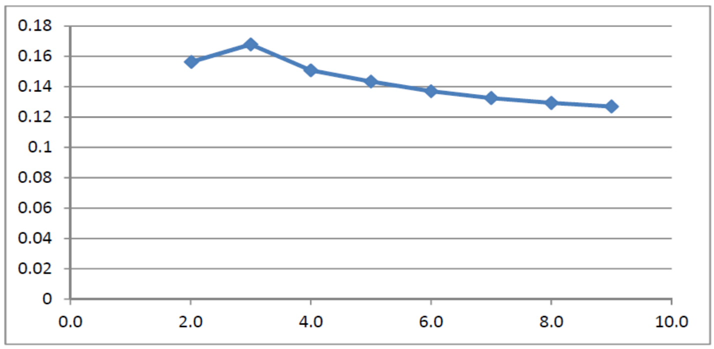

Figure 1: $D(N)\log N/N$ versuslog N

The density of $D(N)/N$ is decreasing as expected but $D(N)\log N/N$ is also decreasing respect to $\log N$ (Figure 1).

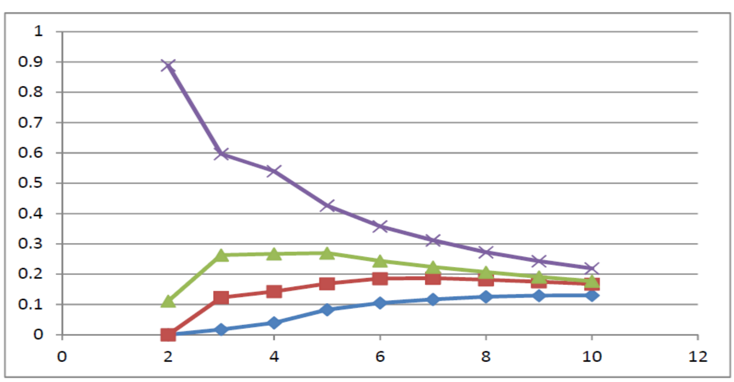

Figure 2: $D_i(N)$ versus $\log N$. From top to bottom, class 2, 8, 14 and 20.

The distribution of each class has a maximum, then the density monotone decreasing as N increases. It is not clear whether it will decrease to zero due to limitation of the computation power. But figure 3 shows that the density of the classes other than 2, 8, 14 and 20 are increasing. This might indicate that the formal four classes might eventually converge to zero.

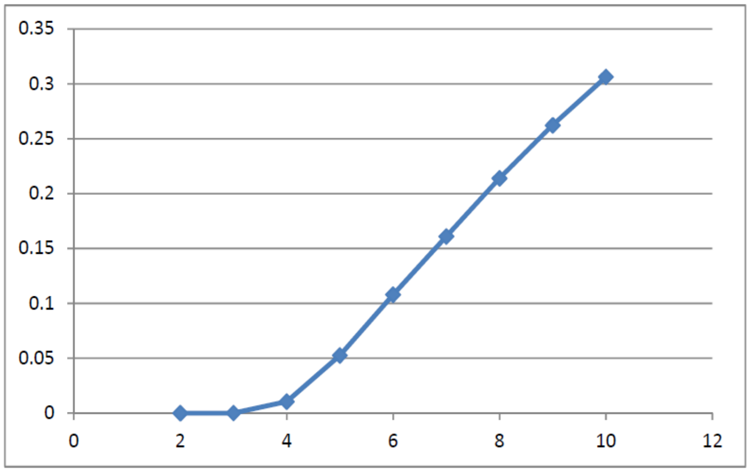

Figure 3: $1 - \sum_{2,8,14,20} \alpha_i$ versus $\log N$

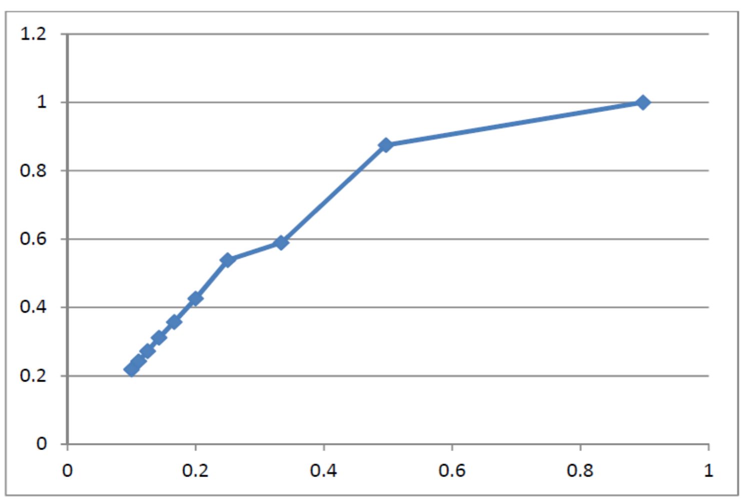

Figure 4: α_2 versus $1/\log N$

From figure 4, one can observe that the curve is close to linear when N is large, where $1/\log N$ is small. From the slope, one can conjecture that, for large N,

$$\alpha_2 \sim \frac{1}{\log N}.$$

Consequently, $D(N)$ is conjectured to have the order

$$\frac{N}{\log N}$$

4. Open Problems

From these observations, we would like to post a few questions:

1. Is there any class of $6k+2$ could be empty? It is non empty for at least for i=2, 8, 14, 20. Since D is an infinite set, it is either some class is infinite or there are infinite classes of $6k+2$.
2. What is the limit of $D(N)/N$? A fraction of $1/\log N$?
3. Is each class infinite or are all densities converging to zero? Or some nonzero constant?
4. Is the density of class i always greater than the previous class j, where $i > j$?
5. When is the first occurrence for each class?
6. Does the density of each class have a maximum and when it happens?
7. Are both $D(N)$ and $D_2(N)$ are at the same order $N/(\log N)^2$? If so, α_2 should be approaching a non-zero constant. This obviously would lead to the proof of the twin prime conjecture.
8. If answer to question above is negative, then is it possible to construct another infinite set S, which $S \subset D$, in such the density of D_2 over S would approaches some non-zero constant?

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