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Hopf and Marcus-Wyse Topologies on $\mathbb{Z}^2$

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ABSTRACT

The two conditions 1^2 and 2^2 are so that any digital topology on $\mathbb{Z}^2$ satisfies them is topologically connected whenever it is graphically connected. In this paper, we show that the two digital topologies on $\mathbb{Z}^2$ satisfy 1^2 and 2^2 are precisely the Hopf and the Marcus-Wyse Topologies. We prove that the Hopf topology is the product of two Khalimsky topologies on $\mathbb{Z}$. We also prove that the Hopf topology is homeomorphic to the cellular-complex topology on $\mathbb{F}^2$ while the Marcus-Wyes topology is homeomorphic to $\mathbb{F}^2_{\{0,2\}}$.

Keywords: Digital spaces, Alexandroff spaces, Hopf topology, Marcus-Wyes topology, cellular-complex topology.

1. Introduction and Preliminaries

For a topological space (X, τ), a subset A is called a *semi-open* in Levine (1963) (resp. a *preopen* (Mashhour et al. (1982)), an α-*open* (Njastad (1965))) if $A \subseteq \overline{A^\circ}$ (resp. $A \subseteq \overline{A}^\circ$, $A \subseteq \overline{A^\circ}^\circ$). It is called a *semi-closed* set Crossley and Hildebrand (1971) (resp. a *preclosed* set in El-Deep et al. (1983)), an α-*closed* in Dontchev (1998a)) if A^c is semi-open (resp. preopen, α-open).

We denote $SO(X)$ (resp. τ_α, $PO(X)$) to be the family of all semi-open (resp. α-open, preopen) sets in X.

A topological space X is called a *semi-*T_o in Maheshwari and Prasad (1975) (resp. an α-T_o in Maki et al. (1993), a *pre-*T_o)-space if whenever x and y are distinct points in X, there is a semi-open (resp. an α-open, a pre-open) set which contains one of x, y and not the other.

X is $T_{\frac{1}{4}}$ in Arenas et al. (1997) (resp. $T_{\frac{1}{3}}$ in Arenas et al. (2000))-space if for every finite (resp. compact) subset F of X and every $y \notin F$, there exists a set A_y containing F and disjoint from $\{y\}$ such that A_y is either open or closed. X is $T_{\frac{1}{2}}$ in Dunham (1977) (resp. $T_{\frac{3}{4}}$ in Dontchev (1998b), *semi-*$T_{\frac{1}{2}}$ in Cueva and Saraf (2000), α-$T_{\frac{1}{2}}$, a *feebly* T_1 (Jankovic and Reilly (1985)), a *semi-*T_D in Jankovic and Reilly (1985)) if every singleton is either open (resp. regular open, semi-open, α-open, nowhere dense, open) or closed (resp. closed, semi-closed, α-closed, clopen, nowhere dense). X is *semi-*T_1 in Maheshwari and Prasad (1975) (resp. α-T_1 in Maki et al. (1993)) if every singleton is semi-closed (resp. α-closed) set. X is T_1^**-space* Ganster et al. (1992) if every nowhere dense subset of X is union of closed sets. X is a *submaximal space* (Reilly and Vamanamurthy (1990)) if every dense subset is open, or equivalently every preopen subset is open. A *nodec space* in Van Mill and Mills (1980) is a space where all nowhere dense sets are closed. A *locally finite* space X is a space such that for any element $x \in X$ there exists a finite open set U_x such that $x \in U_x$.

For a given poset $(X, \leq)$, we define M to be the set of all maximal elements and the set m to be the set of minimal elements in X.

For $x \in X$ we define the down set $\downarrow x := \{y \in X : y \leq x\}$ and the up set $\uparrow x := \{y \in X : y \geq x\}$. For a point $x \in X$, we define $\hat{x} = \uparrow x \cap M$. A poset X satisfies the *ascending chain conditions*, (ACC) if any increasing sequence is finally constant. A poset X satisfies the *descending chain conditions*, (DCC) if any decreasing sequence is finally constant. If X satisfies ACC (resp. DCC), then the set M (resp. m) is nonempty set. There is a very useful way to depict posets using the so called *Hasse diagrams.*

For posets $(P_i, \leq_i)$, $i = 1, 2, \cdots, n$, we can formulate many types of partial orders the cartesian product $\prod_i^n P_i = P_1 \times P_2 \cdots \times P_n$. The most famous order is the *coordinatewise order* $\leq_c$. For two elements $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ in $\prod_i^n P_i$, we have that $a \leq_c b$ iff $a_i \leq_i b_i$ $\forall$ $i = 1, 2, \cdots, n$.

A topological space X is called *an Alexandroff space* (In short, A-space) if the intersection of any collection of open sets is open. Equivalently, any element x in X has a minimal neighborhood base $V(x)$ which is the intersection of all open sets containing x. If X is an A-space and $x, y \in X$, then $x \in \overline{\{y\}}$ if and only if $y \in V(x)$.

If an A-space (X, τ) is T_0, then we define a *(Alexandroff) specialization order* as $a \leq_\tau b$ if $a \in \overline{\{b\}}$, (equivalently $b \in V(a)$). If $(X, \leq)$ is a poset, then the collection $\mathfrak{B} = \{\uparrow x : x \in X\}$ is a base for a T_o A-topology on X denoted by $\tau_{\leq}$.

From now on, we denote $(X, \tau(\leq))$ to be a T_o A-space (X, τ) where $(X, \leq)$ is its corresponding poset. If $(\mathrm{X}, \tau(\leq))$ is a T_o A-space, then $\forall x \in X$, $V(x)$ equals $\uparrow x$. If (X, τ) is an $A-$space with the collection $\mathcal{F}$ of closed sets, then $\mathcal{F}$ is itself an Alexandroff topology on X called *the dual* of τ on X and usually denoted by τ^d. If $(X, \tau(\leq))$ is a T_0 $A-$space, then the Alexandroff dual is also T_0-space, and the induced order $\leq_d$ is the reverse order of the order $\leq$; that is $x \leq_d y$ iff $y \leq x$.

Thus, for $x \in X$, we have $V(x) = cl_d(x)$ and $V_d(x) = cl(x)$. Two distinct points x and y in X are called *adjacent* if the subspace $\{x, y\}$ is (topologically) connected.

Definition 1.1. *(Mahdi and EL-Mabhouh (2011)). A* T_0 $A-$*space* $(X, \tau(\leq))$ *is called Artinian (resp. Noetherian) Mahdi and Elatrash (2005) if the corresponding poset satisfies the ACC (resp. DCC). If a* T_0 $A-$*space is both Artinian and Noetherian, then it is called a generalized locally finite (g-locally finite).*

Theorem 1.2. *(Mahdi (2015)). Let* X *and* Y *be two* T_0 $A-$*spaces. Then* X *and* Y *are homeomorphic iff there exists a bijective function* $f : X \longrightarrow Y$ *such that* $f(V_X(x)) = V_Y(f(x))$, $\forall x \in X$.

Theorem 1.3. *(Mahdi and Elatrash (2006)). Let* X *be a* T_0 $A-$*space. Then all the following are equivalent:*
(i) X *is* $T_{\frac{1}{2}}$*-space.*
(ii) X *is* $T_{\frac{1}{3}}$*-space.*

(iii) X is $T_{\frac{1}{4}}$-space.
(iv) Each element of X is either open or closed. Equivalently, the elements in the corresponding poset is either minimal or maximal.
(v) X is submaximal.
(vi) PO(X)= $\tau_\alpha = \tau(\leq)$; that is, every preopen set is open.
(vii) X is T_1^*-space.

Theorem 1.4. *(Mahdi and Elatrash (2006)). Let X be a T_0 A−space. Then X is $T_{\frac{3}{4}}$ iff the following two conditions are satisfied:*
(a) X is $T_{\frac{1}{2}}$-space.
(b) $\forall x \notin M$, $|\hat{x}| \geq 2$; where $|\hat{x}|$ is the cardinality of the set $\hat{x}$.

Theorem 1.5. *(Mahdi and Elatrash (2006)). Let X be an Artinian T_0 A−space. Then X is both α-$T_{\frac{1}{2}}$−space and semi-$T_{\frac{1}{2}}$−space.*

Theorem 1.6. *(Mahdi and Elatrash (2006)). If X is an Artinian T_0 A−space, then the following statements are equivalent:*
(1) X is a semi-T_2−space.
(2) X is a semi-T_1−space.
(3) $\forall x \notin M$, $|\hat{x}| \geq 2$.

Theorem 1.7. *(Mahdi (2010)). If $(X, \tau_x(\leq_x))$ and $(Y, \tau_y(\leq_y))$ are two T_0 A−spaces with corresponding posets $(X, \leq_x)$, $(Y, \leq_y)$ respectively, then $X \times Y$ is a T_0 A−space induces a specialization order $\leq_p$ coincides with the coordinatewise order of the product of the corresponding posets.*

2. Properties of Digital Spaces on $\mathbb{Z}^2$

The *digital space* $\mathbb{Z}^2$ is the set of all tuples (x_1, x_2) of the Euclidean space having integer coordinates. Let $a = (a_1, a_2)$, $b = (b_1, b_2) \in \mathbb{Z}^2$. The *length* of the i'th coordinate between a and b is $r_i(a, b) = |a_i - b_i|$, and the *total coordinates length* between a and b is $R(a, b) = |a_1 - b_1| + |a_2 - b_2|$. For a given point $x \in \mathbb{Z}^2$, the 8-*neighborhood* $N_8(x)$ of x is the set of all points $y \in \mathbb{Z}^2$ such that $x \neq y$ and $r_i(x, y) \leq 1$ for $i = 1, 2$. The elements in $N_8(x)$ are called 8−*neighbors* (*or* 8−*adjacent*) of x. The 4-*neighborhood* $N_4(x)$ of x is the set of all points $y \in \mathbb{Z}^d$ such that $R(x, y) = 1$. The elements in $N_4(x)$ are called 4−*neighbors* (*or* 4−*adjacent*) of x. An *n-path* (n = 4, 8) from x to y is a list of elements $x = x_1, x_2, \cdots, x_k = y$ satisfy that for $1 < i \leq k$, x_i is n-adjacent of x_{i-1}. Let $X \subseteq \mathbb{Z}^2$. For n = 4, 8 if for each points $x, y \in X$, there is n-path contained in X from x to y, X is called *n-connected*. Eckhardt and Latecki (2003) suggest the following two conditions so that any topology on $\mathbb{Z}^2$ satisfies these conditions will be topologically connected whenever it is graphicly connected:

(1^2) If a set in $\mathbb{Z}^2$ is 4-connected, then it is topologically connected.

(2^2) If a set in $\mathbb{Z}^2$ is not 8−connected, then it is not topologically connected.

Henceforth, we will consider the digital topology on $\mathbb{Z}^2$ to mean any topology satisfying 1^2 and 2^2. In Mahdi and Hegazy (2016) we proved that there are only two digital topologies τ_s and τ_m on $\mathbb{Z}^2$,which are g-locally finite T_0 $A-$spaces. With respect to any digital topology on $\mathbb{Z}^2$, if $x, y \in \mathbb{Z}^2$, define $x \to y$ (or $y \leftarrow x$) if for each open set U containinig x, we have $y \in U$.

The notation $x \to y$ denotes the negation of $x \to y$ Eckhardt and Latecki (2003). the relation " $\to$ " is a partial order on $\mathbb{Z}^2$. With respect to this order, a point $x \in \mathbb{Z}^2$ is called a *saddle point* (Eckhardt and Latecki (2003)) if it is neither maximal nor minimal points. The set of all saddle points is denoted by SD. We prove that $(\mathbb{Z}^2, \tau_s)$ contains saddle points, while $(\mathbb{Z}^2, \tau_m)$ has no saddle point.

Definition 2.1. *(Mahdi and Hegazy (2016)). Define two subsets EV_2, OD_2 of $\mathbb{Z}^2$ as follows :*

(i) $EV_2 = \{(a, b) : a + b$ *is even number* $\}$.

(ii) $OD_2 = \{(a, b) : a + b$ *is odd number* $\}$.

Theorem 2.2. *(Mahdi and Hegazy (2016)). If x is a maximal (resp. a minimal, a saddle) point in $\mathbb{Z}^2$, then no point in $N_4(x)$ is a maximal (resp. a minimal, a saddle) point.*

Lemma 2.3. *(Mahdi and Hegazy (2016)). In the digital space $(\mathbb{Z}^2, \tau_m)$ if x is a maximal (resp. a minimal) point, then all points in $N_8(x) \setminus N_4(x)$ are maximal (resp. minimal) points.*

Theorem 2.4. *(Mahdi and Hegazy (2016)). In the digital space $(\mathbb{Z}^2, \tau_m)$, if $x \in X$ is a minimal point, then the minimal open set $V(x) = N_4(x) \cup \{x\}$.*

Theorem 2.5. *(Mahdi and Hegazy (2016)). Let τ_s be the digital topology on $\mathbb{Z}^2$ with saddle points and $x \in \mathbb{Z}^2$. Then:*

(1) if x is a maximal point or a minimal point, then $N_4(x)$ is a set of saddle points.

(2) if x is a saddle point, then $\forall y \in N_4(x)$, y is either maximal or minimal point.

(3) if x is a maximal (resp. a minimal) point, then $\forall\ y \in N_8(x) \setminus N_4(x)$, y is a minimal (resp. a maximal) point.

(4) if x is a saddle point, then $\forall\ y \in N_8(x) \setminus N_4(x)$, y is a saddle point.

Theorem 2.6. *(Mahdi and Hegazy (2016)). In the digital space $(\mathbb{Z}^2, \tau_s)$ with saddle points, if $x \in X$ is a minimal point, then the minimal open set $V(x) = N_8(x) \cup \{x\}$. if $y \in X$ is a saddle point, then the minimal open set $V(y) = (M \cap N_4(y)) \cup \{y\}$.*

3. Marcus-Wyse Topology on $\mathbb{Z}^2$

The Marcus-Wyse topology is a special connected $T_{\frac{1}{2}}$ $A-$space on $\mathbb{Z}^2$. This topology was described by Marcus and Wyse (1970). They defined this topology by its minimal neighbourhood as follows: for any point $x = (a, b) \in \mathbb{Z}^2$, $V(x) = \{x\} \cup N_4(x)$ if $a+b$ is even and $V(x) = \{x\}$ otherwise.

Hence the digital topology τ_m on $\mathbb{Z}^2$ is exactly the Marcus-Wyse topology. This topology is a T_0 $A-$space, and its specialization order is denoted by $(\leq_m)$. If $u \in EV_2$, then $N_4(u) \subseteq OD_2$ and $N_8(u) \setminus N_4(u) \subseteq EV_2$. If $u \in OD_2$, then $N_4(u) \subseteq EV_2$ and $N_8(u) \setminus N_4(u) \subseteq OD_2$. Moreover if $u, v \in EV_2$ (resp. OD_2), then there exists a finite sequence $u = u_0,\ u_1,\ u_2, \cdots,\ u_n = v$ in EV_2 (resp. OD_2) such that $u_{i-1} \in N_8(u_i) \setminus N_4(u_i)$ $\forall\ i = 1, 2, \cdots$, n.

Theorem 3.1. *Let $(\mathbb{Z}^2, \tau_m)$ be the Marcus-Wyse space. If there exists a minimal (resp. a maximal) point u such that $u \in EV_2$, then $EV_2 = m$ and $OD_2 = M$ (resp. $EV_2 = M$ and $OD_2 = m$).*

Proof. Let $u \in EV_2 \cap m$ and $v \in EV_2$. Then there exists a finite sequence $u = u_0, u_1, \cdots, u_n = v$ in EV_2 such that $u_{i-1} \in N_8(u_i) \setminus N_4(u_i)$, $i = 1, 2, \cdots, n$. Hence v is a minimal point. Conversely, if $w = (a, b) \in OD_2$, then $(a + 1, b) \in EV_2$ and so $(a+1, b)$ is minimal point. By Theorem 2.2, w is maximal point. $\square$

Hence, in the Marcus-Wyse topology on $\mathbb{Z}^2$, we have two possible cases: "$m = EV_2$ and $M = OD_2$" or "$M = OD_2$ and $m = EV_2$". It is clearly that they are homeomorphic. By convention, we will take the topology in the first case to be the Marcus -Wyse topology $(\mathbb{Z}^2, \tau_m)$ and the topology τ_m^d in the second case will be its homeomorphic dual. The up of a point $x \in \mathbb{Z}^2$ in τ_m is denoted by $\uparrow_m x$ and in τ_m^d by $\uparrow_m^d x$.

Using a net diagram in (a) or a Hasse diagram in (b) a part of the Marcus-Wyse topology on $\mathbb{Z}^2$ is shown in Figure 1:

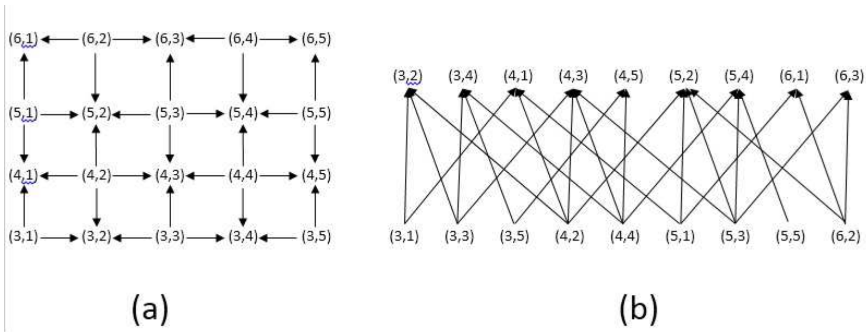

Figure 1: Part of Marcus-Wyse Topology on $\mathbb{Z}^2$

The specialization order of the Marcus-Wyse topology on $\mathbb{Z}^2$ denote by $\leq_m$. Using Lemma 2.4, if $x \in EV_2 = m$, then $\uparrow_m x = N_4(x) \cup \{x\}$ and if $x \in OD_2 = M$, then $\uparrow_m x = \{x\}$. Then we have the following theorem.

Theorem 3.2. *If x and y are two distinct points in $\mathbb{Z}^2$, then $x \leq_m y$ iff $x \in EV_2$ and $y \in N_4(x)$.*

Theorem 3.3. *Let $(\mathbb{Z}^2, \tau_m)$ be the Marcus-Wyse topology on $\mathbb{Z}^2$, then*

(1) $\mathbb{Z}^2$ is submaximal.

(2) $\mathbb{Z}^2$ is nodec.

(3) $PO(\mathbb{Z}^2)=\tau_\alpha = \tau(\leq)$; that is, every preopen set is open.

(4) $\mathbb{Z}^2$ is T_1^*-space.

(5) $\mathbb{Z}^2$ is $T_{\frac{3}{4}}$.

(6) $\mathbb{Z}^2$ is semi-T_2-space.

Proof. Any element of $\mathbb{Z}$ is either maximal or minimal, so by Theorem 1.3, we get the parts (1),(2),(3) and (4). Since $\mathbb{Z}$ is submaximal, then by Theorem 1.3, it is $T_{\frac{1}{2}}$-space. Moreover, since $\mid \hat{x} \mid = 2 \ \forall x \notin M$, by Theorem 1.4, $\mathbb{Z}$ is $T_{\frac{3}{4}}$-space. Part(6) is coming directly from Theorem 1.6. □

4. Alexandroff Hopf Topology on $\mathbb{Z}^2$

In Eckhardt and Latecki (2003) said that the digital topology τ_s on $\mathbb{Z}^2$ which has saddle points (τ_s) is homeomorphic to the cellular-complex topology (Alexandroff Hopf topology). We will latter define a useful function and using it to prove this fact.

Let O and V be the odd and the even numbers in $\mathbb{Z}$ respectively. We define the subsets of $\mathbb{Z}^2$ EE, OO, EO and OE as : $EE = E \times E$, $OO = O \times O$, $EO = E \times O$ and $OE = O \times E$. Hence $OD_2 = EO \cup OE$ and $EV_2 = EE \cup OO$.

Let $\leq_s$ be the induced order on $\mathbb{Z}^2$ by the Hopf topology τ_s. The up of a point $x \in \mathbb{Z}^2$ in τ_s is denoted by $\uparrow_s$.

Lemma 4.1. *Let $(\mathbb{Z}^2, \tau_s)$ be the Hopf space and let $x, y \in \mathbb{Z}^2$.*

(1) If x is a maximal (resp. a minimal) point and if $r_1(x, y) = 2k_1$ and $r_2(x, y) = 2k_2$ for some integers k_1, k_2, then y is a maximal (resp. a minimal) point.

(2) If x is a saddle point and $R(x, y) = 2$, then y is a saddle point.

Theorem 4.2. *Let $(\mathbb{Z}^2, \tau_s)$ be the Hopf space, and $u = (x, y) \in EE$.*

(i) If u is a minimal point, then $m = EE$, $M = OO$, and $SD = OD_2$.

(ii) If u is a maximal point, then $m = OO$, $M = EE$, and $SD = OD_2$.

(iii) If u is a saddle point, then $SD = EV_2$, and either "$M = OE$ and $m = EO$" or "$M = EO$ and $m = OE$".

Proof. (i) Let $(a, b) \in EE$. Then $a = x + 2s$ and $b = y + 2r$ for some integers s, r. Using the induction and Lemma 4.1, we have $(a, b) \in m$. Hence $EE \subseteq m$. If $(a, b) \in OO$, then $(a+1, b+1) \in EE \subseteq m$ and $(a+1, b+1) \in N_8(a, b) \setminus N_4(a, b)$. Hence $(a, b) \in M$ and so $OO \subseteq M$. If $(a, b) \in OD_2$, then $(a + 1, b) \in EV_2 \subseteq m \cup M$. So $(a, b) \in SD$ and hence $OD_2 \subseteq SD$. Thus $m = EE$, $M = OO$, and $SD = EO \cup OE = OD_2$.

(ii) Similar to (i).

(iii) Let $(a, b) \in EE$. Similarly to (i), $EE \subseteq SD$. Assume that $(a, b) \in OO$. Hence $a = x + 2s + 1$ and $b = y + 2r + 1$ for some integers s, r. By Lemma

4.1, $(x+2n, y+2r) \in SD$ and so $(a,b) \in SD$. Therefore, $OO \subseteq SD$ and thus $EV_2 \subseteq SD$. Let $(a,b) \in OE$. Then $(a+1,b) \in EV_2 \subseteq SD$. By Theorem 2.5, either $(a,b) \in M$ or $(a,b) \in m$. Hence $OE \subseteq M \cup m$. Assume that $OE \subseteq M$ (resp. $OE \subseteq m$). Let $(a,b) \in EO$. Then $(a+1,b) \in SD$ and $(a,b) \notin SD$. Suppose to contrary that $(a,b) \in M$ (resp. $(a,b) \in m$). So $N_4(a+1,b) \subseteq M$ (resp. $N_4(a+1,b) \subseteq m$) which is a contradiction since $(a+1,b) \in SD$. $\square$

Corollary 4.3. *Let $(\mathbb{Z}^2, \tau_s)$ be the Hopf space.*

(a) If $m = EE$, then $M = OO$ and $SD = EO \cup OE$.

(b) If $m = EO$, then $M = OE$ and $SD = EE \cup OO$.

(c) If $m = OE$, then $M = EO$ and $SD = EE \cup OO$.

(d) If $m = OO$, then $M = EE$ and $SD = EO \cup OE$.

Proof. (a) Direct from Theorem 4.2.
(b) Let $(x,y) \in EO$, then $(x, y+1) \in SD \cap EE$. Hence $M = OE$, $SD = EE \cup OO$. Parts (c) and (d) are similar to (b). $\square$

It is clearly that if $\mathbb{Z}^2$ has the topology τ_s and $(x,y) \in \mathbb{Z}^2$, such that (x,y) is a maximal or a minimal point, then we can determine the type (maximal, minimal, or saddle) of any point in $\mathbb{Z}^2$. Moreover, if (x,y) is a saddle point in $\mathbb{Z}^2$, then we have at most two cases for any point in $\mathbb{Z}^2$. In fact there exist four topologies in $\mathbb{Z}^2$. All of them can be considered as a τ_s topology, and all of them are homeomorphic to each others. They are considered as one topology on $\mathbb{Z}^2$. By convention, we will consider the Hopf topology to be such that $m = EE$, $M = OO$ and $SD = EO \cup OE$. The other three types of homeomorphic topologies on $\mathbb{Z}^2$ are also called Hopf topology on $\mathbb{Z}^2$, but if we take any one of the three other types as the Hopf topology on $\mathbb{Z}^2$, we must say so.

Theorem 4.4. *Let $\mathbb{Z}^2$ be the Hopf topology and $(a,b) \in \mathbb{Z}^2$.*

(a) If $(a,b) \in OO$, then $\uparrow (a,b) = \{(a,b)\} = \{a\} \times \{b\}$.

(b) If $(a,b) \in EE$, then $\uparrow (a,b) = N_8(a,b) \cup \{(a,b)\} = \{a-1, a, a+1\} \times \{b-1, b, b+1\}$.

(c) If $(a,b) \in EO$, then $\uparrow (a,b) = \{(a-1,b), (a,b), (a+1,b)\} = \{a-1, a, a+1\} \times \{b\}$.

(d) *If* $(a,b) \in OE$, *then* $\uparrow(a,b) = \{(a,b-1),(a,b),(a,b+1)\} = \{a\} \times \{b-1, b, b+1\}$.

Definition 4.5. *(Melin (2003)). The Khalimsky topology on the set of integer numbers* $\mathbb{Z}$ *is a* T_o $A-$*space where the smallest neighborhoods* $V(i) = \{i\}$ *if* i *is odd and* $V(i) = \{i-1, i, i+1\}$ *if* i *is even.*

Definition 4.6. *Let* $(\mathbb{Z}, \tau_{kh})$ *be the Khalimsky space. The product of two Khalimsky spaces on* $\mathbb{Z}^2$ *denoted by* τ_p *which is* T_0 *A-space with minimal neighbourhood* $\uparrow_{kh}(a,b) = \uparrow_{kh} a \times \uparrow_{kh} b$ *for all* $(a,b) \in \mathbb{Z}^2$.

Theorem 4.7. *Let* $(\mathbb{Z}, \tau_{kh})$ *be the Khalimsky space and* $(\mathbb{Z}^2, \tau_s)$ *be the Hopf space. Then* $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$. *That is* $\tau_s = \tau_p$.

Proof. Firstly the two topologies τ_s and τ_p are T_0 A-spaces on $\mathbb{Z}^2$. By Theorem 4.4, $\uparrow_p(a,b) = \uparrow_{kh} a \times \uparrow_{kh} b = \uparrow_s (a,b)$ for all $(a,b) \in \mathbb{Z}^2$. Hence by Theorem 1.7 and Theorem 1.2, $\tau_s = \tau_p$. □

Remark 4.8. *The other three homeomorphic types of the Hopf topologies are the product of* $(\mathbb{Z}, \tau_{kh}) \times (\mathbb{Z}, \tau^d_{kh})$, $(\mathbb{Z}, \tau^d_{kh}) \times (\mathbb{Z}, \tau_{kh})$ *and* $(\mathbb{Z}, \tau^d_{kh}) \times (\mathbb{Z}, \tau^d_{kh})$. *For example, in the Hopf topology* $(\mathbb{Z}, \tau^d_{kh}) \times (\mathbb{Z}, \tau_{kh})$, $m = OE$.

Using a net diagram in (a) or a Hasse diagram in (b), a part of Hopf topology on $\mathbb{Z}^2$ looks like as in Figure 2.

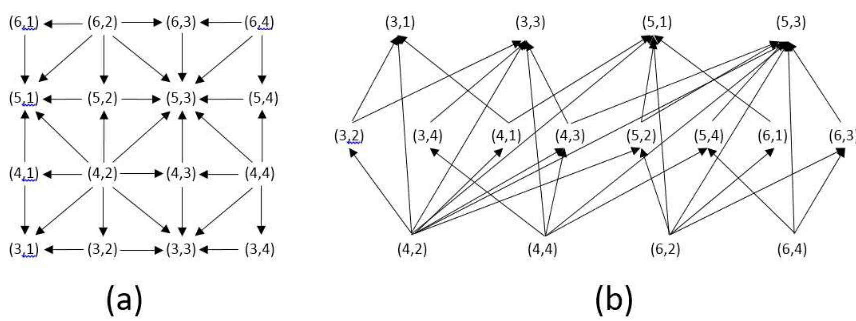

Figure 2: Part of the Hopf topology on $\mathbb{Z}^2$

Remark 4.9. *Let* $(\mathbb{Z}^2, \tau_s)$ *be the Hopf topology. We describe* $\leq_s$ *as follows : if* $(x,y) \in OO$, *then* $(x,y) \leq_s (a,b)$ *iff* $x = a$ *and* $y = b$. *If* $(x,y) \in EE$, *then*

$(x,y) \leq_s (a,b)$ iff $a \in \{x-1,x,x+1\}$ and $b \in \{y-1,y,y+1\}$. If $(x,y) \in EO$, then $(x,y) \leq_s (a,b)$ iff $a \in \{x-1,x,x+1\}$ and $b = y$. Finaly if $(x,y) \in OE$, then $(x,y) \leq_s (a,b)$ iff $a = x$ and $b \in \{y-1,y,y+1\}$.

Theorem 4.10. *If $A \in \{OO, EE\}$ and $B = \mathbb{Z}^2 \setminus A$, then the relative topology on B with respect to the Hopf topology is equal to the relative topology on B with respect to the Marcus-Wyse topology. That is, $(B, \leq_s |_B) = (B, \leq_m |_B)$.*

Proof. Let $A = OO$ and suppose that U is a nonempty open set in (B, τ_m). Let $x \in U$. If $x \in EE$, then $(\uparrow_s x) \cap B = N_4(x) = (\uparrow_m x) \cap B \subseteq U$. If $x \in EO \cup OE$, then $(\uparrow_s x) \cap B = \{x\} = (\uparrow_m x) \cap B \subseteq U$. Thus, U is open in (B, τ_s) and $\tau_m|_B \subseteq \tau_s|_B$. Similarly we prove $\tau_s|_B \subseteq \tau_m|_B$. Hence $\tau_m|_B = \tau_s|_B$.
Let $A = EE$. It suffices to note that the relative topology on B with respect to the τ_s^d is equal to dual relative topology on B with respect to τ_s. ☐

Theorem 4.11. *Let $\mathbb{Z}^2$ be the Hopf space, then:*

(1) $\mathbb{Z}^2$ is a semi-T_2-space.

(2) $\mathbb{Z}^2$ is not submaximal space and hence it is not a $T_{\frac{1}{4}}$-space.

Proof. (1) For any $x = (a,b) \in \mathbb{Z}^2 \setminus OO$, we have $|\hat{x}| \geq 2$. Then by Theorem 1.6, $\mathbb{Z}^2$ is a semi-T_2-space.

(2) Direct from Theorem 1.3.

☐

5. Applications of Digital Topologies on $\mathbb{Z}^2$

Let $\mathbb{F}_0^1 = \{\{a\} : a \in \mathbb{Z}\}$ and $\mathbb{F}_1^1 = \{\{a, a+1\} : a \in \mathbb{Z}\}$. Let $f \subseteq \mathbb{Z}^2$. For $n = 0,1,2$, if f is a cartesian product of n elements of $\mathbb{F}_1^1$ and $2-n$ elements of $\mathbb{F}_0^1$, we say that f is a *an n-face* or simply a face of $\mathbb{Z}^2$, n is the *dimension* of f, and we write $dim(f) = n$. The *space of cubical complexes* $\mathbb{F}^2$ is the set composed of all faces of $\mathbb{Z}^2$. We denote by $\mathbb{F}_k^2$ $(0 \leq k \leq 2)$ the set composed of all k-faces of $\mathbb{Z}^2$. Clearly that $\mathbb{F}_k^2 \subseteq \mathbb{F}^2$. The couple $(\mathbb{F}^2, \subseteq)$ is a poset.

Thus there is a corresponding T_o-Alexandroff space $(\mathbb{F}^2, \tau_{\subseteq})$. Indeed this topology is called the *cellular-complex topology* for the digital plane introduced by Alexandroff and Hopf (1935). Let $F \subseteq \mathbb{F}^2$ be a set of faces, and let $f \in F$ be a face. Then the face f is a *facet* of F if f is a maximal in F. Actually, if $x = (x_1, x_2) \in \mathbb{Z}^2$, the set $\dot{x} = \prod_{i=1}^{2}\{x_i, x_i + 1\}$ is a facet of $\mathbb{F}^2$ and x is called the *leader* of $\dot{x}$ and we write $L(\dot{x}) = x$ in Mazo (2012). Let $\mathbb{F}^2_{\{0,2\}} = \mathbb{F}^2_0 \cup \mathbb{F}^2_2$.

Definition 5.1. *The Marcus-Wyse function* $\kappa : \mathbb{F}^2_{\{0,2\}} \longrightarrow \mathbb{Z}^2$ *is a bijection function define as :*

$$\kappa(f) = \begin{cases} (a+b, a-b), & \text{if } dim(f) = 0 \text{ and } f = \{(a,b)\}; \\ (a+b+1, a-b), & \text{if } dim(f) = 2 \text{ and } L(f) = (a,b). \end{cases}$$

Definition 5.2. *The* 2*-Alexandroff Hopf function* $\psi_2 : \mathbb{F}^2 \longrightarrow \mathbb{Z}^2$ *is a bijection function define as:*

$$\psi_2(f) = \begin{cases} (2a, 2b), & \text{if } dim(f) = 0 \text{ and } f = \{(a,b)\}; \\ (2a+1, 2b), & \text{if } dim(f) = 1 \text{ and } f = \{a, a+1\} \times \{b\}; \\ (2a, 2b+1), & \text{if } dim(f) = 1 \text{ and } f = \{a\} \times \{b, b+1\}; \\ (2a+1, 2b+1), & \text{if } dim(f) = 2 \text{ and } f = \{a, a+1\} \times \{b, b+1\}. \end{cases}$$

Using the 2-Alexandroff Hopf function and Theorem 1.2, we have the two following theorems :

Theorem 5.3. *The Hopf topology* $(\mathbb{Z}^2, \tau_s)$ *is homeomorphic to the cellular-complex topology* $(\mathbb{F}^2, \tau_{\subseteq})$.

Theorem 5.4. *Let* $(\mathbb{Z}^2, \tau_m)$ *be the Marcus-Wyse topology on* $\mathbb{Z}^2$*, then* $(\mathbb{Z}^2, \tau_m)$ *is homeomorphic to* $(\mathbb{F}^2_{\{0,2\}}, \tau_{\subseteq})$.

6. Conclusion

In this paper we proved there are two topologies on $\mathbb{Z}^2$ that are satisfying the two conditions 1^2, 2^2 which are the Hopf and the Marcus-Wyse Topologies. We studied their properties. We hope this study will be a facilitating component of the study of Digital Topology and its applications through our findings about the minimal neighbourhoods of this topologies and represent them graphically.

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