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Exact Confidence Interval for the Generalized Inverted Exponential Distribution with Progressively Censored Data

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ABSTRACT

In this article, the estimation of parameters for a generalized inverted exponential (*GIE*) distribution based on progressively Type-II censored sample is studied. The maximum likelihood estimators (MLEs) are developed for estimating the unknown parameters. An exact confidence interval and exact joint confidence region are constructed. Monte Carlo simulations are then performed for comparing the confidence intervals based on complete and censored samples. Furthermore, because of the great importance of prediction in engineering data, conditional median predictor method is considered to obtain the point prediction of future observation based on progressively Type-II censored sample. Finally, a real data set is used to illustrative purposes.

Keywords: Exact Confidence Interval, Generalized Inverted Exponential, Joint Confidence Region, Maximum Likelihood Estimates, Progressively Type II Censoring.

1. Introduction

Censoring occurs when exact survival times are known only for a portion of the individuals or items under study since complete survival times may not have been observed by the experimenter. Due to the form of the likelihood function, the estimation methods for censored data are more complex than the complete data. The most common censoring schemes are Type I censoring, where the experiment stops at a predetermined time T, and Type II censoring, where the experiment stops when predetermined number (r) are observed to have failed. The mixture of theses censoring schemes is called hybrid censoring. Conventional Type I, Type II and hybrid censoring schemes have been studied in detail by many authors; see Balakrishnan and Kundu (2013), Childs et al. (2003), Kundu and Pradhan (2009), Kundu and Howlader (2010), Singh Yadav et al. (2016), Panahi and Sayyareh (2014) and Panahi (2017).

However, the conventional Type I, Type II and hybrid censoring schemes do not have the flexibility of allowing removal of units at points other than the terminal point of the experiment. For this reason, a more general censoring scheme called progressively Type-II censoring is proposed. Briefly, it can be described as follows. Suppose n units are placed on a life test and the experimenter decides beforehand the quantity m, the number of units to be failed. Now at the time of the first failure, R_1 of the remaining $n-1$ surviving units are randomly removed from the experiment. Continuing on, at the time of the second failure, R_2 of the remaining $n-2-R_1$ units are randomly with-drawn from the experiment. Finally, at the time of the m^{th} failure, all the remaining $R_m = n - m - R_1 - R_2 - ... - R_{m-1}$ surviving units are removed from the experiment. The $R_i{}'s$ are fixed prior to the study.

Some of the earlier work on progressive censoring was conducted by Ghitany et al. (2014), Kim et al. (2011) and Rastogi and Tripathi (2014). In this article, we consider the analysis of progressively Type II censored lifetime data when the lifetime of each experimental unit follows a two-parameter generalized inverted exponential (GIE) distribution. The two-parameter GIE distribution with the shape and scale parameters $\alpha > 0$ and $\beta > 0$, respectively, has the probability density function (pdf) as:

$$f(x;\alpha,\beta) = \frac{\alpha\beta}{x^2}\exp(-\beta/x)[1-\exp(-\beta/x)]^{\alpha-1};\ x>0,\alpha>0,\beta>0. \quad (1)$$

and the corresponding cumulative distribution function (cdf) is given by

$$F(x;\alpha,\beta) = 1-[1-\exp(-\beta/x)]^{\alpha};\ x>0,\alpha>0,\beta>0. \quad (2)$$

We denote a two-parameter *GIE* distribution by $GIE(\alpha, \beta)$. Recent past, *GIE* distribution has been focus of investigation for many authors, see for example, Abouammoh and Alshingiti (2009), Dey and Pradhan (2014), Krishna and Kumar (2013) and Dey and Dey (2014). It is clear that the $GIE(\alpha, \beta)$ is reduced to the Inverted exponential distribution for $\alpha = 1$. Also, the distribution of $Y = 1/X$ has a generalized exponential distribution. The hazard function of *GIE* distribution is either increasing or decreasing, but is not constant, depending on α. The *GIE* distribution has a unimodal and right-skewed *pdf* for the shape parameter $\alpha > 4$. The aim of this article is twofold. We first try to earn the MLE's of the unknown parameters. It is observed that the maximum likelihood estimators can be obtained implicitly by solving two nonlinear equations, but they cannot be obtained in closed form. So using the iteration method, the MLE's of parameters are derived. The second aim of this article is to provide the exact confidence interval for the parameter β and the confidence region for the two unknown parameters of *GIE* distribution.

Moreover, we present the point prediction of the future observation based on the Type II progressive censored sample. The rest of the paper is organized as follows. In Section 2, MLEs of the unknown parameters are obtained using the iteration method. An exact confidence interval for the parameter β is provided in Section 3. The exact joint confidence region for the parameters α and β is presented in Section 4. In Section 5, a numerical comparison is made using Monte Carlo simulations. In Section 6, point prediction method is presented for obtaining the future data. An analysis of real data set appears in section 7 and finally we conclude the paper in Section 8.

2. Maximum Likelihood Estimation

Let $X_{1:m:n}, ..., X_{m:m:n}$ denote the progressively Type II order statistics observed from an experimental test involving n units taken from a $GIE(\alpha, \beta)$ distribution and $(R_1, ..., R_m)$ being the censoring scheme. To simplify the notation, we will use X_i in place of $X_{i:m:n}$.Then, the log-likelihood function from $GIE(\alpha, \beta)$, dropping terms that do not involve α and β is

$$L(\alpha, \beta) = m \ln \alpha + m \ln \beta - 2\sum_{i=1}^{m} \ln x_i - \beta \sum_{i=1}^{m} \frac{1}{x_i} - \sum_{i=1}^{m} \ln(1 - e^{-\beta/x_i})$$
$$+ \alpha \sum_{i=1}^{m} (R_i + 1) \ln(1 - e^{-\beta/x_i}). \tag{3}$$

Taking derivatives with respect to α and β in Equation (3) and equating them to zero, we obtain the likelihood equations as

$$\frac{\partial L}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m} (R_i + 1) \ln(1 - e^{-\beta/x_i}) = 0, \tag{4}$$

and

$$\frac{\partial L}{\partial \beta} = \frac{m}{\beta} - \sum_{i=1}^{m} \frac{1}{x_i} - \sum_{i=1}^{m} \frac{e^{-\beta/x_i}}{x_i(1 - e^{-\beta/x_i})} + \alpha \sum_{i=1}^{m} \frac{(R_i + 1)e^{-\beta/x_i}}{x_i(1 - e^{-\beta/x_i})} = 0. \tag{5}$$

The maximum-likelihood estimate of the parameter α is

$$\hat{\alpha}_{ML} = \frac{-m}{\sum\limits_{i=1}^{m} (R_i + 1) \ln(1 - e^{-\hat{\beta}_{ML}/x_i})}, \tag{6}$$

where $\hat{\beta}_{ML}$ can be obtained by solving the nonlinear equation

$$\frac{m}{\beta} - \sum_{i=1}^{m} \frac{1}{x_i} - \sum_{i=1}^{m} \frac{e^{-\beta/x_i}}{x_i(1 - e^{-\beta/x_i})} - \frac{m\hbar(\beta)}{\wp(\beta)} = 0, \tag{7}$$

where, $\hbar(\beta) = \sum\limits_{i=1}^{m} \frac{(R_i+1)e^{-\beta/x_i}}{x_i(1-e^{-\beta/x_i})}$ and $\wp(\beta) = \sum\limits_{i=1}^{m} (R_i + 1) \ln(1 - e^{-\beta/x_i})$.

Equation (7) can be written as

$$u(\beta) = \beta \tag{8}$$

where,

$$u(\beta) = \frac{m}{\sum\limits_{i=1}^{m} \frac{1}{x_i} + \sum\limits_{i=1}^{m} \frac{e^{-\beta/x_i}}{x_i(1-e^{-\beta/x_i})} + \frac{m\hbar(\beta)}{\Im(\beta)}}.$$

We propose a simple iterative scheme to solve (8). Start with an initial guess of β, say $\beta^{(0)}$, compute $\beta^{(1)} = u(\beta^{(0)})$. Continue the process until it converges. Once we obtain the MLE of β, then the MLE of α, say $\hat{\alpha}_{ML}$ can be obtained as from Equation (6).

3. Exact Confidence Interval for the Scale Parameter

Suppose that $Y_{i:m:n} = \alpha \ln\left(1 - e^{-\beta/X_i}\right)^{-1}$; $i = 1, 2, ..., m$. It can be seen that $Y_{1:m:n} < Y_{2:m:n} < ... < Y_{m:m:n}$ is a progressively Type II censored order statistics from an exponential distribution with mean 1. From Thomas and Wison (1972) , we have the generalized spacings,

$$\Upsilon_1 = nY_{1:m:n}, \quad \Upsilon_2 = (n - R_1 - 1)(Y_{2:m:n} - Y_{1:m:n}),$$
$$\Upsilon_3 = (n - R_1 - R_2 - 2)(Y_{3:m:n} - Y_{2:m:n}), ...,$$
$$\Upsilon_m = (n - R_1 - R_2 - R_{m-1} - m + 1)(Y_{m:m:n} - Y_{m-1:m:n}).$$

which are independent and identically random variables from a standard exponential distribution. Therefore, $\xi_1 = 2\Upsilon_1$ and $\xi_2 = 2\sum_{i=2}^{m} \Upsilon_i$ are independently chi-square distributed with 2 and $2m - 2$ degrees of freedom, respectively. We consider $\iota_1^* = \frac{\xi_2}{(m-1)\xi_1}$ and $\iota_2^* = \xi_1 + \xi_2$ as pivotal quantities. It is clear that ι_1^* has an F distribution with $2m - 2$ and 2 degrees of freedom and ι_2^* has chi-squared distribution with $2m$ degree of freedom. Moreover, ι_1^* and ι_2^* are independent.
In addition, the following lemma is needed for constructing the exact confidence interval.

Lemma. Based on the $X_1, X_2, ..., X_m$, let

$$\Im(X_1, ..., X_m, \beta) = \frac{\sum_{i=1}^{m} (R_i + 1)\left\{\frac{ln(1-e^{-\beta/X_i})}{ln(1-e^{-\beta/X_1})} - 1\right\}}{n(m-1)}. \quad (9)$$

Then, the function $\Im(X_1, ..., X_m, \beta)$ is strictly increasing of β; $\beta > 0$. Also, the $\Im(X_1, ..., X_m, \beta) = a$ has a unique solution for $a > 0$.

Proof. Based on the function $\Im(X_1, ..., X_m, \beta)$, it is observed that $ln(1 - e^{-\beta/X_i})$ is an increasing function of β and hence $\Im(X_1, ..., X_m, \beta)$ is a strictly increasing function of β. Furthermore,

$$\lim_{\beta \to 0} \Im(X_1, ..., X_m, \beta) = 0 \ , \ \lim_{\beta \to \infty} \Im(X_1, ..., X_m, \beta) = \infty.$$

Thus, there is a unique solution for the given equation $\Im(X_1, ..., X_m, \beta) = a$ when $a > 0$.

Now, let $F_\gamma(\lambda_1, \lambda_2)$ denotes the percentile of F distribution with the right-tail probability γ and λ_1 and λ_2 degrees of freedom.

Theorem 1. Suppose $X_1, ..., X_m$ be a progressively Type II censored order sample from a *GIE* distribution. Then the $100(1-\gamma)\%$ confidence interval for β is

$$\eta\left(X_1, ..., X_m, F_{1-\gamma/2}(2m-2, 2)\right) < \beta < \eta\left(X_1, ..., X_m, F_{\gamma/2}(2m-2, 2)\right).$$

Proof. It is observed that $\Im(X_1, ..., X_m, \beta)$has an F distribution with $2m-2$ and 2 degrees of freedom. So, we have

$$F_{1-\gamma/2}(2m-2, 2) < \frac{\sum_{i=1}^{m}(R_i+1)\left\{\frac{ln(1-e^{-\beta/X_i})}{ln(1-e^{-\beta/X_1})} - 1\right\}}{n(m-1)} < F_{\gamma/2}(2m-2, 2). \quad (10)$$

We know that equation $\Im(X_1, ..., X_m, \beta) = a$ has a unique solution for $a > 0$ (Lemma). Therefore, the Equation (10) is equivalent to the event

$$\eta\left(X_1, ..., X_m, F_{1-\gamma/2}(2m-2, 2)\right) < \beta < \eta\left(X_1, ..., X_m, F_{\gamma/2}(2m-2, 2)\right).$$

This complete the proof.

4. Exact Confidence Region for the Unknown Parameters

In this section, we discuss an exact joint confidence region for the parameters α and β.

Let $X_1, X_2, ..., X_m$ be a progressively Type II censored order sample from a *GIE* distribution with parameters α and β. Then the $100(1-\gamma)\%$ confidence region for α and β is given by

$$\begin{cases} \eta\left(X_1, ..., X_m, F_{h_1}(2m-2, 2)\right) < \beta < \eta\left(X_1, ..., X_m, F_{h_2}(2m-2, 2)\right) \\ \frac{\chi^2_{h_1}(2m)}{\iota_2^*} < \alpha < \frac{\chi^2_{h_2}(2m)}{\iota_2^*}; \ \iota_2^* = 2\alpha \sum_{i=1}^{m}(R_i+1)\ln\left(1-e^{-\beta/X_i}\right)^{-1}. \end{cases}$$

where, $h_1 = (1+\sqrt{1-\gamma})/2$ and $h_2 = (1-\sqrt{1-\gamma})/2$.

Proof. Since two pivotal quantities ($\iota_1^*$and ι_2^*) are independent and also, $\iota_1^*$and ι_2^* have F distribution and chi-square distribution respectively(defined before). So, we can write

$$P\left\{F_{h_1}(2m-2,2) < \iota_1^* < F_{h_2}(2m-2,2), \chi^2_{h_1}(2m) < \iota_2^* < \chi^2_{h_2}(2m)\right\} = \gamma^*, \tag{11}$$

where, $\gamma^* = \sqrt{1-\gamma} \times \sqrt{1-\gamma} = 1-\gamma$ (Due to independence of ι_1^* and ι_2^*). The proof is thus obtained, because, the Equation (11) is equivalent by

$$P\left\{ \eta\left(X_1, ..., X_m, F_{h_1}(2m-2,2)\right) < \beta < \eta\left(X_1, ..., X_m, F_{h_2}(2m-2,2)\right), \right.$$

$$\left. \frac{\chi^2_{h_1}(2m)}{\iota_2^*} < \alpha < \frac{\chi^2_{h_2}(2m)}{\iota_2^*} \right\} = \gamma * . \tag{12}$$

5. Simulations

In this section, we present some simulation results to examine the behavior of the different confidence intervals for different sample sizes (n), different effective sample sizes (m) and different sampling schemes (i.e., different Ri values). We generate the Type II progressive censored sample from the GIE distribution with parameters . We compute the MLEs and the 95% exact confidence intervals for β and joint confidence region of (α, β). A mean square error (MSE) criterion is used for comparison the MLEs. MSEs are computed based on 10000 progressively Type II censored samples generated from a $GIE(\alpha, \beta)$ distribution of size m.

The simulation programs are written in R and the results are presented in Tables 1 and 2 . From this simulation results, it is observed that, the coverage probabilities (CPs) of the exact intervals of β and the exact confidence region for(α, β), are all close to the desired level of 0.95. For a fixed n and m, one can determine the censoring scheme that is most efficient. For almost all choices, the $R_1 = n-m,\ R_i = 0;\ i \neq 1$ seems to provide the smallest MSEs for the estimators. We further observed that the censoring schemes that censored all $n-m$ items at the last observed failure (Schemes no. [2], [6], [9]) does not work well in the MSE senses.

Table 1: Progressive censoring schemes used in the Monte Carlo simulation study

n	m	$(R_1, R_2, ..., R_m)$	$SchemeNo.$
20	16	$R_1 = 4, R_i = 0$ for $i \neq 1$	[1]
20	16	$R_{16} = 4, R_i = 0$ for $i \neq 16$	[2]
20	16	$R_5 = 4, R_i = 0$ for $i \neq 5$	[3]
20	20	$R_i = 0$ for all i	[4]
60	50	$R_1 = 10, R_i = 0$ for $i \neq 1$	[5]
60	50	$R_{50} = 10, R_i = 0$ for $i \neq 50$	[6]
60	50	$R_1 = R_{50} = 5, R_i = 0$ for $i \neq 1, 50$	[7]
60	30	$R_1 = 30, R_i = 0$ for $i \neq 1$	[8]
60	30	$R_{30} = 30, R_i = 0$ for $i \neq 30$	[9]
60	60	$R_i = 0$ for all i	[10]

Table 2: The MSEs of the MLEs and average length and coverage probability (CP) of α and β.

$SchemeNo.$	$(MSE(\alpha), MSE(\beta))$	$Exact(\beta)$	$CP(\beta)$	$CP(\alpha, \beta)$
[1]	(0.0985, 0.1994)	1.74850	0.947	0.951
[2]	(0.1388, 0.2452)	1.98222	0.954	0.947
[3]	(0.1139, 0.1743)	1.75387	0.947	0.948
[4]	(0.0487, 0.1246)	1.72025	0.951	0.950
[5]	(0.0215, 0.0996)	0.89376	0.952	0.951
[6]	(0.0345, 0.1022)	0.91570	0.948	0.952
[7]	(0.0279, 0.1025)	0.91846	0.948	0.948
[8]	(0.0522, 0.1240)	0.92589	0.954	0.956
[9]	(0.0758, 0.1299)	0.95300	0.957	0.944
[10]	(0.0126, 0.0975)	0.90018	0.950	0.951

6. Point Prediction Method

In this section, we consider the method for predicting the future observation. Suppose that $X_{1:m:n}, ..., X_{m:m:n}$ is a progressively Type II order statistics from the *GIE* distribution with the censoring scheme $(R_1, ..., R_m)$.Our aim is to make point prediction about the $Z = X_{i,d}$; $(d = 1, 2, ..., R_i,\ i = 1, 2, ..., m)$. It is noted that Z denotes the d^{th} order statistic out of R_i removed units at stage *i*. The probability density function of $Z\,|X_i = x_i;\ z > x_i$ is given by

$$f(z\,|X_i = x_i) = \frac{R_i!}{(d-1)!(R_i-d)!}\frac{(F(z)-F(x_i))^{d-1}(1-F(z))^{R_i-d}f(z)}{(1-F(x_i))^{R_i}}. \tag{13}$$

We consider the Conditional median predictor method (CMPM) for predicting the censored data. A statistic $\hat{Z}$ is called a conditional median predictor, if

$$P(Z \leq \hat{Z}\,|X_i = x_i) = P(Z \geq \hat{Z}\,|X_i = x_i),$$

also

$$P(Z \leq \hat{Z}\,|X_i = x_i) = P(1-(\frac{\overline{F}(Z)}{\overline{F}(x_i)}) \geq 1-(\frac{\overline{F}(Z)}{\overline{F}(x_i)})\,|X_i = x_i).$$

Here, $\overline{F}(.)$ denotes the survival function of *GIE* distribution. From Equation (13), it is clear that the distribution of $1-(\frac{\overline{F}(Z)}{\overline{F}(x_i)})$ given $X_i = x_i$ is a $Beta(d,\ R_i-d+1)$ distribution with pdf

$$f(w) = \frac{w^{d-1}(1-w)^{n-s-d}}{Beta(d,\ n-s-d+1)}; \qquad 0 < w < 1.$$

So, the conditional median predictor can be written as

$$\hat{Y}_{CMP} = \overline{F}^{-1}\left[(1-Median(B))\ \overline{F}(x_i)\right]. \tag{14}$$

where, B has Beta distribution with shape parameters d and R_i-d+1 respectively.

7. Real Data Analysis

In this section, we present the data analysis of micro-droplet data which are obtained by Kang and Ng (2006), for illustrative purposes. Micro-droplet data is important data in coating process which in practice may be removed during the experiments. Therefore in this section, we consider the real data analyzing under different censoring schemes. First we verify whether *GIE* distribution is a valid model for this data set. For this purpose, we compute the Kolmogorov-Smirnov statistic and it is 0.1113 with the corresponding p-value = 0.7939.

Therefore, the high p-value clearly indicates that *GIE* model can be used to analyze this data set. For more comparison, we consider fitting of four other distributions such as generalized inverted Weibull (GIW), inverse Rayleigh (*IR*), inverse exponential (*IE*), inverse Weibull (IW) and exponential (E) distributions. We start with the GIW defined as:

$$f(x;\alpha,\beta,\delta) = \delta\beta\alpha^{\beta}x^{-(\beta+1)}\exp\left[-\delta(\frac{\alpha}{x})^{\beta}\right];\ x>0;\alpha>0,\beta>0,\delta>0.$$

The maximum likelihood estimator of (α,β,δ) is (6.42391, 7.87638, 0.03661). The log likelihood (LL), Akaike's information criterion ($AIC = 2*$number of parameters$-\ 2*\log L$), the associated second-order information criterion ($AICc = AIC + \frac{2*(\text{number of parameters})(\text{number of parameters}+1)}{n-\text{number of parameters}-1}$) and Bayesian information criterion ($BIC =$ (number of parameters)*($\log n) -$ $2*\log L$) are -32.3858, 70.7716, 71.5716 and 75.3506 respectively. Then we fit the *IR*, *IE*, IW and E distributions to the given data set.

First, we employ the maximum likelihood approach to estimate associated unknown parameters of these distributions. Then the different criteria of these distributions are obtained as:

IR: (LL, AIC, $AICc$, BIC)=(-62.7028, 127.4056, 127.5306, 128.9319).
IE: (LL, AIC, $AICc$, BIC)=(-85.4412, 172.8825, 173.0075, 174.4088).
IW: (LL, AIC, $AICc$, BIC)=(-32.3857, 68.7715, 0.691586, 71.8243).
E: (LL, AIC, $AICc$, BIC)=(-85.4318, 172.8636, 172.9886, 174.3900).

Additional model fitting to the GIE distribution yields:

GIE: (LL, AIC, $AICc$, BIC)=(-32.3857, 70.7715, 71.1586, 75.3506).

Therefore, different criteria such that maximum LL criterion and minimum AIC, $AICc$, BIC criteria suggest that *GIE* distribution can be considered as an adequate model for the given data set. For comparison purposes, we want to use the tracking interval (Panahi and Sayyareh (2016)) for comparing the two models. This interval helps us to evaluate proposed models in comparison with each other. In other words, if the calculated distance includes zero, it can be concluded that based on the predetermined confidence, both proposed models are equivalent.

An interval which does not contain zero, indicates that one model is better than the other one. Based on complete sample, the tracking interval of the two rival models $F(x;\theta) = \{f(x;\theta),\ \theta \in \Theta\}$ and $G(x,\gamma) = \{g(x,\gamma),\ \gamma \in \Gamma\}$ can be written as:

$$\left[D\left(f(x;\hat{\theta}), g(x;\hat{\gamma})\right) - n^{-1/2}\, z_{\alpha/2}\, \hat{\omega}\ ,\ D\left(f(x;\hat{\theta}), g(x;\hat{\gamma})\right) + n^{-1/2}\, z_{\alpha/2}\, \hat{\omega}\right] \tag{15}$$

where,

$$D(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) = \frac{1}{2n}\left[AIC(f^{\hat{\alpha}_n}) - AIC(g^{\hat{\beta}_n})\right] = -\frac{1}{n}\left[\ L_n^{f/g}\,(\hat{\alpha}_n, \hat{\beta}_n) - (p-q)\ \right]$$
$$= -\frac{1}{n}\left[\ \sum_{i=1}^{n} \ln \frac{f^{\hat{\alpha}_n}(x_i)}{g^{\hat{\beta}_n}(x_i)} - (p-q)\ \right].$$

Here, $\hat{\theta}$ and $\hat{\gamma}$ are the maximum likelihood estimators for the parameters θ and γ respectively. Also, p and q are the number of parameters in two models and

$$\hat{\omega}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\ln \frac{f(x_i;\hat{\theta})}{g(x_i;\hat{\gamma})}\right)^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\left(\ln \frac{f(x_i;\hat{\theta})}{g(x_i;\hat{\gamma})}\right)\right)^2.$$

First, we have estimated the unknown parameters using the MLEs and then constructed the tracking intervals for the rival models as:

I: *GIE*(g) and *IE*(f).
II: *GIE*(g) and *GIW*(f).

The tracking interval for cases I and II are $(8.6155\times10^{-2},\ 1.7621\times10^{-1})$ and $(0.077894, 0.091551)$ respectively. As we expected, both limits of the tracking interval are positive, which indicates that the *GIE* is better than the *IE* and *GIW* distributions to estimate the true model for micro-droplet dataset.

Now, we generated a progressively Type II censored sample from the original measurements. In this case, $n = 34$, and we take $m = 10$, $T = 5$, $R_1 = R_2 = ... = R_9 = 2$ and $R_{10} = 6$. In this case, the maximum likelihood estimates of α and β are 9.4369×10^3 and 40.8945 respectively. Using the above estimates and the percentiles $F_{0.025}(66, 2) = 0.2562$, $F_{0.975}(66, 2) = 39.4827$, $F_{0.025}(18, 2) = 0.2193$ and $F_{0.975}(18, 2) = 39.4422$, the 95% exact confidence interval of β based on complete and censored data are respectively computed as (27.54434 , 81.49943) and (26.62069 , 90.38516). Also, using the appropriate percentiles and based on the complete and progressively Type II censoring data, the 95% joint confidence region for α and β are determined by the following inequalities:

$$\begin{cases} 25.4277 < \beta < 88.32613 \\ \frac{44.632}{2\sum_{i=1}^{34}(R_i+1)\ln(1-e^{-\beta/X_i})^{-1}} < \alpha < \frac{96.6828}{2\sum_{i=1}^{34}(R_i+1)\ln(1-e^{-\beta/X_i})^{-1}}, \end{cases}$$

and

$$\begin{cases} 14.1767 < \beta < 82.2085 \\ \frac{8.5779}{2\sum_{i=1}^{10}(R_i+1)\ln(1-e^{-\beta/X_i})^{-1}} < \alpha < \frac{36.7027}{2\sum_{i=1}^{10}(R_i+1)\ln(1-e^{-\beta/X_i})^{-1}}. \end{cases}$$

Figures 1 and 2 show the shape of the 95% joint confidence regions for α and β under censored and complete samples respectively. It is easy to see that the region is large when β is large. Now we consider the prediction of the order statistics $(X_{i,d}(d = 1, 2, ..., R_i,\ i = 1, 2, ..., m))$, which are missing. The results are displayed in Table 3. It is observed that the prediction method works well.

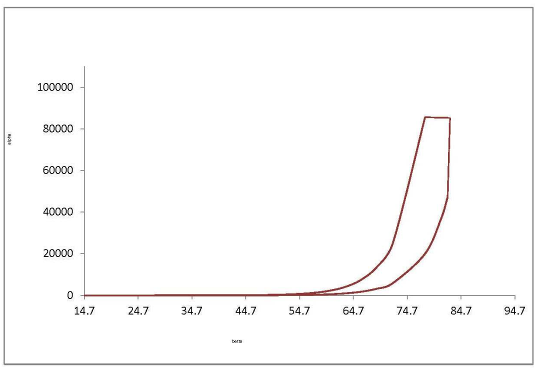

Figure 1: Joint confidence region for α and β under censored sample

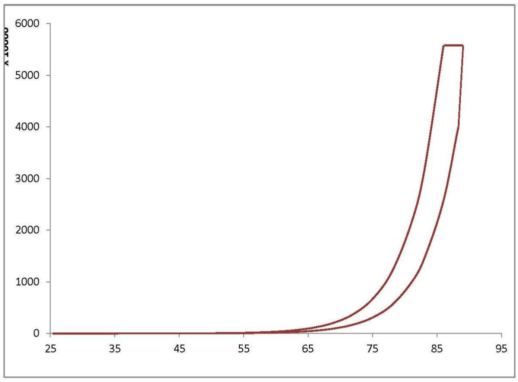

Figure 2: Joint confidence region for α and β under complete sample

Table 3: The point prediction for $Z = X_{i,d};\ (d = 1, 2, ..., R_i,\ i = 1, 2, ..., m)$ and their real values

	$X_{1,1}$	$X_{1,2}$	$X_{2,1}$	$X_{2,2}$	$X_{3,1}$
Predicted	3.724	3.765	3.769	3.899	4.023
Real	3.696	3.751	3.764	3.872	3.994

	$X_{3,2}$	$X_{4,1}$	$X_{4,2}$	$X_{5,1}$	$X_{5,2}$
Predicted	4.335	4.335	4.523	4.408	4.525
Real	4.319	4.319	4.495	4.386	4.495

	$X_{6,1}$	$X_{6,2}$	$X_{7,1}$	$X_{7,2}$	$X_{8,1}$
Predicted	4.613	4.614	4.809	4.824	4.837
Real	4.603	4.605	4.806	4.820	4.833

	$X_{8,2}$	$X_{9,1}$	$X_{9,2}$	$X_{10,1}$	$X_{10,2}$
Predicted	4.877	4.948	5.341	5.082	5.086
Real	4.874	4.941	5.334	5.050	5.052

	$X_{10,3}$	$X_{10,4}$	$X_{10,5}$	$X_{10,6}$	
Predicted	5.152	5.189	5.229	5.700	
Real	5.131	5.172	5.226	5.686	

8. Conclusion

In this paper, we considered the estimation of the unknown parameters of a $GIE(\alpha, \beta)$ distribution when items lie under the well-known progressively Type II censoring scheme. It is observed that the maximum likelihood estimators of the unknown parameters can be obtained by solving a simple iterative procedure. Two pivotal quantities for constructing an exact confidence interval and an exact confidence region for the parameters are provided. Theoretical results are studied through the Monte Carlo simulations. The results show that the coverage probabilities for intervals are better and closer to the nominal level of 95% when the proportion of uncensored data is larger. The point prediction of the future order statistics are developed based on the observed progressively Type II censored data. Finally, real example is presented to illustrate all the inferential results established here.

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