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Functions Defined by Coefficient Inequalities

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ABSTRACT

In this paper, sharp radii constants are obtained for the analytic functions satisfying some coefficient inequalities. For such functions, growth and distortion estimates are determined. In addition, it is proved that functions in these classes are closed under Hadamard product with convex functions.

Keywords: Coefficient inequality; starlike functions; α-convex function; radius of univalence; convolution; convex combination.

1. Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. The well-known Bieberbach conjecture of 1916, proved by Branges (1985) states that an analytic univalent function $f : \mathbb{D} \to \mathbb{C}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

satisfies the sharp inequality $|a_n| \leq n$ for all $n \geq 2$. The converse does not hold. For example, the function $f(z) = 2z - z/(1-z)^2 = z - \sum_{n=2}^{\infty} nz^n$ is non-univalent. Therefore, the inequality $|a_n| \leq n$ for $n \geq 2$ is not a sufficient condition for a function f to be univalent. In 1970, for the analytic functions f of the form (1), satisfying the coefficient inequality $|a_n| \leq n$, Gavrilov (1970) has shown that the radius of univalence is r_0, where r_0 is the real root of the equation $2(1-r)^3 - (1+r) = 0$. This result is sharp for function $f(z) = 2z - z/(1-z)^2$. In addition, Gavrilov showed that the radius of univalence of the function f satisfying $|a_n| \leq M$ $(M > 0)$ is $1 - \sqrt{M/(1+M)}$. In 1982, Yamashita (1982) proved that the radius of univalence, obtained by Gavrilov is equal to radius of starlikeness for the corresponding function. Kalaj et al. (2014) determined the radius of univalence, starlikeness, and convexity for harmonic functions. For $0 \leq b \leq 1$, let $\mathcal{A}_b$ denote the class of functions given by (1) with fixed second coefficient $|a_2| = 2b$. Ravichandran (2014) obtained the sharp radius of starlikeness and convexity of order α for the functions $f \in \mathcal{A}_b$, satisfying the coefficient inequalities $|a_n| \leq n$, M, or M/n $(M > 0)$ for $n \geq 3$. Nargesi et al. (2014) obtained similar sharp radius constants for function $f \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq cn + d$ and c/n $(c > 0)$. Sharma and Ankita (2015) also determined radius estimates for functions in a Janowski type class, satisfying certain coefficient inequalities. Recently, Mendiratta et al. (2015) determined the sharp radii of starlikeness of order α, convexity of order α $(0 \leq \alpha \leq 1)$, parabolic starlikeness and uniform convexity when $|a_n| \leq M/n^2$ or Mn^2 $(M > 0)$ for $n \geq 3$.

A function f of the form (1) is α-convex (see Goodman (1983)) if it is analytic, $f(z)f'(z)/z \neq 0$ and satisfies the inequality

$$\operatorname{Re}\left(\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{zf'(z)}{f(z)}\right) > 0$$

for all $z \in \mathbb{D}$. In particular, for $\alpha = 1$ and $\alpha = 0$, α-convex functions become convex and starlike respectively. For details of α-convex functions, see Miller et al. (1973), Mocanu and Reade (1975), Noor (1996).

In the present paper, we consider the class $\mathcal{S}_k$ of analytic functions defined as

$$\mathcal{S}_k := \left\{ f : f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ is analytic in } \mathbb{D} \text{ and } |a_n| \leq cn^k \right\}, \qquad (2)$$

for $k = 0, \pm 1, \pm 2$ and some $c > 0$. The radius of α-convexity is determined for the class $\mathcal{S}_k$. For any value of β, the sharp radii constants are obtained for a function $f \in \mathcal{S}_k$ that satisfies the inequality $\operatorname{Re}(f'(z) + \beta z f''(z)) > 0$. For functions $f \in \mathcal{S}_k$, bounds for $|f(z)/z|$ and $|f'(z)|$ are determined. Further, closure properties of the class $\mathcal{S}_k$ under convolution with convex function are investigated.

2. Radius constants

First, the sharp radius of α- convexity is determined for analytic functions f of the form (1) to be in the classes $\mathcal{S}_k$, for $k = 0, \pm 1, \pm 2$.

Theorem 2.1. *Let $0 \leq \alpha \leq 1$ and let the function f be defined by* (1).

(a) If $f \in \mathcal{S}_1$, then f is α-convex in the disk $|z| < r_1$, where r_1 is a real root of the equation

$$-1 + 2(\alpha c + 4c + 3)r - \left(16c^2 + 30c + 15\right) r^2 + 2(-3\alpha c^2 - 3\alpha c + 12c^2 + 22c + 10)r^3 + \left(4\alpha c^2 + 4\alpha c - 17c^2 - 32c - 15\right) r^4 + (c+1)^2 (6 - r) r^5 = 0$$

(b) If $f \in \mathcal{S}_0$, then f is α-convex in the disk $|z| < r_0$, where r_0 is a real root of the equation

$$1 - r((3\alpha + 2)c + 5) + r^2 \left(4\alpha c^2 + 5(\alpha + 2)c + 10\right) - r^3(c+1)((\alpha + 6)c + 10) - (c+1)r^4((\alpha - 5)c - 5) - (c+1)^2 r^5 = 0$$

(c) If $f \in \mathcal{S}_{-1}$, then f is α-convex in the disk $|z| < r_{-1}$, where r_{-1} is a real root of the equation

$$(1 - r) + \frac{c(1-\alpha)(r + (1-r)\log(1-r))}{(1+c)r + c\log(1-r)} + \frac{c\alpha r}{(c+1)(1-r) - c} = 0.$$

(d) If $f \in \mathcal{S}_2$, then f is α-convex in the disk $|z| < r_2$, where r_2 is a root of

the equation

$$-1+4((\alpha+4)c+2)r-2((37-2\alpha)c+32c^2+14)r^2-4((9\alpha-20)c^2 +4(2\alpha-9)c-14)r^3+((32\alpha-89)c^2+2(16\alpha-79)c-70)r^4-4(c+1) ((\alpha-14)c-14)r^5-2(c+1)((2\alpha+13)c+14)r^6+8(c+1)^2r^7 -(c+1)^2r^8=0.$$

(e) *If $f\in\mathcal{S}_{-2}$, then f is α-convex in the disk $|z|<r_{-2}$, where r_{-2} is a real root of the equation*

$$(1-r)+\frac{c(1-\alpha)(1-r)(\log(1-r)+Li_2(r))}{(1+c)r-cLi_2(r)}-\frac{c\alpha(r+(1-r)\log(1-r))}{(1+c)r+c\log(1-r)}=0$$

where $Li_2:\mathbb{D}\to\mathbb{C}$ is the polylogarithm function of order 2 given by

$$Li_2(z)=\sum_{n=1}^{\infty}z^n/n^2. \tag{3}$$

The results are sharp.

Proof. Set $\mathcal{Q}_{\alpha ST}=\alpha(1+zf''(z)/f'(z))+(1-\alpha)zf'(z)/f(z)$ and $\mathbb{D}(a,R)=\{\,z\in\mathbb{C}:\,|z-a|<R\,\}$. In particular, $\mathbb{D}(0,1)=\mathbb{D}$.

(a) Since $f\in\mathcal{S}_1$, we have

$$\begin{aligned}|\mathcal{Q}_{\alpha ST}-1|&\leq(1-\alpha)\frac{\sum_{n=2}^{\infty}(n-1)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}|a_n||z|^{n-1}}+\alpha\frac{\sum_{n=2}^{\infty}n(n-1)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty}n|a_n||z|^{n-1}}\\ &\leq(1-\alpha)\frac{c\sum_{n=2}^{\infty}n^2r^{n-1}-c\sum_{n=2}^{\infty}nr^{n-1}}{1-c\sum_{n=2}^{\infty}nr^{n-1}}\\ &\qquad+\alpha\frac{c\sum_{n=2}^{\infty}n^3r^{n-1}-c\sum_{n=2}^{\infty}n^2r^{n-1}}{1-c\sum_{n=2}^{\infty}n^2r^{n-1}}\\ &=(1-\alpha)\frac{2cr}{(1-r)[(1+c)r^2-(2+2c)r+1]}\\ &\qquad+\alpha\frac{2cr^2+4cr}{(1-r)[(-1-c)r^3+3(1+c)r^2+(-3-4c)r+1]}=R_1(r).\end{aligned}$$

Therefore, it follows that $\operatorname{Re}(\mathcal{Q}_{\alpha ST})>1-R_1(r_1)=0$ for $0\leq r\leq r_1$. The result is sharp for the function defined by

$$f_0(z)=(1+c)z-\frac{cz}{(1-z)^2}. \tag{4}$$

For the function $f_0(z)$, we have $\mathcal{Q}_{\alpha ST} = 1 - R_1(z)$ and so, at $z = r_1$, $\operatorname{Re}(\mathcal{Q}_{\alpha ST}) = \mathcal{Q}_{\alpha ST} = 1 - R_1(r_1) = 0$ and hence the radius of α- convexity is sharp.

(b) Since $f \in \mathcal{S}_0$, $\mathcal{Q}_{\alpha ST} \in \mathbb{D}(1, R_0)$, where

$$R_0 = \frac{c(1-\alpha)r}{(1-r)((1-r(1+c))} + \frac{2cr\alpha}{(1-r)((1-r)^2 - cr(2-r))},$$

we see that $\operatorname{Re}(\mathcal{Q}_{\alpha ST}) > 1 - R_0 = 0$ for $0 \leq r \leq r_0$. This result is sharp for the function

$$f(z) = z - \frac{cz^2}{1-z}.$$

(c) Since $f \in \mathcal{S}_{-1}$, $\mathcal{Q}_{\alpha ST} \in \mathbb{D}(1, R_{-1})$, where

$$R_{-1} = \frac{c(1-\alpha)(r + (1-r)\log(1-r))}{(1-r)((1+c)r + c\log(1-r))} + \frac{\alpha cr}{(1-r)((1+c)(1-r) - c)},$$

it is easily seen that $\operatorname{Re}(\mathcal{Q}_{\alpha ST}) > 1 - R_{-1} = 0$ for $0 \leq r \leq r_{-1}$. The result is sharp by considering the function defined by

$$f_0(z) = z - cz - c\log(1-z). \tag{5}$$

(d) Since $f \in \mathcal{S}_2$, $\mathcal{Q}_{\alpha ST} \in \mathbb{D}(1, R_2)$, where

$$R_2 = \frac{2c(1-\alpha)r(r+2)}{(1-r)((1+c)(1-r)^3 - c(1+r))} + \frac{c\alpha(2r^3 + 14r^2 + 8r)}{(1-r)((1+c)(1-r)^4 - c(1+4r+r^2))},$$

it is easily seen that $\operatorname{Re}(\mathcal{Q}_{\alpha ST}) > 1 - R_2 = 0$ for $0 \leq r \leq r_2$. The result is sharp by considering the function defined by

$$f_0(z) = z - cz^2(z^2 - 3z + 4)/(1-z)^3. \tag{6}$$

(e) Since $f \in \mathcal{S}_{-2}$, $\mathcal{Q}_{\alpha ST} \in \mathbb{D}(1, R_{-2})$, where

$$R_{-2} = -\frac{c(1-\alpha)(\log(1-r) + \mathrm{Li}_2(r))}{((1+c)r - c\mathrm{Li}_2(r))} + \frac{c\alpha(r + (1-r)\log(1-r))}{(1-r)((1+c)r + c\log(1-r))},$$

it is easy to deduce that $\operatorname{Re}(\mathcal{Q}_{\alpha ST}) > 1 - R_{-2} = 0$ for $0 \leq r \leq r_2$. The result is sharp for the function

$$f_0(z) = z - c(\mathrm{Li}_2(z) - z). \tag{7}$$

□

Motivated by the work of Mendiratta et al. (2015), we close this section by radii constants for functions belonging to $\mathcal{S}_k$ for $k = 0, \pm 1, \pm 2$.

Theorem 2.2. *Let the function f be defined by* (1)*.*

(a) If $f \in \mathcal{S}_1$, then f satisfies the inequality $\operatorname{Re}(f'(z) + \beta z f''(z)) > 0$ in the disk $|z| < r_{\beta_1}$, where r_{β_1} is a real root of the equation

$$(1 + c(|\beta| + |1 - \beta|))(1 - r)^4 - c|\beta|(1 + 4r + r^2) - c|1 - \beta|(1 - r^2) = 0.$$

(b) If $f \in \mathcal{S}_{-1}$, then f satisfies the inequality $\operatorname{Re}(f'(z) + \beta z f''(z)) > 0$ in the disk $|z| < r_{\beta_{-1}}$, where $r_{\beta_{-1}}$ is a real root of the equation

$$(1 + c(|\beta| + |1 - \beta|))(1 - r)^2 - c|1 - \beta|(1 - r) - c|\beta| = 0.$$

(c) If $f \in \mathcal{S}_2$, then f satisfies the inequality $\operatorname{Re}(f'(z) + \beta z f''(z)) > 0$ in the disk $|z| < r_{\beta_2}$, where r_{β_2} is a real root of the equation

$$\begin{aligned}(1 + c(|\beta| + |1 - \beta|))(1 - r)^5 - c|\beta|(1 + 11r + 11r^2 + r^3)& \\ - c|1 - \beta|(1 - 3r - 3r^2 - r^3) &= 0.\end{aligned}$$

(d) If $f \in \mathcal{S}_{-2}$, then f satisfies the inequality $\operatorname{Re}(f'(z) + \beta z f''(z)) > 0$ in the disk $|z| < r_{\beta_{-2}}$, where $r_{\beta_{-2}}$ is a real root of the equation

$$(1 + c|1 - \beta| r(1 - r) - c|\beta| r^2 + c|1 - \beta|(1 - r)\log(1 - r)) = 0.$$

The results are sharp.

Proof. Set $\mathcal{F}_\beta = f'(z) + \beta z f''(z)$.

(a) If $f \in \mathcal{S}_1$, then a simple calculation yields $|\mathcal{F}_\beta - 1| \leq R_{\beta_1}$ where

$$R_{\beta_1} = -c(|\beta| + |1 - \beta|) + \frac{c|\beta|(1 + 4r + r^2)}{(1 - r)^4} + \frac{c|1 - \beta|(1 + r)}{(1 - r)^3},$$

so that $\operatorname{Re}(\mathcal{F}_\beta) > 1 - R_{\beta_1} = 0$ for $0 \leq r \leq r_{\beta_1}$. The result is sharp for the function defined by (4).

Let $f \in \mathcal{S}_1$. Then f is univalent in the disk $|z| < r_0$, where r_0 is a root of the equation $(1 + c)(1 - r)^3 - c(1 + r) = 0$ and f satisfies the inequality $\operatorname{Re}(f'(z) + \beta f''(z)) > 0$ in the disk $|z| < r_0^*$, where r_0^* is a root of the equation $(1 + c)(1 - r)^4 - c(1 + 4r + r^2) = 0$. If we take $\beta = 0$ and $c = 1$, Theorem 2.2(a) simplifies to (Yamashita, 1982, Theorem 1, p. 454).

(b) If $f \in \mathcal{S}_{-1}$, then a simple computation gives $\mathcal{F}_\beta \in \mathbb{D}(1, R_{\beta_{-1}})$ where

$$R_{\beta_{-1}} = -c(|\beta| + |1-\beta|) + \frac{c|\beta|}{(1-r)^2} + \frac{c|1-\beta|}{(1-r)}.$$

Therefore it is easily seen that $\mathrm{Re}(\mathcal{F}_\beta) > 1 - R_{\beta_{-1}} = 0$ for $0 \le r \le r_{\beta_{-1}}$. Sharpness follows from the function defined by (5).

Further, a calculation shows if $f \in \mathcal{S}_{-1}$, then function f is univalent in the disk $|z| < 1/(1+c)$.

(c) If $f \in \mathcal{S}_2$, then a calculation gives $\mathcal{F}_\beta \in \mathbb{D}(1, R_{\beta_2})$ where

$$R_{\beta_2} = -c(|\beta|+|1-\beta|) + \frac{c|\beta|(1+11r+11r^2+r^3)}{(1-r)^5} + \frac{c|1-\beta|(1+4r+r^2)}{(1-r)^4},$$

and, therefore, $\mathrm{Re}(\mathcal{F}_\beta) > 1 - R_{\beta_2} = 0$ for $0 \le r \le r_{\beta_2}$. Sharpness follows from the function defined by (6).

(d) If $f \in \mathcal{S}_{-2}$, we have $\mathcal{F}_\beta \in \mathbb{D}(1, R_{\beta_{-2}})$ where

$$R_{\beta_{-2}} = \frac{c|\beta|}{(1-r)} - \frac{c|1-\beta|\log(1-r)}{r} - c|1-\beta|,$$

and therefore it is easy to deduce that $\mathrm{Re}(\mathcal{F}_\beta) > 1 - R_{\beta_{-2}} = 0$ for $0 \le r \le r_{\beta_{-2}}$. For sharpness, we consider the function defined by (7). □

If $f \in \mathcal{S}_2$, then f is univalent in the disk $|z| < r_0$ where r_0 is a real root of the equation $(1+c)(1-r)^4 - c(1+4r+r^2) = 0$ and if $f \in \mathcal{S}_{-2}$, we note that f is univalent in the disk $|z| < r_0^*$ where r_0^* is a real root of the equation $r + c(r + \log(1-r)) = 0$. Similarly it is easily seen that if $f \in \mathcal{S}_0$, then f satisfies the inequality $\mathrm{Re}(f'(z) + f''(z)) > 0$ in the disk $|z| < r_c$, where r_c is a real root of the equation $(1+c)(1-r)^3 - c(1+r) = 0$. This result is sharp and we observe that the radius r_c is equal to the radius of univalence, obtained by Gavrilov (1970) for the function f with coefficient inequality $|a_n| \le n$.

3. Growth and distortion estimates

In this section, for the analytic function f in the the class $\mathcal{S}_k$, we compute sharp estimates of $|f(z)/z|$ and $|f'(z)|$.

Theorem 3.1. *Let $|z| = r < 1$ and the function f be given by* (1).

(a) If $f \in \mathcal{S}_1$, then

(i) $(1+c) - \frac{c}{(1-r)^2} \leq \operatorname{Re}\left(\frac{f(z)}{z}\right) \leq \left|\frac{f(z)}{z}\right| \leq (1-c) + \frac{c}{(1-r)^2}$;

(ii)

$$((c+1)r^3 - 3(c-1)r^2 + (4c+3)r - 1)/(r-1)^3 \leq \operatorname{Re}(f'(z)) \\ \leq |f'(z)| \leq ((1-c)r^3 + 3(c-1)r^2 + (3-4c)r - 1)/(r-1)^3.$$

(b) If $f \in \mathcal{S}_{-1}$, then

(i) $(1+c) + \frac{c}{r}\log(1-r) \leq \operatorname{Re}\left(\frac{f(z)}{z}\right) \leq \left|\frac{f(z)}{z}\right| \leq (1-c) - \frac{c}{r}\log(1-r)$;

(ii) $(1-(1+c)r)/(1-r) \leq \operatorname{Re}(f'(z)) \leq |f'(z)| \leq (1+(c-1)r)/(1-r)$.

(c) If $f \in \mathcal{S}_2$, we have

(i) $(1+c) - \frac{c(1+r)}{(1-r)^3} \leq \operatorname{Re}\left(\frac{f(z)}{z}\right) \leq \left|\frac{f(z)}{z}\right| \leq (1-c) + \frac{c(1+r)}{(1-r)^3}$;

(ii) $(1+c) - c\frac{(1+4r+r^2)}{(1-r)^4} \leq \operatorname{Re}(f'(z)) \leq |f'(z)| \leq (1-c) + c\frac{(1+4r+r^2)}{(1-r)^4}$.

(d) If $f \in \mathcal{S}_{-2}$, we have the following estimates in terms of polylogarithm function of order 2 defined by (3)*:*

(i) $(1+c) - cLi_2(r)/r \leq \operatorname{Re}(f(z)/z) \leq |f(z)/z| \leq (1-c) + cLi_2(r)/r$;

(ii) $(1+c) + c\log(1-r)/r \leq \operatorname{Re}(f'(z)) \leq |f'(z)| \leq (1-c) - c\log(1-r)/r$.

(e) If $f \in \mathcal{S}_0$, then

(i) $(1+c) - c/(1-r) \leq \operatorname{Re}(f(z)/z) \leq |f(z)/z| \leq (1-c) + c/(1-r)$;

(ii) $(1+c) - c/(1-r)^2 \leq \operatorname{Re}(f'(z)) \leq |f'(z)| \leq (1-c) + c/(1-r)^2$.

All the estimates are sharp.

Proof. Let $|z| = r < 1$.

(a) If $f \in \mathcal{S}_1$, then the upper bound for $|f(z)/z|$ is given by

$$\left|\frac{f(z)}{z}\right| \leq 1 + c\sum_{n=2}^{\infty} nr^{n-1} = (1-c) + \frac{c}{(1-r)^2}$$

and the lower bound for $\mathrm{Re}(f(z)/z)$ is given by

$$\mathrm{Re}\left(\frac{f(z)}{z}\right) \geq 1 - c\sum_{n=2}^{\infty} nr^{n-1} = (1+c) - \frac{c}{(1-r)^2},$$

which completes part (i). Similarly, $|f'(z)| \leq (1-c) + c(1+r)/(1-r)^3$ and $|f'(z)| \geq \mathrm{Re}\,(f'(z)) \geq (1+c) - c(1+r)/(1-r)^3$ prove second part. For the upper bound, the sharpness follows for the function $f_0(z) = (1-c)z + cz/(1-z)^2$ and for the lower bound, the sharpness follows for the function defined by (4).

(b) If $f \in \mathcal{S}_{-1}$, then $|f(z)/z| \leq 1+c\sum_{n=2}^{\infty}(r^{n-1}/n) = 1+c(-1-\log(1-r)/r)$ and $\mathrm{Re}(f(z)/z) \geq 1 - c(-1 - \log(1-r)/r)$ prove part (i). Similarly $|f'(z)| \leq 1 + cr/1 - r$ and $|f'(z)| \geq \mathrm{Re}(f'(z)) \geq 1 - cr/(1-r)$ prove part (ii). For the lower bound, the result is sharp for the function defined by (5) and for upper bound, result is sharp for the function $g_0(z) = z + cz + c\log(1-z)$.

(c) If $f \in \mathcal{S}_2$, then $|f(z)/z| \leq 1 + c(-1+1+r/(1-r)^3)$ and $\mathrm{Re}(f(z)/z) \geq 1 - c(-1+1+r/(1-r)^3)$ yield part (i). Similarly

$$|f'(z)| \leq 1 + c\left(\frac{1+4r+r^2}{(1-r)^4} - 1\right)$$

and $|f'(z)| \geq \mathrm{Re}(f'(z)) \geq 1 - c(1+4r+r^2/(1-r)^4 - 1)$ give the part (ii). For upper bound, sharpness follows for the function $f_0(z) = z + cz^2(z^2-3z+4)/(1-z)^3$ and for lower bound, sharpness follows for the function defined by (6).

(d) If $f \in \mathcal{S}_{-2}$, we have $|f(z)/z| \leq 1 + c(\mathrm{Li}_2(r)/r - 1)$ and $\mathrm{Re}(f(z)/z) \geq 1 - c(\mathrm{Li}_2(r)/r - 1)$, part (i) follow. A similar calculation leads to the upper and lower bounds for $|f'(z)|$. For upper bound, sharpness follows for the function $f_0(z) = z + c(\mathrm{Li}_2(z) - z)$ and for lower bound, sharpness follows for the function defined by (7).

(e) Proof is similar to previous parts. □

4. Closure theorems

For two analytic functions $f(z) = z+\sum_{n=1}^{\infty} a_n z^n$ and $g(z) = z+\sum_{n=2}^{\infty} b_n z^n$, the Hadamard product or convolution of f and g, is defined by $(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,\ z \in \mathbb{D}$ (see Ruscheweyh (1982)). The following theorem proves that the class $\mathcal{S}_k$ is closed under Hadamard product with convex functions.

Theorem 4.1. *Let $f \in \mathcal{S}_k$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ satisfy the coefficient inequality $|b_n| \leq 1$ for $n \geq 2$. Then $f * g \in \mathcal{S}_k$. In particular, if $f \in \mathcal{S}_k$ and g belongs to the class of convex functions, then $f * g \in \mathcal{S}_k$.*

Proof. If $f \in \mathcal{S}_k$, then $|a_n| \leq cn^k$. Since $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$ where $|a_n b_n| \leq cn^k$, it follows that $f * g \in \mathcal{S}_k$. □

The last theorem shows that the class $\mathcal{S}_k$ is closed under convex combinations of its members.

Theorem 4.2. *Let $0 \leq \lambda_j \leq 1$ for $j = 1, 2, 3, \ldots m$ and $\sum_{j=1}^{m} \lambda_j = 1$. Let the functions f_j defined by $f_j(z) = z + \sum_{n=2}^{\infty} a_{n,j} z^n \quad (j = 1, 2, 3, \ldots m)$ belong to the class $\mathcal{S}_k$. Then $\sum_{j=1}^{m} \lambda_j f_j \in \mathcal{S}_k$.*

Proof. Since $f_j \in \mathcal{S}_k$, $|a_{n,j}| \leq cn^k$ for $j = 1, 2, 3, \ldots m$, then

$$\begin{aligned}
\sum_{j=1}^{m} \lambda_j f_j(z) = \sum_{j=1}^{m} \lambda_j \left(z + \sum_{n=2}^{\infty} a_{n,j} z^n \right) \\
= z + \sum_{n=2}^{\infty} (\lambda_1 a_{n,1} + \lambda_2 a_{n,2} + \cdots + \lambda_m a_{n,m}) z^n
\end{aligned}$$

where $|\lambda_1 a_{n,1} + \lambda_2 a_{n,2} + \cdots + \lambda_m a_{n,m}| \leq \lambda_1 |a_{n,1}| + \lambda_2 |a_{n,2}| + \cdots + \lambda_m |a_{n,m}| \leq cn^k$. □

Corollary 4.1. *The class $\mathcal{S}_k$ is closed under convex combinations.*

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References

Branges, L. d. (1985). A proof of the bieberbach conjecture. *Acta Math.*, 154(1-2):137–152.

Gavrilov, V. I. (1970). Remarks on the radius of univalence of holomorphic functions. *Mat. Zametki*, 7:295–298.

Goodman, A. W. (1983). *Univalent Functions.* Mariner Tampa, FL, vol 1 edition.

Kalaj, D., Ponnusamy, S., and Vuorinen, M. (2014). Radius of close-to-convexity and fully starlikeness of harmonic mappings. *Complex Var. Elliptic Equ.*, 59(4):539–552.

Mendiratta, R., Nagpal, S., and Ravichandran, V. (2015). Radii of starlikeness and convexity for analytic functions with fixed second coefficient satisfying certain coefficient inequalities. *Kyungpook Math. J.*, 55(2):395–410.

Miller, S. S., Mocanu, P., and Reade, M. O. (1973). All α-convex functions are univalent and starlike. *Proc. Amer. Math. Soc.*, 37:553–554.

Mocanu, P. T. and Reade, M. O. (1975). The radius of α-convexity for the class of starlike univalent functions, α-real. *Proc. Amer. Math. Soc.*, 51:395–400.

Nargesi, M. M., Ali, R. M., and Ravichandran, V. (2014). Radius constants for analytic functions with fixed second coefficient. *Scientific World J.*, 2014(2014)(Art. ID 898614,):6 pp.

Noor, K. I. (1996). On strongly alpha-convex and alpha-quasi convex functions. *J. Natur. Geom.*, 10(2):111–118.

Ravichandran, V. (2014). Radii of starlikeness and convexity of analytic functions satisfying certain coefficient inequalities. *Math. Slovaca*, 64(1):27–38.

Reade, M. O. (1955). On close-to-convex univalent functions. *Michigan Math. J.*, 3:59–62.

Ruscheweyh, S. (1982). *Convolutions in Geometric Function Theory.* Séminaire de Mathématiques Supérieures, 83, Presses Univ. Montréal.

Sharma, P. and Ankita (2015). Radius constants for a class of analytic functions associated with a multiplier linear operator. *Facta Univ. Ser. Math. Inform.*, 30(5):693–705.

Yamashita, S. (1982). Radii of univalence, starlikeness, and convexity. *Bull. Austral. Math. Soc.*, 25(3):453–457.