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Model Reduction of Non-minimal Discrete-Time Linear-Time-Invariant Systems

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ABSTRACT

Model reduction is a method for reducing the order of mathematical models such that the behavior of reduced system is similar with the original system. Many models in real systems are non minimal. However, to the best of our knowledge, there is no literature that discusses the model reduction of non-minimal systems. Therefore in this paper, we propose a procedure for model reduction of non-minimal discrete-time linear-time-invariant systems by using balanced truncation methods. In this paper, we generate an algorithm to reduce non-minimal discrete-time linear-time-invariant systems. From the simulation results, we obtain a reduced system with similar behavior with the original system. Furthermore, we also conclude that the behavior of the reduced system is very close to the original system in high frequency.

Keywords: Balanced truncation methods, model reduction, non-minimal discrete-time linear-time-invariant systems.

1. Introduction

Model reduction is a method to reduce the order of mathematical models. As we already know, many real-life systems have a large order. Thus the computational time to study the behavior of such systems takes a long time. Therefore, we need a method to reduce the order of mathematical models.

There are many methods in the literature for model reduction, for example balanced truncation and balanced singular perturbation approximation Green and Limebeer (2012). Arif et al. (2014) have combined Kalman filter and balanced truncation methods. Zhou et al. (1999) have developed some new controllability and observability gramians for unstable systems. In Kumar et al. (2011b), an algorithmic approach for reduction of a stable system based on hankel norm approximation technique is extended. Then the reduction of large-scale systems using extended balanced truncation approach has been discussed in Kumar et al. (2011a). Finally, Magruder et al. (2010) discussed the potential of an input-output map associated with an unstable system to represent a bounded map from $\mathcal{L}_2(\mathbb{R})$ to itself and then develop criteria for optimal reduced order approximations to the original (unstable) system with respect to an $\mathcal{L}_2$-induced Hilbert-Schmidt norm. Recently, Sari et al. (2017) and Kartika et al. (2017) have studied model reduction of discrete-time and continuous-time systems using balanced truncation approach. Then, Lesnussa et al. (2017) and Mustaqim et al. (2017) applied balanced truncation methods to estimation of state variables and shallow water equations, respectively. Finally, Arif et al. (2017) studied the model reduction of unstable systems using singular perturbation approximation.

To the best of our knowledge, the model reduction of non-minimal systems has not been discussed in the above literature. Furthermore, many real-life systems are non-minimal. Thus in this work, we develop a model reduction procedure for non-minimal discrete-time linear-time-invariant systems. The procedure is as follows. First we check whether the original system is stable, controllable and observable. If the original system is not asymptotically stable, we decompose the original system into two subsystems: asymptotically stable subsystem and stable or unstable subsystem. Then we check whether the asymptotically stable subsystem is controllable and observable. If the asymptotically stable subsystem is uncontrollable or unobservable, then we transform the asymptotically stable subsystem into its minimal realization. Finally the total reduced system is obtained by combining the minimal asymptotically stable subsystem and the stable or unstable subsystem.

This manuscript is structured as follows. Section 2 discusses some models and algorithms related to model reduction. Section 3 presents the model reduction procedure for non-minimal discrete-time linear-time-invariant systems. Then the procedure is applied on an example, which is discussed in Section 4. Finally, the conclusions are written in Section 5.

2. Model and Preliminaries

This section consists of three parts. In the first part, we introduce discrete-time linear systems and its properties. Then, we describe the balanced systems. Finally, we discuss the model reduction using balanced truncation methods.

2.1 Discrete-Time Linear-Time-Invariant Systems

A discrete-time linear-time-invariant system is given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) + Du(k) \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state variables, $u(k) \in \mathbb{R}^m$ is the input variables, $y(k) \in \mathbb{R}^p$ is the output variables and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$ are constant matrices. In this paper, such system is written as (A, B, C, D) for simplicity.

In order to directly relate the input and output variables, we can use the transfer function that can be obtained by using the following formula:

$$G(z) = C(zI - A)^{-1}B + D. \tag{2}$$

For the purpose of model reduction, we need to analyze the behavior of the system, such as stability, controllability and observability. Those properties have to be satisfied by the original system so that the original system can be reduced by using balanced truncation methods. Next we will discuss those properties in detail.

Stability means the solution remains in a neighborhood of the equilibrium point and asymptotic stability means the solution converges to the equilibrium point (under the condition that the initial point is sufficiently close to this equilibrium point).

Definition 2.1 (Stability). *Given an autonomous system $x(x + 1) = Ax(k)$ the solution of which, with initial condition $x(0) = x_0$, will be indicated by*

$x(k, x_0)$. A vector $\bar{x}$ which satisfies $A\bar{x} = 0$ is called an equilibrium point. An equilibrium point $\bar{x}$ is called stable if for every $\varepsilon > 0$, a $\delta > 0$ exists such that, if $\|x_0 - \bar{x}\| < \delta$ then $\|x(k, x_0) - \bar{x}\| < \varepsilon$ for all $k \geq 0$. An equilibrium point x is called asymptotically stable if it is stable and, moreover, a $\delta_1 > 0$ exists such that $\lim_{k\to\infty} \|x(k, x_0) - \bar{x}\| = 0$ provided that $\|x_0 - \bar{x}\| < \delta_1$. An equilibrium point x is unstable if it is not stable.

Checking the stability of a system by using Definition 2.1 is difficult in practice. The following theorem describes a method to check the stability of a system.

Theorem 2.2 (Stability). *Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A in (1). The origin is asymptotically stable if $|\lambda_i| < 1$ for $i = 1, \ldots, n$. The origin is stable if $|\lambda_i| \leq 1$ for $i = 1, \ldots, n$.*

Then we discuss the controllability of a system. A system is called controllable if, starting from an arbitrary initial state, we can go to a desired arbitrary state by using a controller in a finite time. The formal definition of controllability is as follows:

Definition 2.3 (Controllability). *The dynamical system described by (1) is said to be controllable if, for any initial state $x(0) = x_0$, $t_1 > 0$ and final state x_1, there exists an input $u(\cdot)$ such that the solution of (1) satisfies $x(t_1) = x_1$. Otherwise, the system is said to be uncontrollable.*

Checking the controllability by using Definition 2.3 is quite difficult. The following theorem describes an easier way to determine whether a system is controllable or not.

Theorem 2.4 (Controllability). *The discrete-time system in (1) is controllable if the rank of controllability matrix M_c equals n, where controllability matrix $M_c = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}$.*

A system is observable if the initial state $x(0)$ can be constructed from the observation $y(k)$ for a finite time. The formal definition of observability is as follows:

Definition 2.5 (Observability). *The dynamical system described by (1) is said to be observable if, for any $k_1 > 0$, the initial state $x(0) = x_0$ can be determined from the time history of the input $u(k)$ and the output $y(k)$ in the interval of $[0, k_1]$. Otherwise, the system is said to be unobservable.*

Checking the observability of a system by using Definition 2.5 is in general difficult. The following theorem provides an alternative way to check the observability of a system.

Theorem 2.6 (Observability). *The discrete-time system in* (1) *is observable if the rank of observability matrix* $M_o = \begin{bmatrix} C^T & A^T C^T & \dots & (A^{n-1})^T C^T \end{bmatrix}^T$ *equals* n.

A system is called minimal if the system is asymptotically stable, controllable and observable. The level of controllability of each state can be described by the controllability gramian and the level of observability of each state can be described by the observability gramian. The formal definition of controllability and observability gramians is as follows.

Definition 2.7 (Controllability and Observability Gramians). *The controllability gramian of discrete-time system in* (1) *is* $W = \sum_{k=0}^{\infty} A^k B B^T (A^T)^k$. *The observability gramian of discrete-time system in* (1) *is* $M = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k$.

The stability property is used to guarantee the existence of controllability and observability gramians. Controllability property is used to guarantee that the controllability gramian is positive definite. Observability property is used to guarantee that the observability gramian is positive definite.

Fortunately, to compute the controllability and observability gramians of a system, it is not necessary to perform infinite summations as in Definition 2.7. The following theorem can be used to determine the controllability and observability gramians.

Theorem 2.8 (Green and Limebeer (2012)). *The controllability and observability gramians are the solution of the following Lyapunov equations:*

$$\begin{aligned} AW + WA^T + BB^T = 0 \\ A^T M + MA + C^T C = 0 \end{aligned} \tag{3}$$

2.2 Balanced Realizations

A balanced realization is an asymptotically stable and minimal realization in which the controllability and observability gramians are equal and in the form of a diagonal matrix (Green and Limebeer, 2012, p. 325). Let the gramian of a balanced realization be defined by Σ[1], where $\Sigma = diag(\sigma_1, \dots, \sigma_n)$. The

[1]In the literature, notation Σ is commonly used to denote the gramian of a balanced realization. We choose to use this commonly used notation, even if it is the same with the notation for the summation.

values $\sigma_1, \ldots, \sigma_n$ are called Hankel singular values, where $\sigma_1 > \sigma_2 > \cdots > \sigma_n$. The Hankel singular values can be computed from the controllability and observability gramians, as follows:

$$\sigma_i = \sqrt{\lambda_i(WM)}$$

Since W and M are positive definite matrices, then σ_i is always a positive number for all i.

A given system (A, B, C, D) can be transformed by matrix transformation T to a balanced realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ if and only if the given system is asymptotically stable, controllable and observable (Green and Limebeer, 2012, Lemma 9.3.1). Balanced realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ can be expressed in the following form:

$$\begin{aligned} \tilde{x}(k+1) &= \tilde{A}\tilde{x}(k) + \tilde{B}\tilde{u}(k) \\ \tilde{y}(k) &= \tilde{C}\tilde{x}(k) + \tilde{D}\tilde{u}(k) \end{aligned} \tag{4}$$

where $\tilde{x}(k) \in \mathbb{R}^n$ is the state variables of a balanced realization, $\tilde{u}(k) \in \mathbb{R}^m$ is the input variables of a balanced realization, $\tilde{y}(k) \in \mathbb{R}^p$ is the output variables of a balanced realization and $\tilde{A} \in \mathbb{R}^{n\times n}$, $\tilde{B} \in \mathbb{R}^{n\times m}$, $\tilde{C} \in \mathbb{R}^{p\times n}$, $\tilde{D} \in \mathbb{R}^{p\times m}$ are constant matrices of a balanced realization. The state variables $\tilde{x}$ of a balanced realization is obtained from the transformation of state variables x via a matrix transformation T, that can be written as $\tilde{x} = Tx$. Thus, the relationship between original system (A, B, C, D) and balanced realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, $\tilde{C} = CT^{-1}$, $\tilde{D} = D$.

2.3 Model Reduction by using Balanced Truncation Methods

Model reduction by using balanced truncation simply applies the truncation operation to a balanced realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ in (4). Remember that the gramian of a balanced realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is denoted by Σ and the diagonal elements of Σ are ordered. First we partition the gramian Σ into Σ_1 and Σ_2, where $\Sigma_1 = diag(\sigma_1, \ldots, \sigma_r)$, $\Sigma_2 = diag(\sigma_{r+1}, \ldots, \sigma_n)$ and $\sigma_r >> \sigma_{r+1}$. As a consequence, the state variables are also partitioned into $\tilde{x} = [\tilde{x}_{(1)}^T, \tilde{x}_{(2)}^T]^T$, where $\tilde{x}_{(1)}$ corresponds to Σ_1 and $\tilde{x}_{(2)}$ corresponds to Σ_2. State variables $\tilde{x}_{(1)}$ are important from the point of controllability and observability. More precisely, the energy required to move from one point to another is small (controllability) and the output energy generated by these states is large, thus it is easier to observe (observability). On the other hand, state variables $\tilde{x}_{(2)}$ represent the states that are not important. More precisely, the energy required to move from one point to another is large (controllability) and the output energy generated by these states is small, thus it is hard to observe (observability).

Model reduction by using balanced truncation method is done by truncating the states associated with Σ_2. We obtain a reduced system that is expressed as follows

$$\begin{aligned} \tilde{x}_r(k+1) &= \tilde{A}_r\tilde{x}_r(k) + \tilde{B}_r\tilde{u}_r(k) \\ \tilde{y}_r(k) &= \tilde{C}_r\tilde{x}_r(k) + \tilde{D}_r\tilde{u}_r(k) \end{aligned} \tag{5}$$

where

$$\tilde{A}_r = \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1r} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{r1} & \tilde{a}_{r2} & \dots & \tilde{a}_{rr} \end{pmatrix}, \quad \tilde{B}_r = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_r \end{pmatrix}, \quad \tilde{C}_r = \begin{pmatrix} \tilde{c}_1 & \tilde{c}_2 & \dots & \tilde{c}_r \end{pmatrix},$$

and $\tilde{D}_r = \tilde{D}$. The transfer function of reduced system (5) is

$$G_r(z) = \tilde{C}_r(zI - \tilde{A}_r)^{-1}\tilde{B}_r + \tilde{D}_r. \tag{6}$$

The following proposition discusses the error bound between the transfer function of the original model and the transfer function of the reduced model.

Proposition 2.9 (Error Bound (Green and Limebeer, 2012, Th. 9.4.3))**.** *The upper bound on the error between the transfer function of the original system (A, B, C, D) and reduced system $(\tilde{A}_r, \tilde{B}_r, \tilde{C}_r, \tilde{D}_r)$ is given by*

$$\|G - G_r\|_\infty \leq 2(\sigma_{r+1} + \dots + \sigma_n).$$

3. Procedure to Reduce Non-Minimal Discrete-Time Linear Systems by using Balanced Truncation Methods

In this section, we discuss a model reduction procedure of non-minimal discrete-time linear-time-invariant systems (A, B, C, D). The procedure is as follows:

1. Check whether the original system (A, B, C, D) is asymptotically stable, controllable and observable.
2. If the system is not asymptotically stable, decompose the original system (A, B, C, D) into asymptotically stable subsystem (A_s, B_s, C_s, D_s) and stable or unstable subsystem (A_u, B_u, C_u, D_u), as described in Nagar and Singh (2004).

3. We focus on asymptotically stable subsystem (A_s, B_s, C_s, D_s). First check whether the subsystem is controllable and observable. If the subsystem is uncontrollable or unobservable, then we transform the subsystem into its minimal realization $(A_{sm}, B_{sm}, C_{sm}, D_{sm})$.

4. Since the asymtotically stable subsystem is minimal, then we reduce the minimal subsystem $(A_{sm}, B_{sm}, C_{sm}, D_{sm})$ into the reduced system $(\tilde{A}_{smr}, \tilde{B}_{smr}, \tilde{C}_{smr}, \tilde{D}_{smr})$ (cf. Section 2.3).

5. The total reduced system (A_r, B_r, C_r, D_r) is obtained by combining the reduced asymptotically stable subsystem $(\tilde{A}_{smr}, \tilde{B}_{smr}, \tilde{C}_{smr}, \tilde{D}_{smr})$ and stable or unstable subsystem (A_u, B_u, C_u, D_u). Notice that $\tilde{D}_{smr} = D_u$. The reduced system is given by

$$A_r = \left(\begin{array}{c|c} \tilde{A}_{smr} & 0 \\ \hline 0 & A_u \end{array}\right), B_r = \left(\begin{array}{c} \tilde{B}_{smr} \\ \hline B_u \end{array}\right), C_r = \left(\begin{array}{c|c} \tilde{C}_{smr} & C_u \end{array}\right), D_r = \tilde{D}_{smr}$$

6. Check whether the properties of the total reduced system is the same with the properties of the original system. The properties that need to be checked are stability, controllability and observability.

4. Numerical Examples

We want to investigate whether the model reduction procedure discussed in Section 3 is consistent, i.e. when order of the reduced model is increased, the error is smaller. Our investigation is done by using numerical experiments. In this case, the error is defined as maximum difference of the transfer function corresponding to the original and reduced systems.

Given original system (A, B, C, D), where

$$A = \begin{pmatrix} 0.8465 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.03114 & 0.998 & -0.0653 & -0.0006526 & 0.06533 & 0.001958 & -0.06538 & -0.003264 & 0.0327 & -0.9968 \\ 0 & 0 & 1 & 0.03332 & -0.001306 & -0.06665 & 0.002613 & 0.06669 & -0.001633 & 0.003187 \\ 0 & 0 & 0.03265 & 0.9983 & -0.06531 & -0.001305 & 0.06535 & 0.002611 & -0.03269 & 0.9968 \\ 0 & 0 & 0 & 0 & 1 & 0.03332 & -0.001306 & -0.06665 & 0.0009796 & 0.01675 \\ 0 & 0 & 0 & 0 & 0.03265 & 0.9983 & -0.06531 & -0.001305 & 0.03266 & -0.9969 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.03332 & -0.0003265 & -0.03669 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.03265 & 0.9983 & -0.03265 & 0.9974 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.02333 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B = (0\ 1.003\ -0.07005\ -1.002\ 0.05002\ 1.001\ -0.03\ -1\ 0.01\ 1)^T,$$

$$C = (0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0),$$

$$D = (0).$$

The transfer function of the original system is as follows:

$$G(z) = \frac{1.003z^8 - 7.998z^7 + 27.93z^6 - 55.8z^5 + 69.75z^4 - 55.86z^3 + 27.99z^2 - 8.023z + 1.007}{z^9 - 7.994z^8 + 27.95z^7 - 55.85z^6 + 69.74z^5 - 55.72z^4 + 27.82z^3 - 7.938z^2 + 0.9907z} \tag{7}$$

Let us check the properties of the system. First we check whether the system is asymptotically stable. The magnitudes of all eigenvalues are $|\lambda_1| = 0.9980$, $|\lambda_2| = 1.0323$, $|\lambda_3| = 1.0323$, $|\lambda_4| = 1.0323$, $|\lambda_5| = 0.9663$, $|\lambda_6| = 0.9663$, $|\lambda_7| = 0.9663$, $|\lambda_8| = 0.8465$, $|\lambda_9| = 1.0000$, $|\lambda_{10}| = 0$. Since some magnitudes of eigenvalues are larger than one, the system is unstable (cf. Theorem 2.2).

Then we check whether the system is controllable or observable by computing the rank of controllability and observability matrices (cf. Theorems 2.4 and 2.6). The controllability and observability matrices are given by

$$M_c = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.003 & 0.02026 & 0.05405 & 0.08801 & 0.1223 & 0.1572 & 0.1927 & 0.2292 & 0.2668 & 0.3058 \\ -0.07005 & -0.2339 & -0.2357 & -0.2396 & -0.2455 & -0.2534 & -0.2635 & -0.2756 & -0.2898 & -0.3063 \\ -1.002 & -0.01568 & -0.04183 & -0.06808 & -0.09455 & -0.1214 & -0.1486 & -0.1765 & -0.2051 & -0.2345 \\ 0.05002 & 0.1669 & 0.1675 & 0.1689 & 0.171 & 0.1738 & 0.1774 & 0.1817 & 0.1867 & 0.1925 \\ 1.001 & 0.007837 & 0.0209 & 0.03397 & 0.04709 & 0.0603 & 0.07361 & 0.08707 & 0.1007 & 0.1146 \\ -0.03 & -0.1 & -0.1002 & -0.1004 & -0.1009 & -0.1014 & -0.1021 & -0.103 & -0.104 & -0.1052 \\ -1 & -0.002611 & -0.006962 & -0.01131 & -0.01566 & -0.02001 & -0.02438 & -0.02876 & -0.03317 & -0.03759 \\ 0.01 & 0.03333 & 0.03333 & 0.03333 & 0.03333 & 0.03333 & 0.03333 & 0.03333 & 0.03333 & 0.03333 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_o = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.03114 & 0.998 & -0.0653 & -0.0006526 & 0.06533 & 0.001958 & -0.06538 & -0.003264 & 0.0327 & -0.9968 \\ 0.05744 & 0.996 & -0.1305 & -0.003478 & 0.1307 & 0.01044 & -0.1312 & -0.01741 & 0.06573 & -0.9966 \\ 0.07964 & 0.994 & -0.1957 & -0.008471 & 0.1966 & 0.02543 & -0.1983 & -0.04244 & 0.09971 & -1.016 \\ 0.09837 & 0.992 & -0.261 & -0.01563 & 0.2632 & 0.04694 & -0.2677 & -0.07843 & 0.1353 & -1.055 \\ 0.1142 & 0.99 & -0.3263 & -0.02494 & 0.331 & 0.07498 & -0.3404 & -0.1255 & 0.1731 & -1.113 \\ 0.1275 & 0.9881 & -0.3919 & -0.03642 & 0.4003 & 0.1096 & -0.4171 & -0.1838 & 0.2138 & -1.191 \\ 0.1387 & 0.9861 & -0.4577 & -0.05006 & 0.4714 & 0.1509 & -0.4989 & -0.2537 & 0.2581 & -1.288 \\ 0.1481 & 0.9841 & -0.5239 & -0.06587 & 0.5448 & 0.1988 & -0.5868 & -0.3354 & 0.3065 & -1.406 \\ 0.156 & 0.9822 & -0.5905 & -0.08386 & 0.6207 & 0.2536 & -0.6816 & -0.4293 & 0.3599 & -1.543 \end{pmatrix},$$

Without going into the details, the rank of controllability matrix is 9 and the rank of observability matrix is 10. Thus the system is observable but not controllable. In summary, properties of the original system (A, B, C, D) are unstable, uncontrollable and observable. Equivalently, the system is not minimal.

Since the original system is unstable, we decompose the original system into two subsystems. The first subsystem is asymptotically stable. The second subsystem is either stable but not asymptotically stable or unstable. In this numerical example, the unstable subsystem (A_u, B_u, C_u, D_u) is given by

$$A_u = \begin{pmatrix} 1.032 & -0.07161 & 0.04578 & -0.01091 \\ 0 & 1.032 & -0.04608 & 0.005945 \\ 0 & 0 & 1.032 & -0.02189 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_u = \begin{pmatrix} -0.02462 & 0.1002 & -0.02508 & 0.07876 \end{pmatrix}^T,$$

$$C_u = \begin{pmatrix} -2.278 & -1.587 & -0.8062 & -2.599 \end{pmatrix},$$

$$D_u = (0).$$

The decomposition of the original system also produces an asymptotically stable subsystem given by (A_s, B_s, C_s, D_s) where

$$A_s = \begin{pmatrix} 0.998 & -2.236e-07 & -0.0189 & -0.01616 & -0.02771 & -0.2601 \\ 0 & 0.8465 & -0 & -0 & -0 & -0 \\ 0 & 0 & 0.9663 & -0.06629 & -0.04701 & -0.2818 \\ 0 & 0 & 0 & 0.9663 & -0.06123 & -0.4557 \\ 0 & 0 & 0 & 0 & 0.9663 & -0.3595 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_s = \begin{pmatrix} 0.2512 & 0 & 0.2607 & 0.4329 & 0.3278 & 1 \end{pmatrix}^T,$$
$$C_s = \begin{pmatrix} 17 & 2.094e-21 & -7.035 & 8.665 & -18.34 & 1.114 \end{pmatrix},$$
$$D_s = (0).$$

As mentioned previously, this subsystem will be reduced by using balanced truncation method. In order to apply the method, the system has to be minimal, i.e. stable, controllable and observable. Now, we check whether the asymptotically stable subsystem is controllable and observable. We can check the controllability and observability from the rank of controllability and observability matrices, respectively. The controllability and observability matrices are given by

$$M_{sc} = \begin{pmatrix} 0.2512 & -0.03041 & -0.02684 & -0.02354 & -0.02051 & -0.01771 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2607 & -0.07404 & -0.06573 & -0.05807 & -0.05102 & -0.04455 \\ 0.4329 & -0.05741 & -0.05286 & -0.04855 & -0.04447 & -0.04061 \\ 0.3278 & -0.04276 & -0.04132 & -0.03993 & -0.03858 & -0.03728 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{so} = \begin{pmatrix} 17 & 2.094 \times 10^{-21} & -7.035 & 8.665 & -18.34 & 1.114 \\ 16.97 & -3.802 \times 10^{-6} & -7.12 & 8.565 & -18.39 & 0.2057 \\ 16.93 & -7.012 \times 10^{-6} & -7.201 & 8.475 & -18.43 & 0.3032 \\ 16.9 & -9.722 \times 10^{-6} & -7.279 & 8.393 & -18.46 & 0.3907 \\ 16.87 & -1.201 \times 10^{-5} & -7.353 & 8.32 & -18.48 & 0.469 \\ 16.83 & -1.394 \times 10^{-5} & -7.424 & 8.255 & -18.49 & 0.5387 \end{pmatrix},$$

The rank of controllability matrix is 5 and the rank of observability matrix is 6. Thus the system is observable but not controllable. This means that the system is not minimal, i.e. we cannot apply the balanced truncation method to this subsystem. In order to obtain a system that can be reduced, we have to eliminate the uncontrollable or unobservable states in the subsystem. In the end, we obtain a system in its minimal realization as follows:

$$A_{sm} = \begin{pmatrix} 0.998 & -0.0189 & -0.01616 & -0.02771 & -0.2601 \\ 0 & 0.9663 & -0.06629 & -0.04701 & -0.2818 \\ 0 & 0 & 0.9663 & -0.06123 & -0.4557 \\ 0 & 0 & 0 & 0.9663 & -0.3595 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_{sm} = \begin{pmatrix} 0.2512 & 0.2607 & 0.4329 & 0.3278 & 1 \end{pmatrix}^T,$$
$$C_{sm} = \begin{pmatrix} 17 & -7.035 & 8.665 & -18.34 & 1.114 \end{pmatrix},$$
$$D_{sm} = (0).$$

The magnitude of eigenvalues of A_{sm} is $|\lambda_1| = 0.9980$, $|\lambda_2| = 0.9663$, $|\lambda_3| = 0.9663$, $|\lambda_4| = 0.9663$, $|\lambda_5| = 0$. Since all absolute values are less than one, then the system is asymptotically stable. Next we determine the controllability and observability matrices, as follows:

$$M_{smc} = \begin{pmatrix} 0.2512 & -0.03041 & -0.02684 & -0.02354 & -0.02051 \\ 0.2607 & -0.07404 & -0.06573 & -0.05807 & -0.05102 \\ 0.4329 & -0.05741 & -0.05286 & -0.04855 & -0.04447 \\ 0.3278 & -0.04276 & -0.04132 & -0.03993 & -0.03858 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M_{smo} = \begin{pmatrix} 17 & -7.035 & 8.665 & -18.34 & 1.114 \\ 16.97 & -7.12 & 8.565 & -18.39 & 0.2057 \\ 16.93 & -7.201 & 8.475 & -18.43 & 0.3032 \\ 16.9 & -7.279 & 8.393 & -18.46 & 0.3907 \\ 16.87 & -7.353 & 8.32 & -18.48 & 0.469 \end{pmatrix},$$

The rank of controllability matrix and the rank of observability matrix are 5. Thus the system is controllable and observable.

Now, we reduce the minimal asymptotically stable subsystem of order 5 into a reduced system of order 4 by using balanced truncation method. The reduced system is as follows:

$$A_{smr4} = \begin{pmatrix} 0.9982 & -0.001244 & -0.00192 & -0.004537 \\ 0.001244 & 0.9939 & -0.0212 & -0.04362 \\ -0.00192 & 0.0212 & 0.899 & -0.3005 \\ -0.004537 & 0.04362 & -0.3005 & 0.07414 \end{pmatrix},$$

$$B_{smr4} = \begin{pmatrix} 0.4319 & -0.148 & 0.2371 & 0.5583 \end{pmatrix}^T,$$

$$C_{smr4} = \begin{pmatrix} 1.08 & 0.3699 & 0.5927 & 1.396 \end{pmatrix},$$

$$D_{smr4} = (0).$$

In order to obtain a total reduced of the original system (A_r, B_r, C_r, D_r), we combine the reduced minimal asymptotically stable subsystem which is denoted by $(A_{smr4}, B_{smr4}, C_{smr4}, D_{smr4})$ and the unstable subsystem (A_u, B_u, C_u, D_u), as follows:

$$A_r = \begin{pmatrix} 0.9982 & -0.001244 & -0.00192 & -0.004537 & 0 & 0 & 0 & 0 \\ 0.001244 & 0.9939 & -0.0212 & -0.04362 & 0 & 0 & 0 & 0 \\ -0.00192 & 0.0212 & 0.899 & -0.3005 & 0 & 0 & 0 & 0 \\ -0.004537 & 0.04362 & -0.3005 & 0.07414 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.032 & -0.07161 & 0.04578 & -0.01091 \\ 0 & 0 & 0 & 0 & 0 & 1.032 & -0.04608 & 0.005945 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.032 & -0.02189 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_r = \begin{pmatrix} 0.4319 & -0.148 & 0.2371 & 0.5583 & -0.02462 & 0.1002 & -0.02508 & 0.07876 \end{pmatrix}^T,$$

$$C_r = \begin{pmatrix} 1.08 & 0.3699 & 0.5927 & 1.396 & -2.278 & -1.587 & -0.8062 & -2.599 \end{pmatrix},$$

$$D_r = (0).$$

Finally, we check whether the properties of total reduced system is the same with the original system. First, we determine the stability of total reduced system. The magnitude of eigenvalues of A_r is $|\lambda_1| = 0.0215$, $|\lambda_2| = 0.9944$, $|\lambda_3| = 0.9944$, $|\lambda_4| = 0.9981$, $|\lambda_5| = 1.0323$, $|\lambda_6| = 1.0323$, $|\lambda_7| = 1.0323$, $|\lambda_8| = 1.0000$. Since some absolute values are greater than one, then the total reduced system is unstable. Next we compute the controllability and observability matrices as follows:

$$M_{rc} = \begin{pmatrix} 0.4319 & 0.4283 & 0.4279 & 0.4273 & 0.4268 & 0.4263 & 0.4258 & 0.4253 \\ -0.148 & -0.1759 & -0.1735 & -0.1717 & -0.17 & -0.1682 & -0.1665 & -0.1648 \\ 0.2371 & 0.04138 & 0.04415 & 0.04267 & 0.0413 & 0.03995 & 0.0386 & 0.03728 \\ 0.5583 & -0.03827 & -0.02489 & -0.02462 & -0.02408 & -0.02355 & -0.02302 & -0.0225 \\ -0.02462 & -0.03459 & -0.04536 & -0.05695 & -0.06944 & -0.08286 & -0.09729 & -0.1128 \\ 0.1002 & 0.105 & 0.1102 & 0.1156 & 0.1213 & 0.1273 & 0.1337 & 0.1404 \\ -0.02508 & -0.02761 & -0.03023 & -0.03293 & -0.03572 & -0.0386 & -0.04157 & -0.04463 \\ 0.07876 & 0.07876 & 0.07876 & 0.07876 & 0.07876 & 0.07876 & 0.07876 & 0.07876 \end{pmatrix},$$

$$M_{ro} = \begin{pmatrix} 1.08 & 0.3699 & 0.5927 & 1.396 & -2.278 & -1.587 & -0.8062 & -2.599 \\ 1.071 & 0.4397 & 0.1034 & -0.09569 & -2.351 & -1.476 & -0.8634 & -2.566 \\ 1.07 & 0.4337 & 0.1104 & -0.06222 & -2.427 & -1.355 & -0.931 & -2.531 \\ 1.068 & 0.4294 & 0.1067 & -0.06156 & -2.506 & -1.225 & -1.01 & -2.492 \\ 1.067 & 0.425 & 0.1033 & -0.0602 & -2.587 & -1.085 & -1.101 & -2.45 \\ 1.066 & 0.4206 & 0.09986 & -0.05888 & -2.67 & -0.9348 & -1.205 & -2.404 \\ 1.065 & 0.4162 & 0.09651 & -0.05756 & -2.757 & -0.7738 & -1.323 & -2.354 \\ 1.063 & 0.4119 & 0.0932 & -0.05626 & -2.846 & -0.6014 & -1.456 & -2.299 \end{pmatrix},$$

The rank of controllability matrix and observability matrix are both 8. This means the total reduced system is controllable and observable. In summary, the model reduction of an original system that is unstable, uncontrollable and observable results in a total reduced system that is unstable, controllable and observable.

Now we compute the error between the transfer function of the original system and the transfer function of the total reduced system. The transfer function of the original system has been mentioned earlier in (7). The transfer function of the total reduced system is

$$G_r(z) = \frac{1.044z^7 - 7.274z^6 + 21.73z^5 - 36.09z^4 + 35.98z^3 - 21.53z^2 + 7.162z - 1.021}{z^8 - 7.062z^7 + 21.35z^6 - 35.8z^5 + 35.91z^4 - 21.49z^3 + 7.033z^2 - 0.9242z - 0.02334}$$

The error of the total reduced system w.r.t. the original system is given by

$$\|G - G_r\|_\infty = 0.8609$$

The above steps can be applied to obtain a total reduced system of order 7, 6 and 5. Without writing the results, we have verified that the properties of total reduced system of order 7, 6 and 5 are unstable, controllable and observable. On the other hand, the error between the transfer function of the total reduced system and the transfer function of the original system is given in Table 1.

From Table 1, when the order of total reduced system is larger, the error is not always decreasing. As an example, the error when $r = 2$ is less than the error when $r = 3$. Furthermore, it is not always the case that the computational time increases when the order of reduced minimal system increases. We will investigate this issue in a future research.

We have also drawn the frequency response of the original system and the total reduced system of order 8, 7, 6, 5 in Figure 1. From Figure 1, we can see that the error is small in high frequency and the error is high in small frequency.

Table 1: Relationship between order and infinity-norm error of the difference between total reduced system and original system.

Order of the reduced minimal system (r)	Order of the total reduced model	Error of the total reduced model	Computational time (seconds)
1	5	2.6445	1.205904
2	6	1.3638	1.174199
3	7	1.6027	1.192290
4	8	0.8609	1.202717

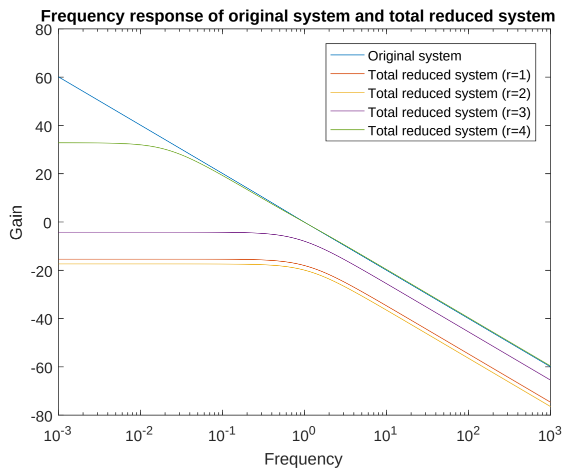

Figure 1: Error of the total reduced system w.r.t. the original system.

5. Conclusions

We have developed a procedure to reduce a non-minimal discrete-time linear-time-invariant system. According to the simulation results, when the properties of the original system are unstable, uncontrollable and observable, then we obtain a total reduced system with the following properties: unstable, controllable and observable. Furthermore when the order of total reduced system is larger, the error is not always decreasing. From the plot of frequency response, the error of total reduced system is small in high frequency and the error of total reduced system is high in small frequency. This means the balanced truncation method for non-minimal systems is accurate in high frequency. From simulation results, it is not always the case that the computational time increases when the order of total reduced system increases.

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