



On Conditioned Limit Structure of the Markov Branching Process without Finite Second Moment

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ABSTRACT

Consider the continuous-time Markov Branching Process. In the critical case we consider a situation when the generating function of intensity of transformation of particles has the infinite second moment, but its tail regularly varies in sense of Karamata. First we discuss limit properties of transition functions of the process. We prove local limit theorems and investigate ergodic properties of the process. Further we investigate limiting probability function conditioned to be never extinct. Hereupon, we obtain a new stochastic population process as a continuous-time Markov chain called the Markov Q-Process. We study main properties of Markov Q-Process.

Keywords: Markov Branching process, transition function, slowly varying function, λ -classification, invariant measures, ergodic chain, Tauberian theorem, Markov Q-process, q-matrix, limit theorems.

1. Introduction and Preliminaries

Introducing the population of monotype individuals that are capable to perish and transforms into individuals of random number of the same type, we are interested in its evolution. These individuals may be biological kinds, molecules in chemical reactions etc. The most primitive mathematical model of population growth initiated by famous English statisticians H.Watson and F.Galton (1874) which is called now the Galton-Watson process. Yule (1924), considering the birth-and-death process, studied an evolution of biologic individuals. Feller (1939) used this model in problem of "struggle for existence". The Feller's problem was discussed by Neyman (1956), (1961) in situation of epidemic spread. The birth-and-death process was also studied by D'Ancona (1954) and Kendall (1948a), (1948b). In the book of Bharucha-Reid (1960) applications of models of particles evolution processes with Markov properties in the physics and biology were discussed. Kolmogorov and Dmitriev (1947) considered a population process which is an extension on the continuous-time case of definition of the Galton-Watson process and called the Markov Branching Process (MBP).

Letting $Z(t)$ be the population size at the moment $t \in \mathcal{T} = [0; +\infty)$ in MBP, we have the homogeneous continuous-time Markov chain $\{Z(t), t \in \mathcal{T}\}$ with the state space $\mathcal{S}_0 = \{0\} \cup \mathcal{S}$, where $\mathcal{S} \subset \mathbb{N} = \{1, 2, \dots\}$. Evolution of the process occurs by the following scheme. Each individual existing at epoch $t \in \mathcal{T}$, independently of his history and of each other for a small time interval $(t; t+\varepsilon)$ transforms into $j \in \mathcal{S}_0 \setminus \{1\}$ individuals with probability $a_j\varepsilon + o(\varepsilon)$ and, with probability $1 + a_1\varepsilon + o(\varepsilon)$ each individual survives or makes evenly one descendant (as $\varepsilon \downarrow 0$). Here $\{a_j\}$ are intensities of individuals' transformation that $a_j \geq 0$ for $j \in \mathcal{S}_0 \setminus \{1\}$ and

$$0 < a_0 < -a_1 = \sum_{j \in \mathcal{S}_0 \setminus \{1\}} a_j < \infty.$$

Appeared new individuals undergo transformations under same way as above. The Markovian nature of the process yields that its transition functions

$$P_{ij}(t) = \mathbb{P}_i \{Z(t) = j\} := \mathbb{P} \{Z(t + \tau) = j | Z(\tau) = i\}$$

satisfy the Kolmogorov-Chapman equation

$$P_{ij}(t) = \sum_{k \in \mathcal{S}_0} P_{ik}(u) \cdot P_{kj}(t - u) \quad \text{for } u \leq t, \tag{1}$$

and the following branching property holds for all $i, j \in \mathcal{S}_0$:

$$P_{ij}(t) = \sum_{j_1 + j_2 + \dots + j_i = j} P_{1j_1}(t) \cdot P_{1j_2}(t) \cdots P_{1j_i}(t); \tag{2}$$

see Athreya and Ney (1972, Ch. III). Thus, for studying of evolution of $\{Z(t), t \in \mathcal{T}\}$ is suffice to set the transition functions $P_{1j}(t)$. These functions in turn, can be expressed using the local densities $\{a_j\}$ by relation

$$P_{1j}(\varepsilon) = \delta_{1j} + a_j\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \downarrow 0, \tag{3}$$

where δ_{ij} represents Kronecker's delta function. A probability generating function (GF) is a main analytical tool in our discussions on MBP. A GF version of relation (3) is

$$F(\tau; s) = s + f(s) \cdot \tau + o(\tau) \quad \text{as } \tau \downarrow 0,$$

for all $0 \leq s < 1$, where $F(t; s) = \sum_{j \in \mathcal{S}_0} P_{1j}(t)s^j$ and $f(s) = \sum_{j \in \mathcal{S}_0} a_j s^j$.

GF $F(t; s)$ satisfies to the functional equation

$$F(t + \tau; s) = F(t; F(\tau; s)), \tag{4}$$

for any $t, \tau \in \mathcal{T}$ with the boundary condition $F(0; s) = s$. Moreover it satisfies the equation

$$\frac{\partial F(t; s)}{\partial t} = f(F(t; s)), \tag{5}$$

the backward Kolmogorov equation. It follows from theory of differential equations that the solution of this equation represents unique GF which satisfies the equation (4); see Athreya and Ney (1972, p.106).

If the offspring mean $a := \sum_{j \in \mathcal{S}} ja_j = f'(s \uparrow 1)$ is finite then $F(t; 1) = 1$; see Asmussen and Hering (1983, p.119). By using equations (1), (2) and (5) it can be computed that $\mathbb{E}_i Z(t) := \sum_{j \in \mathcal{S}_0} j P_{ij}(t) = i e^{at}$. The last formula shows that long-term properties of MBP seem variously depending on value of parameter a . Hence the MBP is classified as critical if $a = 0$ and sub-critical or supercritical if $a < 0$ or $a > 0$ respectively.

Further we write everywhere $\mathbb{P}\{*\}$ and $\mathbb{E}[*]$ instead of $\mathbb{P}_1\{*\}$ and $\mathbb{E}_1[*]$ respectively.

Let a random variable $\mathcal{H} = \inf \{t \in \mathcal{T} : Z(t) = 0\}$ be the extinction time of MBP. The fundamental extinction theorem states that $\mathbb{P}_i \{\mathcal{H} < \infty\} = q^i$, where $q = \inf \{s \in (0, 1] : f(s) = 0\}$ is the extinction probability that $q = 1$ if the process is non-supercritical. An asymptote of probability of \mathcal{H} has first been observed by Sevastyanov (1951). Exertions of this variable were treated also by Heatcote *et al.* (1967), Nagaev and Badalbaev (1967), Zolotarev (1957). Put the conditioned distribution

$$\mathbb{P}_i^{\mathcal{H}(t)}\{*\} := \mathbb{P}_i \{* | t < \mathcal{H} < \infty\}.$$

Sevastyanov (1951) proved that in the sub-critical case there exists a limiting distribution law $P_j^* = \lim_{t \rightarrow \infty} \mathbb{P}^{\mathcal{H}(t)} \{Z(t) = j\}$ with GF

$$\sum_{j \in \mathcal{S}} P_j^* s^j = 1 - \exp \left\{ a \int_0^s \frac{dx}{f(x)} \right\}, \tag{6}$$

if and only if $\sum_{j \in \mathcal{S}} a_j j \ln j < \infty$. In the critical situation he also proved that if $2b := f''(s \uparrow 1)$ is finite, then $Z(t)/bt$ has a limiting exponential law.

In the discrete-time situation $\mathbb{P}_i^{\mathcal{H}(t+\tau)} \{*\}$ converge as $\tau \rightarrow \infty$ to a probability measure, which defines homogeneous Markov chain called the Q-process; see Athreya and Ney (1972, pp.56–60). The Q-process was considered first by Lamperti and Ney (1968). Some properties of it were discussed by Pakes (1971, 1999, 2010), Imomov (2001, 2002, 2014b, 2014c).

Similarly in the MBP case a limit $\lim_{\tau \rightarrow \infty} \mathbb{P}_i^{\mathcal{H}(t+\tau)} \{Z(t) = j\}$ has an honest probability measures $\{Q_{ij}(t)\}$ which defines a homogeneous continuous-time process as a Markov chain and, called the Markov Q-Process; see Imomov (2012). Let $W(t)$ be the state size at the moment $t \in \mathcal{T}$ in Markov Q-Process. Then $W(0) \stackrel{d}{=} Z(0)$ and $\mathbb{P}_i \{W(t) = j\} = Q_{ij}(t)$. In the paper Imomov (2012) some asymptotic properties of the chain $\{W(t), t \in \mathcal{T}\}$ are observed. Namely it was proved that if the associated MBP is critical and $f''(s \uparrow 1)$ is finite, then $W(t)/\mathbb{E}W(t)$ has a limiting Erlang's law. In this case there is an invariant measure. In the non-critical situation under the condition when (6) holds, an invariant distribution exists for the process $\{W(t), t \in \mathcal{T}\}$.

In this paper we consider MBP without further power moments. In the non-critical case we rest satisfied only with the condition $\sum_{j \in \mathcal{S}} a_j j \ln j < \infty$. In the critical case our reasoning is bound up with elements of the theory of regularly varying functions in sense of Karamata; see Karamata (1933). We remember that real-valued, positive and measurable function $\ell(x)$ is said to be slowly varying (SV) at α if it belongs to a class

$$\mathfrak{S}_\alpha := \left\{ \ell(x) \in \mathbb{R}_+ : \lim_{x \rightarrow \alpha} \frac{\ell(\lambda x)}{\ell(x)} = 1, \quad \forall \lambda \in \mathbb{R}_+ \right\}.$$

A function $V(x)$ is said to be regularly varying (RV) at α with index of regular variation $\rho \in \mathbb{R}_+$ if it in the form $V(x) = x^\rho \ell(x)$, where $\ell(x) \in \mathfrak{S}_\alpha$. We denote \mathfrak{R}_α^ρ be the class of RV-functions. It is evidently that $\mathfrak{S}_\alpha \equiv \mathfrak{R}_\alpha^0$.

Throughout the paper in the critical case we suppose that the infinitesimal GF $f(s)$ satisfies the condition

$$f(s) = (1 - s)^{1+\nu} \mathfrak{L}\left(\frac{1}{1-s}\right), \tag{7}$$

for $0 \leq s < 1$, where $0 < \nu \leq 1$ and $\mathfrak{L}(x) \in \mathfrak{S}_\infty$. By the assumed criticality

$$\frac{f''(s \uparrow 1)}{2} = \lim_{s \uparrow 1} \frac{f(s)}{(1-s)^2} = \lim_{x \downarrow 0} \frac{1}{x^{1-\nu}} \mathfrak{L}\left(\frac{1}{x}\right) = \infty.$$

If $f''(s \uparrow 1) < \infty$, then condition (7) holds with $\nu = 1$ and $\mathfrak{L}(t) \rightarrow f''(s \uparrow 1)/2$ as $t \rightarrow \infty$. Thus our process contains a process with the finite second moment.

Section 2 is devoted to auxiliary lemmas these will be useful for our purpose. First we take assertions about asymptotical decay of the functions $F(t; s)$ and $\partial F(t; s)/\partial s$. One of important facts of this section is the Monotone ratio Lemma on limit properties of the transition functions $P_{ij}(t)$. In consequence of this lemma we take complete accounts on asymptotic behaviors of states of MBP. In Section 3 we observe invariance properties of states of MBP. In the non-critical case we hold on to results from the paper of Imomov (20104a).

In Section 4 we consider the Markov Q-Process and discuss properties concerning its construction and asymptotic properties of transition functions $\{Q_{ij}(t)\}$. In particular we compute the q-matrix and the GF version of the Kolmogorov backward equation implied by $\{Q_{ij}(t)\}$. We also observe ergodic properties of the Markov Q-Process.

2. Auxiliary Results

Let

$$R(t; s) := q - F(t; s).$$

Lemma 1. *The following assertions are true for all $0 \leq s < 1$.*

- *Let $a \neq 0$. Then*

$$R(t; s) = \mathcal{A}(t; s) \cdot \beta^t, \tag{8}$$

where $\beta := \exp\{f'(q)\} < 1$, and

$$\mathcal{A}(t; s) = (q - s) \exp \left\{ \int_s^{F(t; s)} \left[\frac{1}{u - q} - \frac{f'(q)}{f(u)} \right] du \right\}.$$

- Let $a = 0$. If the condition (7) holds then

$$R(t; s) \sim \frac{\mathcal{N}(t)}{(\nu t)^{1/\nu}} \cdot \left(1 + \frac{\mathcal{M}(s)}{t}\right)^{-1/\nu}, \tag{9}$$

where

$$\mathcal{N}(n) \cdot \mathcal{L}^{1/\nu} \left(\frac{(\nu n)^{1/\nu}}{\mathcal{N}(n)} \right) \rightarrow 1 \quad \text{as } n \rightarrow \infty \tag{10}$$

and

$$\mathcal{M}(s) = \int_1^{1/(1-s)} \frac{dx}{x^{1-\nu} \mathfrak{L}(x)}. \tag{11}$$

Proof. Let's consider first the non-critical case. Multiplying by $f'(q) \cdot (F(t; s) - q)$ the both sides of (5) yields

$$\frac{dF(t; s)}{F(t; s) - q} \cdot \left[1 - \frac{f(F(t; s)) - f'(q) \cdot (F(t; s) - q)}{f(F(t; s))} \right] = f'(q) \cdot dt.$$

After integration on $[0; t] \subset \mathcal{T}$ it follows from this equality

$$\frac{R(t; s)}{R(0; s)} = \beta^t \exp \left\{ \int_s^{F(t; s)} \left[\frac{1}{u - q} - \frac{f'(q)}{f(u)} \right] du \right\},$$

hereinafter $\beta = \exp \{f'(q)\}$. Since $R(0; s) = q - s$, this relation we can write in the form of (8).

To prove the second part we transform the backward Kolmogorov equation (5) to the following integral equation:

$$\int_s^{F(t; s)} \frac{dx}{f(x)} = t. \tag{12}$$

Further we rewrite equation (12) in form of

$$\int_0^{F(t; s)} \frac{dx}{f(x)} = t + \mathcal{M}(s), \tag{13}$$

where

$$\mathcal{M}(s) = \int_0^s \frac{dx}{f(x)}.$$

Seeing the condition (7) and denoting $1 - x = 1/u$ it follows from (13) that

$$\int_1^{1/R(t;s)} \frac{dx}{x^{1-\nu} \mathfrak{L}(x)} = t + \mathcal{M}(s), \tag{14}$$

herein the function $\mathcal{M}(s)$ becomes (11).

Now we recall the following property of SV-functions. If $\ell(x) \in \mathfrak{S}_\infty$ remains locally bounded in $[A; +\infty)$ for some $A \in \mathbb{R}_+$, then

$$\int_A^x u^\lambda \ell(u) du \sim \frac{1}{\lambda + 1} x^{\lambda+1} \ell(x), \tag{15}$$

as $x \rightarrow \infty$ for $\lambda > -1$; see Bingham *et al.* (1987, p.26).

Since the upper bound of the integral in left-hand side of (14) grows to infinity as $t \rightarrow \infty$, it is possible to use the property (15) and we can get the following asymptotic formula:

$$\int_1^{1/R(t;s)} \frac{dx}{x^{1-\nu} \mathfrak{L}(x)} = \frac{1 + o(1)}{\nu R^\nu(t; s) \cdot \mathfrak{L}(1/R(t; s))} \quad \text{as } t \rightarrow \infty.$$

Combining this formula with (14) yields

$$R(t; s) = \frac{\mathfrak{L}^{-1/\nu}(1/R(t; s))}{(\nu t)^{1/\nu}} \cdot \frac{1 + o(1)}{\left(1 + \frac{\mathcal{M}(s)}{t}\right)^{1/\nu}} \quad \text{as } t \rightarrow \infty. \tag{16}$$

Letting $q(t) := R(t; 0) = \mathbb{P}\{Z(t) > 0\}$ and seeing that $\mathcal{M}(0) = 0$, we obtain from (16) that

$$q(t) = \frac{\mathfrak{L}^{-1/\nu}(1/q(t))}{(\nu t)^{1/\nu}} (1 + o(1)) \quad \text{as } t \rightarrow \infty,$$

the surviving probability of the process $\{Z(t)\}$. On the other hand it is clear that $R(t; s)/q(t) \rightarrow 1$. Hence introducing the function $\mathcal{N}(t)$ satisfying property (10), we can write formula (9). □

Since $F(t; s) \rightarrow q$ as $t \rightarrow \infty$, we obtain $\mathcal{A}(t; s) \rightarrow \mathcal{A}(s)$, where

$$\mathcal{A}(s) = (q - s) \exp \left\{ \int_s^q \left[\frac{1}{u - q} - \frac{f'(q)}{f(u)} \right] du \right\}. \tag{17}$$

Therefore it follows from (8) that

$$\mathbb{P} \{n < \mathcal{H} < \infty\} \sim \mathcal{A}(0) \cdot \beta^t \quad \text{as } t \rightarrow \infty$$

if and only if

$$\int_0^q \frac{|\ln \beta| \cdot x - f(q-x)}{x f(q-x)} dx = \ln \frac{q}{\mathcal{A}(0)} < \infty. \tag{18}$$

Integral in the left-hand side in (18) can be transformed to a form

$$q \cdot \int_0^1 \frac{\hat{a} \cdot x - \hat{f}(1-x)}{x \hat{f}(1-x)} dx,$$

where $\hat{f}(s) = \sum_{j \in \mathcal{S}_0} \hat{a}_j s^j$, $\hat{a}_j = a_j q^{j-1}$ and $\hat{a} = \sum_{j \in \mathcal{S}} j \hat{a}_j = f'(q) < 1$. It is known (see Sevastyanov (1951)) that convergence of last integral is equivalent to a convergence of the series $\sum_{j \in \mathcal{S}} \hat{a}_j j \ln j$. As $\hat{a}_j < a_j$, the condition

$$\sum_{j \in \mathcal{S}} a_j j \ln j < \infty \tag{19}$$

is sufficient to be satisfied the condition (18).

Further consider the function $\partial R(t; s)/\partial s$.

Lemma 2. *The following assertions are true for all $0 \leq s < 1$.*

- If $a \neq 0$, then

$$\frac{\partial R(t; s)}{\partial s} = \frac{f'(q)}{f(s)} \mathcal{A}(t; s) \beta^t (1 + o(1)) \quad \text{as } t \rightarrow \infty, \tag{20}$$

where the function $\mathcal{A}(t; s)$ is defined in (8)

- Let $a = 0$. If the condition (7) holds then

$$\frac{\partial R(t; s)}{\partial s} = - \left(\frac{R(t; s)}{1-s} \right)^{1+\nu} \frac{\mathfrak{L}(1/R(t; s))}{\mathfrak{L}(1/(1-s))}. \tag{21}$$

Proof. Transform the backward Kolmogorov equation (5) to (12) and differentiating it with respect s , we have

$$\frac{\partial R(t; s)}{\partial s} = - \frac{f(F(t; s))}{f(s)}. \tag{22}$$

In the non-critical case $f(s) \sim f'(q)(s - q)$ as $s \rightarrow q$. Therefore (22) entails

$$\frac{\partial R(t; s)}{\partial s} = \frac{f'(q)}{f(s)} R(t; s) (1 + o(1)) \quad \text{as } t \rightarrow \infty,$$

for all $0 \leq s < 1$. Using therein (8) we will get (20).

Same way, for critical case, from (22) we obtain the following relation:

$$\frac{\partial R(t; s)}{\partial s} = -\frac{R^{1+\nu}(t; s)}{f(s)} \mathfrak{L} \left(\frac{1}{R(t; s)} \right).$$

This relation together with condition (7) produces (21). □

One can see that the Lemma 2 has a simple appearance, but as it will be visible further, this lemma is important in our discussions. Namely it will easily be computed that

$$\left. \frac{\partial F(t; s)}{\partial s} \right|_{s=0} = P_{11}(t),$$

the probability of return of the process with initial state $Z(0) = 1$ to the one through the time t . As $f(0) = a_0 > 0$, putting $s = 0$ in (20) and (21), we obtain the following local limit theorems.

Theorem 1. *Let $a \neq 0$. Then*

$$\beta^{-t} \cdot P_{11}(t) \sim \frac{|\ln \beta|}{a_0} \mathcal{A}(t; 0) \quad \text{as } t \rightarrow \infty.$$

If the condition (19) holds then $\mathcal{A}(t; 0) \rightarrow \mathcal{A}(0) < \infty$ as $t \rightarrow \infty$.

Theorem 2. *Let $a = 0$. If the condition (7) holds then*

$$(\nu t)^{1+1/\nu} \cdot P_{11}(t) \sim \frac{\mathcal{N}(t)}{a_0} \quad \text{as } t \rightarrow \infty, \tag{23}$$

where the function $\mathcal{N}(t)$ satisfies the property (10).

Further we will use the following Monotone ratio limit property.

Lemma 3. *For all $j \in \mathcal{S}$*

$$\frac{P_{1j}(t)}{P_{11}(t)} \uparrow \mu_j < \infty \quad \text{as } t \rightarrow \infty. \tag{24}$$

Proof. See Imomov (2014a). □

Now observe $P_{ij}(t)/P_{11}(t)$ as $t \rightarrow \infty$ for $i, j \in \mathcal{S}$. We see that

$$\mathbf{M}_i(t; s) := \sum_{j \in \mathcal{S}} \frac{P_{ij}(t)}{P_{11}(t)} s^j \longrightarrow iq^{i-1} \cdot \mathbf{M}(s) \quad \text{as } t \rightarrow \infty, \tag{25}$$

for all $0 \leq s < 1$, where

$$\mathbf{M}(s) = \sum_{j \in \mathcal{S}} \mu_j s^j.$$

By virtue of relation (25) to studying the long-term behavior of $P_{ij}(t)$ is suffice to consider the function $\mathbf{M}(t; s) := \mathbf{M}_1(t; s)$. The transition function version of (25) is

$$\frac{P_{ij}(t)}{P_{11}(t)} \longrightarrow iq^{i-1} \mu_j \quad \text{as } t \rightarrow \infty. \tag{26}$$

Using (26) and from Theorems 1 and 2 we get complete accounts about asymptotic behaviors of transition function $P_{ij}(t)$.

Theorem 3. *Let $a \neq 0$. If the condition (19) holds then*

$$\beta^{-t} \cdot P_{ij}(t) \sim iq^{i-1} \mu_j \frac{|\ln \beta|}{a_0} \mathcal{A}(t; 0) \quad \text{as } t \rightarrow \infty.$$

If the condition (19) holds then $\mathcal{A}(t; 0) \rightarrow \mathcal{A}(0) < \infty$ as $t \rightarrow \infty$.

Theorem 4. *Let $a = 0$. If the condition (7) holds then*

$$(\nu t)^{1+1/\nu} \cdot P_{ij}(t) \sim \frac{i\mu_j}{a_0} \mathcal{N}(t) \quad \text{as } t \rightarrow \infty,$$

where the function $\mathcal{N}(t)$ satisfies the property (10).

3. Invariant Properties of Transition Functions

$$P_{ij}(t)$$

Continuing researches of the asymptote of transition functions $P_{ij}(t)$ we deal with problems of ergodicity and existence of invariant measure. Ergodicity properties of arbitrary continuous-time Markov chain are described in the monograph of Anderson (1991). The invariant (or stationary) measure of chain $\{Z(t), t \in \mathcal{T}\}$ is a set of non-negative numbers $\{\nu_j, j \in \mathcal{S}_0\}$ satisfying to the equation

$$\nu_j = \sum_{k \in \mathcal{S}_0} \nu_k P_{kj}(t), \tag{27}$$

for any $t \in \mathcal{T}$. Equation (27) means an invariant property of the measure $\{\nu_j\}$ concerning to the transition functions $\{P_{ij}(t)\}$. If $\sum_{j \in \mathcal{S}_0} \nu_j < \infty$ (or without loss of generality $\sum_{j \in \mathcal{S}_0} \nu_j = 1$) then it is called an invariant distribution.

Further we will discuss the role of the set $\{\mu_j, j \in \mathcal{S}\}$ defined in (24) as invariant measures. The following theorem holds.

Theorem 5. *Non-negative numbers $\{\mu_j\}$ satisfy the invariant equation*

$$\beta^t \cdot \mu_j = \sum_{k \in \mathcal{S}} \mu_k P_{kj}(t), \tag{28}$$

for $j \in \mathcal{S}$ and for all $t \in \mathcal{T}$. The function $\mathbf{M}(s) = \sum_{j \in \mathcal{S}} \mu_j s^j$ satisfies the functional equation

$$\mathbf{M}(F(t; s)) = \beta^t \cdot \mathbf{M}(s) + \mathbf{M}(F(t; 0)), \tag{29}$$

which converges for $0 \leq s < 1$. Equation (29) has a unique solution that is power series with non-negative coefficients for $0 \leq s < q$.

Proof. See Imomov (2014a). □

In the following two theorems the explicit forms of the function $\mathbf{M}(s)$ will be obtained.

Theorem 6. *Let $a \neq 0$ and the condition (19) holds. Then*

$$\mathbf{M}(s) = \frac{a_0}{|f'(q)|} \cdot \left[1 - \frac{\mathcal{A}(s)}{\mathcal{A}(0)} \right]. \tag{30}$$

Proof. See Imomov (2014a). □

Theorem 7. *Let $a = 0$. If the condition (7) holds then*

$$\mathbf{M}(s) = a_0 \mathcal{M}(s), \tag{31}$$

where $\mathcal{M}(s)$ is form of (11).

Proof. Recall $R(t; s) = 1 - F(t; s)$ and write

$$\mathbf{M}(t; s) = \frac{F(t; s) - F(t; 0)}{P_{11}(t)} = \left(1 - \frac{R(t; s)}{q(t)} \right) \cdot \frac{q(t)}{P_{11}(t)}, \tag{32}$$

where $q(t) := R(t; 0)$. Using the second part of the Lemma 1 and after elementary transformations we find

$$1 - \frac{R(t; s)}{q(t)} = \frac{\mathcal{M}(s)}{\nu t} (1 + o(1)) \quad \text{as } t \rightarrow \infty. \tag{33}$$

According to (23), $q(t)/P_{11}(t) \sim a_0 \nu t$ as $t \rightarrow \infty$. Then considering relations (32) and (33) we get to (31). \square

Previous two theorems along with Lemma 3 allows to judge about asymptotic behavior of the sum $\sum_{j \in \mathcal{S}} \mu_j$. According to Lemma 3 this sum converges for $a < 0$ and diverges if $a > 0$. For the case $a = 0$ formulas (11), (15) and (31) show that

$$\mathbf{M}(s) = \frac{a_0}{\nu(1-s)^\nu} \cdot \mathfrak{L}_\mu \left(\frac{1}{1-s} \right) \quad \text{as } s \uparrow 1, \tag{34}$$

where $\mathfrak{L}(x) \cdot \mathfrak{L}_\mu(x) \rightarrow 1$ as $x \rightarrow \infty$. The ensuing theorem follows from equality (34) according to the Hardy-Littlewood Tauberian theorem.

Theorem 8. *Let $a = 0$. If the condition (7) holds then*

$$\sum_{j=1}^n \mu_j = \frac{a_0}{\nu^2 \Gamma(\nu)} n^\nu \mathfrak{L}_\mu(n),$$

where $\Gamma(*)$ is Euler's Gamma function and $\mathfrak{L}(x) \cdot \mathfrak{L}_\mu(x) \rightarrow 1$ as $x \rightarrow \infty$.

Theorem 3 shows that in the non-critical situation transition functions $P_{ii}(t)$ have an exponential decay behavior as $t \rightarrow \infty$. The limit

$$\lambda_{\mathcal{S}} = - \lim_{t \rightarrow \infty} \frac{\ln P_{ii}(t)}{t}$$

independent on $i \in \mathcal{S}$ and characterizes a decay rate of state space of chain $\{Z(t)\}$. It is called the decay parameter of states of the chain. MBP classified as $\lambda_{\mathcal{S}}$ -transient if $\int_0^{+\infty} e^{\lambda_{\mathcal{S}} t} P_{ii}(t) dt < \infty$ and $\lambda_{\mathcal{S}}$ -recurrent otherwise. In this case invariant measure is called $\lambda_{\mathcal{S}}$ -invariant. According to the general classification MBP is called $\lambda_{\mathcal{S}}$ -positive if $\lim_{t \rightarrow \infty} e^{\lambda_{\mathcal{S}} t} P_{ii}(t) > 0$ and $\lambda_{\mathcal{S}}$ -null if this is zero; see Li *et al.* (2010). Theorems 3 and 6 imply the following theorem.

Theorem 9. *Let $a \neq 0$ and the condition (19) holds. Then $\lambda_{\mathcal{S}} = |\ln \beta|$ and the Markov chain $\{Z(t)\}$ is $\lambda_{\mathcal{S}}$ -positive. The set $\{\mu_j, j \in \mathcal{S}\}$ determined by GF (30) is unique (up to multiplicative constant) $\lambda_{\mathcal{S}}$ -invariant measure.*

In the critical case the set $\{\mu_i\}$ directly enters to a role of invariant measure for MBP. Indeed, in this case $\beta = 1$ and as it has been proved in Theorem 5 that

$$\mu_j = \sum_{k \in \mathcal{S}} \mu_k P_{kj}(t), \quad j \in \mathcal{S},$$

for all $t \in \mathcal{T}$.

As shown in Theorems 3 and 4 hit probability of MBP to any state through the long interval time depends on the initial state. In other words an ergodic property is not carried out. Thereby we will seek an ergodic chain associated to MBP. Recalling \mathcal{H} be the extinction moment of MBP we write

$$\mathbb{P}_i \{t < \mathcal{H} < \infty, Z(t) = j\} = \mathbb{P} \{t < \mathcal{H} < \infty | Z(t) = j\} \cdot P_{ij}(t).$$

Since the probability of extinction of j particles is q^j then it follows that

$$\mathbb{P}_i \{t < \mathcal{H} < \infty, Z(t) = j\} = P_{ij}(t) \cdot q^j. \tag{35}$$

We have also that

$$\mathbb{P}_i \{t < \mathcal{H} < \infty\} = \sum_{j \in \mathcal{S}} \mathbb{P}_i \{Z(t) = j, t < \mathcal{H} < \infty\} = \sum_{j \in \mathcal{S}} P_{ij}(t) q^j.$$

Using the formula (35) from last relation we obtain that

$$\mathbb{P}_i \{t < \mathcal{H} < \infty\} = \sum_{j \in \mathcal{S}} P_{ij}(t) q^j. \tag{36}$$

Now put into consideration the conditional transition function

$$\mathbb{P}_i^{\mathcal{H}(t)} \{*\} := \mathbb{P}_i \{* | t < \mathcal{H} < \infty\}.$$

Let $\tilde{P}_{ij}(t) = \mathbb{P}_i^{\mathcal{H}(t)} \{Z(t) = j\}$ be a transition matrix which defines a new stochastic process $\{\tilde{Z}(t), t \in \mathcal{T}\}$. It is easy to be convinced that $\{\tilde{Z}(t), t \in \mathcal{T}\}$ represents a homogeneous Markov chain. Indeed probabilities $\tilde{P}_{ij}(t)$ satisfy to the Kolmogorov-Chapman equation (1) and have the branching property (2). According to last theorems properties of trajectories of $\{\tilde{Z}(t), t \in \mathcal{T}\}$ lose dependence on the initial state as $t \rightarrow \infty$. Consider an appropriate GF

$$\mathcal{V}_i(t; s) = \sum_{j \in \mathcal{S}} \tilde{P}_{ij}(t) s^j.$$

Theorem 10. *Let $a \neq 0$ and the condition (19) holds. Then limits*

$$\lim_{t \rightarrow \infty} \tilde{P}_{ij}(t) = \nu_j$$

exist for all $i, j \in \mathcal{S}$ and these are determined by GF

$$\mathcal{V}(s) = \frac{\mathbf{M}(qs)}{\mathbf{M}(q)}, \tag{37}$$

where the function $\mathbf{M}(s)$ is defined in (30).

Proof. See Imomov (2014a). □

Remark 1. *Theorem 10 is generalization of Sevastyanov’s result (6) in which corresponding result established for the sub-critical situation only. Indeed it is easy to see that the limit probability GF (6) is the proprietary case of (37). The set $\{\nu_j\}$ represents a probability distribution. In fact setting $s = 1$ in (37) and taking into account equality (30), it follows that $\mathcal{V}(1) = \sum_{j \in \mathcal{S}} \nu_j = 1$. Moreover if the condition (19) holds then*

$$\sum_{j \in \mathcal{S}} j \tilde{P}_{ij}(t) \longrightarrow \frac{q}{\mathcal{A}(0)} \quad \text{as } t \rightarrow \infty,$$

and $\mathcal{V}'(s \uparrow 1) = q/\mathcal{A}(0)$.

Under the condition of Theorem 10 for the MBP $\{\tilde{Z}(t), t \in \mathcal{T}\}$ exists the unique (up to multiplicative constant) set of non-negative numbers $\{\nu_i\}$ which not all are zero and we see without difficulty that the GF $\mathcal{V}(s) = \sum_{j \in \mathcal{S}} \nu_j s^j$ satisfies the invariance equation

$$\beta^t \cdot \mathcal{V}(s) = \mathcal{V}\left(\frac{F(t; qs)}{q}\right) - \mathcal{V}\left(\frac{F(t; 0)}{q}\right).$$

So $\{\nu_i\}$ is invariant measure.

It follows from (35) and (36) that if $a \neq 0$ and the condition (19) holds then

$$\tilde{P}_{ij}(t) = \frac{P_{ij}(t)}{\sum_{k \in \mathcal{S}} P_{ik}(t) q^{k-j}}.$$

In the critical situation $\mathbb{P}\{\mathcal{H} < \infty\} = 1$. Since $1 - F^i(t; s) \sim iR(t; s)$, we write

$$\mathcal{V}_i(t; s) \sim 1 - \frac{R(t; s)}{q(t)} \quad \text{as } t \rightarrow \infty, \tag{38}$$

where $q(t) = R(t; 0)$.

Theorem 11. *Let $a = 0$. If the condition (7) holds then*

$$vt \cdot \mathcal{V}_i(t; s) \longrightarrow \mathcal{M}(s) \quad \text{as } t \rightarrow \infty, \tag{39}$$

where GF $\mathcal{M}(s) = \sum_{j \in \mathcal{S}} \mathbf{m}_j s^j$ is form of (11). Moreover

$$\sum_{j=1}^n \mathbf{m}_j = \frac{1}{\nu^2 \Gamma(\nu)} n^\nu \mathfrak{L}_\mu(n), \tag{40}$$

where $\Gamma(\ast)$ is Euler's Gamma function and $\mathfrak{L}(x) \cdot \mathfrak{L}_\mu(x) \rightarrow 1$ as $x \rightarrow \infty$.

Proof. The convergence (39) directly comes out from (33) and (38). Relation (40) follows from Theorem 7 and Theorem 8. □

4. The Markov Q-Process

In this section we will be interested in a limiting interpretation of the conditioned transition function $\mathbb{P}_i^{\mathcal{H}(t+\tau)} \{Z(t) = j\}$ as $\tau \rightarrow \infty$ for all $t \in \mathcal{T}$. As it was said in first Section this limit is an honest probability measure. This measure defines so-called Markov Q-Process (MQP) be the continuous-time Markov chain $\{W(t), t \in \mathcal{T}\}$ with the state space $\mathcal{E} \subset \mathbb{N}$. The random function $W(t)$ is the state size at the moment $t \in \mathcal{T}$ in MQP. The transition function $\mathcal{Q}_{ij}(t) = \mathbb{P}_i \{W(t) = j\}$ is form of

$$\mathcal{Q}_{ij}(t) = \lim_{\tau \rightarrow \infty} \mathbb{P}_i^{\mathcal{H}(t+\tau)} \{Z(t) = j\} = \frac{j q^{j-i}}{i \beta^t} P_{ij}(t), \tag{41}$$

for $i, j \in \mathcal{E}$, where $\beta = \exp\{f'(q)\}$; see Imomov (2012). It is easy to be convinced that $0 < \beta \leq 1$ decidedly. To wit $\beta = 1$ if $a = 0$ and $\beta < 1$ otherwise. In our presupposition the MBP is honest. Since $F(t; q) = q$ and $F(t; s)/\partial s|_{s=q} = \beta^t$, it follows from (41) that $\sum_{j \in \mathcal{E}} \mathcal{Q}_{ij}(t) = 1$.

Combining equalities (3) and (41) we obtain the following representation:

$$\mathcal{Q}_{1j}(\varepsilon) = \delta_{1j} + \lambda_j \varepsilon + o(\varepsilon), \quad \text{as } \varepsilon \downarrow 0, \tag{42}$$

with probability densities

$$\lambda_0 = 0, \quad \lambda_1 = a_1 - \ln \beta < 0, \quad \text{and} \quad \lambda_j = j q^{j-1} a_j \geq 0 \quad \text{for } j \in \mathcal{E} \setminus \{1\},$$

where $\{a_j\}$ are evolution intensities of MBP $Z(t)$. It follows from (42) that

$$g(s) := \sum_{j \in \mathcal{E}} \lambda_j s^j = s [f'(qs) - f'(q)]. \tag{43}$$

Needles to see that this GF is infinitesimal one because $g(1) = 0$. So the infinitesimal GF $g(s)$ completely defines the process $W(t)$, where $\{\lambda_j\}$ are intensities of process evolution satisfying $\lambda_j > 0$ for $j \in \mathcal{E} \setminus \{1\}$ and

$$0 < -\lambda_1 = \sum_{j \in \mathcal{E} \setminus \{1\}} \lambda_j < \infty.$$

4.1 Construction, Existence and Uniqueness

Let's now discuss basic properties of transition matrix $Q(t) = \{Q_{ij}(t)\}$. Herewith we follow methods and facts from monograph of Anderson (1991). First we prove the following theorem.

Theorem 12. *Let $\{W(t), t \in \mathcal{T}\}$ be the MQP given by infinitesimal GF $g(s)$. Then the transition matrix $Q(t)$ is standard and honest. Its components $Q_{ij}(t)$ are positive and uniformly continuous function with respect to $t \in \mathcal{T}$ for all $i, j \in \mathcal{E}$.*

Proof. According to the branching property (2), we see

$$P_{ij}(\varepsilon) = \delta_{ij} + ia_{j-i+1}\varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \downarrow 0.$$

Hence seeing equality (41) it follows

$$\begin{cases} Q_{ii}(\varepsilon) = 1 + (ia_1 - \ln \beta)\varepsilon + o(\varepsilon), \\ Q_{ij}(\varepsilon) = jq^{j-i}a_{j-i+1}\varepsilon + o(\varepsilon) \end{cases}, \quad \text{as } \varepsilon \downarrow 0, \quad (44)$$

for all $i, j \in \mathcal{E}$. Considering representations (44) we have

$$\begin{aligned} \sum_{j \in \mathcal{E}} |Q_{ij}(\varepsilon) - \delta_{ij}| &= \sum_{j \in \mathcal{E} \setminus \{i\}} Q_{ij}(\varepsilon) + |Q_{ii}(\varepsilon) - 1| \\ &= \sum_{j \in \mathcal{E} \setminus \{i\}} Q_{ij}(\varepsilon) + 1 - Q_{ii}(\varepsilon) \leq 2|1 - Q_{ii}(\varepsilon)| \rightarrow 0, \end{aligned}$$

as $\varepsilon \downarrow 0$. So that $Q_{ij}(t)$ is standard. A positiveness of functions $Q_{ij}(t)$ is obvious owing to (44). The Markovian nature of the process $\{W(t)\}$ implies the Kolmogorov-Chapman equation:

$$Q_{ij}(t + \varepsilon) = \sum_{k \in \mathcal{E}} Q_{ik}(t)Q_{kj}(\varepsilon).$$

Hence supposing $\varepsilon > 0$ it follows that

$$\begin{aligned} \mathcal{Q}_{ij}(t + \varepsilon) - \mathcal{Q}_{ij}(t) &= \sum_{k \in \mathcal{E}} \mathcal{Q}_{ik}(\varepsilon) \mathcal{Q}_{kj}(t) - \mathcal{Q}_{ij}(t) \\ &= \sum_{k \in \mathcal{E} \setminus \{i\}} \mathcal{Q}_{ik}(\varepsilon) \mathcal{Q}_{kj}(t) - \mathcal{Q}_{ij}(t) \cdot [1 - \mathcal{Q}_{ii}(\varepsilon)]. \end{aligned}$$

It follows from here that

$$\begin{aligned} -[1 - \mathcal{Q}_{ii}(\varepsilon)] &\leq -\mathcal{Q}_{ij}(t) \cdot [1 - \mathcal{Q}_{ii}(\varepsilon)] \leq \mathcal{Q}_{ij}(t + \varepsilon) - \mathcal{Q}_{ij}(t) \\ &\leq \sum_{k \in \mathcal{E} \setminus \{i\}} \mathcal{Q}_{ik}(t) \mathcal{Q}_{kj}(\varepsilon) \leq \sum_{k \in \mathcal{E} \setminus \{i\}} \mathcal{Q}_{kj}(\varepsilon) = 1 - \mathcal{Q}_{ii}(\varepsilon), \end{aligned}$$

so $|\mathcal{Q}_{ij}(t + \varepsilon) - \mathcal{Q}_{ij}(t)| \leq 1 - \mathcal{Q}_{ii}(\varepsilon)$. Similarly

$$\begin{aligned} |\mathcal{Q}_{ij}(t - \varepsilon) - \mathcal{Q}_{ij}(t)| &= |\mathcal{Q}_{ij}(t) - \mathcal{Q}_{ij}(t - \varepsilon)| \\ &\leq 1 - \mathcal{Q}_{ii}(t - (t - \varepsilon)) = 1 - \mathcal{Q}_{ii}(\varepsilon). \end{aligned}$$

Therefore we obtain $|\mathcal{Q}_{ij}(t + \varepsilon) - \mathcal{Q}_{ij}(t)| \leq 1 - \mathcal{Q}_{ii}(|\varepsilon|)$ for any $\varepsilon \neq 0$. This relation implies that $\mathcal{Q}_{ij}(t)$ is uniformly continuous function with respect to $t \in \mathcal{T}$ because $\lim_{\varepsilon \downarrow 0} \mathcal{Q}_{ii}(\varepsilon) = 1$. \square

It can easily be verified that a GF version of (44) is

$$G_i(t; s) := \mathbb{E}_i s^{W(t)} = \sum_{j \in \mathcal{E}} \mathcal{Q}_{ij}(t) s^j = \frac{qs}{i\beta^t} \left[\frac{\partial}{\partial x} \left(\frac{F(t; x)}{q} \right)^i \right]_{x=qs},$$

or more obviously

$$G_i(t; s) = \left[\frac{F(t; qs)}{q} \right]^{i-1} G(t; s), \tag{45}$$

where

$$G(t; s) = G_1(t; s) = \frac{s}{\beta^t} \frac{\partial F(t; x)}{\partial x} \Big|_{x=qs}.$$

Theorem 13. *All states of the Markov chain $\{W(t)\}$ are stable. The transition functions $\{\mathcal{Q}_{ij}(t)\}$ are the Feller functions. These functions are differentiable and has a finite and continuous derivative with respect to $t \in \mathcal{T}$. Its q -matrix $\{q_{ij} = \mathcal{Q}'_{ij}(\varepsilon \downarrow 0)\}$ has components*

$$q_{ij} = \begin{cases} i\lambda_1 + (i - 1) \ln \beta, & \text{when } i = j, \\ \frac{j\lambda_{j-i+1}}{j - i + 1}, & \text{when } i \neq j, \end{cases} \tag{46}$$

where λ_i are in (42) and $q_{ij} \geq 0$ when $i \neq j$ and, $q_i := q_{ii} < 0$ for all $i, j \in \mathcal{E}$. Moreover it satisfies the identity

$$\mathcal{Q}'_{ij}(t + \tau) = \sum_{k \in \mathcal{E}} \mathcal{Q}'_{ik}(\tau) \mathcal{Q}_{kj}(t), \quad \text{for any } t, \tau \in \mathcal{T}, \quad (47)$$

the backward Kolmogorov system.

Proof. It follows from the relation (44) that for all $i \in \mathcal{E}$

$$q_i = \lim_{\varepsilon \downarrow 0} \frac{1 - \mathcal{Q}_{ii}(\varepsilon)}{-\varepsilon} < +\infty,$$

that is all states are stable and also the right-sided derivative $\mathcal{Q}'_{ij}(\varepsilon \downarrow 0)$ is finite.

From the relation (45) we have

$$\mathcal{Q}_{ij}(t) < \frac{G(t; s)}{s^j} \cdot [\widehat{F}(t; s)]^{i-1}, \quad \text{for } 0 < s < 1,$$

where $\widehat{F}(t; s) = F(t; qs)/q$ is the GF of a sub-critical MBP and $\widehat{F}(t; s) < 1$, so it converges to one as $i \rightarrow \infty$. Hence $\mathcal{Q}_{ij}(t) \downarrow 0$ as $i \rightarrow \infty$. Last fact implies that $\mathcal{Q}_{ij}(t)$ is the Feller function. Therefore $\mathbb{Q}(t)$ has a stable q-matrix with components $q_{ij} = \mathcal{Q}'_{ij}(\varepsilon \downarrow 0)$; see Anderson (1991, p.43).

Next, since all states are stable then $\mathbb{Q}(t) = \{\mathcal{Q}_{ij}(t)\}$ is differentiable and has a finite and a continuous derivative with respect to $t \in \mathcal{T}$; see Anderson (1991, p.10). Let's compute this derivative. It follows from (27) that

$$\begin{aligned} \Delta \mathcal{Q}_{ij}(t) &= \mathcal{Q}_{ij}(t + \varepsilon) - \mathcal{Q}_{ij}(t) = \frac{j q^{j-i}}{i \beta^t} \left[\frac{P_{ij}(t + \varepsilon)}{\beta^\varepsilon} - P_{ij}(t) \right] \\ &= \frac{j q^{j-i}}{i \beta^t} [\Delta P_{ij}(t) + P_{ij}(t + \varepsilon) \ln \beta \cdot \varepsilon + o(\varepsilon)]. \end{aligned}$$

Hence

$$\frac{\Delta \mathcal{Q}_{ij}(t)}{\varepsilon} = \frac{j q^{j-i}}{i \beta^t} \left[\frac{\Delta P_{ij}(t)}{\varepsilon} + P_{ij}(t + \varepsilon) \cdot \ln \beta + o(1) \right].$$

Taking limit as $\varepsilon \downarrow 0$ here yields

$$\mathcal{Q}'_{ij}(t) = \frac{j q^{j-i}}{i \beta^t} [P'_{ij}(t) - P_{ij}(t) \ln \beta], \quad \text{for all } i, j \in \mathcal{E}. \quad (48)$$

Being that $q_{ij} = \mathcal{Q}'_{ij}(\varepsilon \downarrow 0)$, we should compute $P'_{ij}(\varepsilon \downarrow 0)$. It follows from the general theory of MBP that $P'_{ij}(\varepsilon \downarrow 0) = \lim_{\varepsilon \downarrow 0} P_{ij}(\varepsilon)/\varepsilon = ia_{j-i+1}$. Therefore we get that

$$q_{ij} = \frac{j q^{j-i}}{i} [ia_{j-i+1} - \delta_{ij} \ln \beta].$$

Using the expression for densities $\{\lambda_j\}$ said in (42) from the last formula we obtain (46). Moreover we see that $q_{ij} \geq 0$ when $i \neq j$ for all $i, j \in \mathcal{E}$ and being that both λ_1 and $\ln \beta$ are negative yield that $q_i := q_{ii} < 0$.

Lastly owing to Markovian nature of $W(t)$ it follows from theory of continuous-time Markov chain that equation (47) holds. In particular, at $\tau = 0$

$$\mathcal{Q}'_{ij}(t) = \sum_{k \in \mathcal{E}} q_{ik} \mathcal{Q}_{kj}(t).$$

The proof is completed. □

Let $\mathcal{G}_i(s)$ be the GF of q-matrix $\{q_{ij}\}$ that is

$$\mathcal{G}_i(s) := \sum_{j \in \mathcal{E}} q_{ij} s^j = \sum_{j \in \mathcal{E}} \mathcal{Q}'_{ij}(\varepsilon \downarrow 0) s^j.$$

Using expressions (46) it follows that

$$\begin{aligned} \mathcal{G}_i(s) &= [i\lambda_1 + (i-1)\ln \beta] s^i + \sum_{j \in \mathcal{E} \setminus \{i\}} \frac{j\lambda_{j-i+1}}{j-i+1} s^j \\ &= (i-1)s^{i-1} \left[s \ln \beta + \sum_{j \in \mathcal{E}} \frac{\lambda_j}{j} s^j \right] + s^{i-1} g(s), \end{aligned}$$

where $g(s)$ is defined in (43). On the other hand it is easy to see that

$$\sum_{j \in \mathcal{E}} \frac{\lambda_j}{j} s^j = \int_0^s \frac{g(u)}{u} du.$$

Thence we have that

$$\mathcal{G}_i(s) = \frac{(i-1)m(s) + g(s)}{s} s^i,$$

where

$$m(s) := s \ln \beta + \int_0^s \frac{g(x)}{x} dx.$$

Now more general, consider

$$\mathcal{G}_i(t; s) = \sum_{j \in \mathcal{E}} \mathcal{Q}'_{ij}(t) s^j = \frac{\partial G_i(t; s)}{\partial t}.$$

(The differentiable property of GF $G(t; s)$ will be established in the Theorem 14 below). After standard calculations we make sure that the GF version of (48) is the following identity:

$$\mathcal{G}_i(t; s) = \frac{(i-1)m \left(\widehat{F}(t; s) \right) + g \left(\widehat{F}(t; s) \right)}{\widehat{F}(t; s)} G_i(t; s), \tag{49}$$

for all $t \in \mathcal{T}$, where $\widehat{F}(t; s) = F(t; qs)/q$.

Theorem 14. *The GF $G(t; s)$ is differentiable function with respect to $t \in \mathcal{T}$ uniformly for $0 \leq s < 1$. The transition function $\{\mathcal{Q}_{ij}(t)\}$ is unique solution of the backward Kolmogorov system (47), which is unique GF solution of equation*

$$\frac{\partial G(t; s)}{\partial t} = h \left(\widehat{F}(t; s) \right) G(t; s), \tag{50}$$

with condition $G(0; s) = s$, where $h(s) = g(s)/s$.

Proof. As in proof of the Theorem 13 we have

$$\mathcal{Q}_{ij}(t) - \mathcal{Q}_{ij}(t + \varepsilon) \leq \mathcal{Q}_{ij}(t) \cdot [1 - \mathcal{Q}_{ii}(\varepsilon)]$$

for arbitrary $\varepsilon > 0$. Hence for the difference

$$\Delta_\varepsilon G(t; s) = G(t + \varepsilon; s) - G(t; s)$$

we obtain that

$$\begin{aligned} |\Delta_\varepsilon G(t; s)| &\leq \sum_{j \in \mathcal{E}} |\mathcal{Q}_{1j}(t) - \mathcal{Q}_{1j}(t + \varepsilon)| s^j \\ &\leq [1 - \mathcal{Q}_{11}(\varepsilon)] \cdot \sum_{j \in \mathcal{E}} \mathcal{Q}_{1j}(t) s^j = 2G(t; s) \cdot [1 - \mathcal{Q}_{11}(\varepsilon)]. \end{aligned}$$

Since $\mathcal{Q}_{ij}(t)$ is standard, it follows from last inequality that $\Delta_\varepsilon G(t; s) \rightarrow 0$ as $\varepsilon \downarrow 0$. So $G(t; s)$ is continuous function with respect to $t \in \mathcal{T}$ uniformly for $0 \leq s < 1$. It can easily be seen that a GF version of the relation (42) is

$$G(\varepsilon; s) = s + g(s) \cdot \varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \downarrow 0, \tag{51}$$

for $0 \leq s < 1$. By the way according to formulas (4) and (45) one can see that GF $G(t; s)$ satisfies the following functional equation:

$$G(t + \tau; s) = \frac{G\left(t; \widehat{F}(\tau; s)\right)}{G\left(0; \widehat{F}(\tau; s)\right)} G(\tau; s). \tag{52}$$

We use expressions (51) and (52) to $\Delta_\varepsilon G(t; s)$ and hereupon we get

$$\Delta_\varepsilon G(t; s) = \frac{g\left(\widehat{F}(t; s)\right)}{\widehat{F}(t; s)} \cdot \varepsilon + o(\varepsilon) \quad \text{as } \varepsilon \downarrow 0,$$

for any $t \in \mathcal{T}$ and all $0 \leq s < 1$, which implies that $G(t; s)$ is differentiable.

Equation (50) follows from formula (49) at $i = 1$ and the boundary condition $G(0; s) = s$ follows from (51). The uniqueness of solution of (50) follows from the classical differential equations theory. \square

The following assertion is direct consequence from Theorem 14.

Corollary 1. *The differential equation (50) is equivalent to the following one*

$$\int_0^t h\left(\widehat{F}(\tau; s)\right) d\tau = \ln \frac{G(t; s)}{s} \tag{53}$$

with boundary condition $G(0; s) = s$, where $h(s) = g(s)/s$.

4.2 Classification and Ergodic Behavior

As it has been noticed above, that the parameter $a = f'(s \uparrow 1)$ plays a regulating role for MBP and is subdivided three types of process depending on sign of a . Note that evolution of MQP is regulated in essence by positive parameter $\beta = \exp\{f'(q)\}$. Thus are subdivided two types of process depending on values of this parameter. From equalities (45) and (53) we write

$$G_i(t; s) = s \left[\widehat{F}(t; s)\right]^{i-1} \exp \left\{ \int_0^t h\left(\widehat{F}(\tau; s)\right) d\tau \right\}. \tag{54}$$

If $\alpha := g'(1)$ is finite, then it follows from (14) that

$$\mathbb{E}_i W(t) = (i - 1) \beta^t + \mathbb{E}W(t)$$

and

$$\mathbb{E}W(t) = \begin{cases} 1 + \gamma(1 - \beta^t), & \text{when } \beta < 1, \\ \alpha t + 1 & , \text{ when } \beta = 1, \end{cases} \quad (55)$$

where $\gamma = \alpha/|\ln \beta|$. Moreover we obtain the variance structure

$$\text{Var}_i W(t) = \begin{cases} [\gamma + (i - 1)(1 + \gamma)\beta^t](1 - \beta^t), & \text{when } \beta < 1, \\ \alpha it & , \text{ when } \beta = 1, \end{cases}$$

where $\text{Var}_i W(t) = \text{Var}[W(t) | W(0) = i]$.

The formula (55) implies that when $\beta = 1$

$$\mathbb{E}_i W(t) \sim \alpha t \quad \text{as } t \rightarrow \infty,$$

and if $0 < \beta < 1$ then

$$\mathbb{E}_i W(t) \rightarrow 1 + \gamma \quad \text{as } t \rightarrow \infty.$$

Thereby we classify the MQP as *restrictive* if $\beta < 1$ and *explosive* if $\beta = 1$.

Further in restrictive case we keep on the condition (19). As it was noted in Section 2, this condition is equivalently to $\sum_{j \in \mathcal{S}} a_j q^{j-1} j \ln j < \infty$. Being that $\lambda_j = j q^{j-1} a_j$, for feasibility of (19) it is necessary and sufficient that

$$\sum_{j \in \mathcal{E}} \lambda_j \ln j < \infty. \quad (56)$$

In explosive case we everywhere suppose that the condition (7) holds.

Theorem 15. *The MQP is*

- (i) *positive if it is restrictive and condition (56) holds;*
- (ii) *null if it is explosive.*

Proof. To prove assertion (i) from (53) we get

$$\ln \mathcal{Q}_{11}(t) = \int_0^t h(\widehat{F}(\tau; 0)) d\tau = \int_0^{\widehat{F}(t; 0)} \frac{h(x)}{\widehat{f}(x)} dx \rightarrow \int_0^1 \frac{h(x)}{\widehat{f}(x)} dx,$$

since $\widehat{F}(t; 0) \uparrow 1$ as $t \rightarrow \infty$, where $\widehat{f}(s) = f(qs)/q$. With reference to Yang (1972) we make sure that the condition (56) is sufficient for a converging the

integral in right-hand side. Hence $\lim_{t \rightarrow \infty} \mathcal{Q}_{11}(t) > 0$. For part (ii) we recall that $q = 1$ and $h(s)/s = f'(s)$ if $\beta = 1$. Similarly

$$\ln \mathcal{Q}_{11}(t) = \int_0^t h(F(\tau; 0)) d\tau = \int_0^{F(t; 0)} \frac{h(x)}{f(x)} dx \rightarrow \int_0^1 \frac{f'(x)}{f(x)} dx = -\infty,$$

as $t \rightarrow \infty$, so that $\lim_{t \rightarrow \infty} \mathcal{Q}_{11}(t) = 0$. □

The next two assertions are direct consequences of Lemma 2.

Theorem 16. *Let MQP be restrictive. If condition (56) holds then*

$$G_i(t; s) = \mathcal{U}(s) (1 + o(1)) \quad \text{as } t \rightarrow \infty,$$

for all $0 \leq s < 1$, where the limiting $GF\mathcal{U}(s) = \sum_{j \in \mathcal{E}} u_j s^j$ has a form

$$\mathcal{U}(s) = s \frac{|\ln \beta|}{f(qs)} \mathcal{A}(qs). \tag{57}$$

The numbers $\{u_j\}$ represent an invariant distribution for MQP.

Proof. The convergence of $G_i(t; s)$ to $\mathcal{U}(s)$ follows from assertion (20) and formula (45) because $\widehat{F}(t; s) \uparrow 1$ as $t \rightarrow \infty$, where $\mathcal{U}(s)$ in the form of (57). Taking limit in (52) implies a Schroeder type invariance equation

$$\mathcal{U}(s) = \frac{G(\tau; s)}{\widehat{F}(\tau; s)} \mathcal{U}(\widehat{F}(\tau; s))$$

and hence $u_j = \sum_{i \in \mathcal{E}} u_i \mathcal{Q}_{ij}(\tau)$ for any $\tau \in \mathcal{T}$. Therefore $\{u_j\}$ is an invariant measure. Let now condition (56) holds. Then according to properties of the function $\mathcal{A}(s)$ (see (17))

$$\sum_{j \in \mathcal{E}} u_j = \lim_{s \uparrow 1} \mathcal{U}(s) = \lim_{s \uparrow 1} \frac{\mathcal{A}(qs)}{q(1-s)} = 1.$$

The theorem is proved. □

Theorem 17. *If MQP is explosive, then for all $0 \leq s < 1$*

$$\frac{(\nu t)^{1+1/\nu}}{\mathcal{N}(t)} \cdot G_i(t; s) \rightarrow \pi(s) \quad \text{as } t \rightarrow \infty, \tag{58}$$

where $\mathcal{N}(t)$ satisfies the property (10). The limit GF $\pi(s) = \sum_{j \in \mathcal{E}} \pi_j s^j$ determines an invariant measure $\{\pi_j\}$ and

$$\sum_{j=1}^n \pi_j = \frac{1}{\Gamma(2 + \nu)} n^{1+\nu} \mathfrak{L}_\pi(n), \tag{59}$$

where $\Gamma(*)$ is Euler's Gamma function and $\mathfrak{L}_\pi(n) \cdot \mathfrak{L}(n) \rightarrow 1$ as $n \rightarrow \infty$.

Proof. From second part of Lemma 2 and (45) we will write out

$$G_i(t; s) \sim s \frac{R^{1+\nu}(t; s)}{f(s)} \cdot \mathfrak{L} \left(\frac{1}{R(t; s)} \right) \quad \text{as } t \rightarrow \infty. \tag{60}$$

It follows from (7) and (9) that

$$\frac{R^{1+\nu}(t; s)}{f(s)} = \frac{\mathcal{N}^{1+\nu}(t)}{(\nu t)^{1+1/\nu}} \cdot \frac{\mathfrak{L}^{-1}(1/(1-s))}{(1-s)^{1+\nu}} \cdot \frac{1}{(1 + \mathcal{M}(s)/t)^{1+1/\nu}}, \tag{61}$$

where the function $\mathcal{M}(s)$ is defined in (11). It is cogently that $\mathcal{M}(0) = 0$ and $R(t; s)/R(t; 0) \rightarrow 1$ as $t \rightarrow \infty$ uniformly for $0 \leq s < 1$. Hence according to (10), $\mathcal{N}^\nu(t) \cdot \mathfrak{L}(1/R(t; s)) \rightarrow 1$. Then from (60) and (61) appears

$$G_i(t; s) \sim \frac{\mathcal{N}(t)}{(\nu t)^{1+1/\nu}} \cdot \pi(s) \cdot \frac{1}{(1 + \mathcal{M}(s)/t)^{1+1/\nu}}, \tag{62}$$

as $t \rightarrow \infty$, where

$$\pi(s) = \frac{s}{(1-s)^{1+\nu}} \mathfrak{L}_\pi \left(\frac{1}{1-s} \right). \tag{63}$$

The expansion (58) follows from the relation (62). The invariant equation $\pi_j = \sum_{i \in \mathcal{E}} \pi_i \mathcal{Q}_{ij}(t)$ comes out from the functional equation (52). At last, according to the Hardy-Littlewood Tauberian theorem each of relations (59) and (63) entails another. \square

It undoubtedly that $\lim_{s \downarrow 0} [G_i(t; s)/s] = \mathcal{Q}_{i1}(t)$. Then from Theorems 16 and 17 we get to the following local limit theorems.

Theorem 18. *If MQP is restrictive and condition (56) holds then*

$$\mathcal{Q}_{i1}(t) \longrightarrow \frac{|\ln \beta|}{a_0} \mathcal{A}(0) \quad \text{as } t \rightarrow \infty.$$

Theorem 19. *If MQP is explosive, then*

$$(\nu t)^{1+1/\nu} \cdot Q_{i1}(t) \sim \frac{\mathcal{N}(t)}{a_0} \quad \text{as } t \rightarrow \infty,$$

where the function $\mathcal{N}(t)$ satisfies the property (10).

Further we observe limit properties of $\{Q_{ij}(t)/Q_{11}(t)\}$. Consider the GF

$$\mathcal{W}_i(t; s) = \sum_{j \in \mathcal{E}} \frac{Q_{ij}(t)}{Q_{11}(t)} s^j = \frac{1}{Q_{11}(t)} G_i(t; s) = \left[\widehat{F}(t; s) \right]^{i-1} \mathcal{W}(t; s), \quad (64)$$

where

$$\mathcal{W}(t; s) = \sum_{j \in \mathcal{E}} \frac{Q_{1j}(t)}{Q_{11}(t)} s^j.$$

For general MQP the following ratio limit property holds.

Theorem 20. *The limits*

$$\lim_{t \rightarrow \infty} \frac{Q_{ij}(t)}{Q_{11}(t)} =: \omega_j \quad (65)$$

exist for all $i, j \in \mathcal{E}$. The set $\{\omega_j\}$ is an invariant measure and the GF

$$\mathcal{U}(s) := \sum_{j \in \mathcal{E}} \omega_j s^j = s \exp \left\{ \int_0^s \frac{|h(x)|}{\widehat{f}(x)} dx \right\}, \quad (66)$$

converges for $0 \leq s < 1$, where $h(s) = g(s)/s$ and $\widehat{f}(s) = f(qs)/q$.

Proof. It follows from (64) that it suffice to consider the case $i = 1$ because $\widehat{F}(t; s) \uparrow 1$ as $t \rightarrow \infty$ uniformly for all $0 \leq s \leq r < 1$. So write

$$\mathcal{U}(t; s) = s \exp \left\{ \int_0^t \left[h \left(\widehat{F}(u; s) \right) - h \left(\widehat{F}(u; 0) \right) \right] du \right\}.$$

One can choose $\tau \in \mathcal{T}$ for any $0 \leq s < 1$ so that $s = \widehat{F}(\tau; 0)$. Then considering functional equation (4), we get $\widehat{F}(t; s) = \widehat{F}(t + \tau; 0)$ and hence

$$\begin{aligned} \mathcal{U}(t; s) &= s \exp \left\{ \int_\tau^{t+\tau} h \left(\widehat{F}(u; 0) \right) du - \int_0^t h \left(\widehat{F}(u; 0) \right) du \right\} \\ &= s \exp \left\{ \int_0^\tau \left[h \left(\widehat{F}(t; \widehat{F}(u; 0)) \right) - h \left(\widehat{F}(u; 0) \right) \right] du \right\} \\ &= s \exp \left\{ \int_0^s \frac{h \left(\widehat{F}(t; x) \right) - h(x)}{\widehat{f}(x)} dx \right\}, \end{aligned}$$

where we have used the equation (5) and $\widehat{f}(s) = f(qs)/q$. To get to (66) it suffice to take limit as $t \rightarrow \infty$ being that $\widehat{F}(t; s) \rightarrow 1$ and $h(1) = 0$. Assertion (65) follows from (66) owing to the continuity theorem for GF. It easily to be convinced $\mathcal{W}(s) < \infty$ for all $0 \leq s < 1$.

Now we observe that the set $\{\omega_j\}$ to be the invariant measure for MQP. First using the Kolmogorov-Chapman equation we obtain that

$$\frac{\mathcal{Q}_{ij}(t + \tau)}{\mathcal{Q}_{11}(t + \tau)} \cdot \frac{\mathcal{Q}_{11}(t + \tau)}{\mathcal{Q}_{11}(t)} = \sum_{k \in \mathcal{E}} \frac{\mathcal{Q}_{ik}(t)}{\mathcal{Q}_{11}(t)} \mathcal{Q}_{kj}(\tau).$$

Setting $s = 0$ in (52) we can see that $\mathcal{Q}_{11}(t + \tau)/\mathcal{Q}_{11}(t) \rightarrow 1$ as $t \rightarrow \infty$. Hence we get the following invariant equation for $\{\omega_j\}$

$$\omega_j = \sum_{k \in \mathcal{E}} \omega_k \mathcal{Q}_{kj}(t) \quad \text{for any } t \in \mathcal{T}. \tag{67}$$

The GF version of (67) is

$$\mathcal{W}(\widehat{F}(t; s)) = \frac{\widehat{F}(t; s)}{G(t; s)} \mathcal{W}(s),$$

for $0 \leq s < 1$, the functional equation of generalized Schroeder form. □

We complete the paper with stating of the following limit theorem.

Theorem 21. *Let MQP is explosive and the function $\mathcal{N}(t)$ satisfies the property (10). Then for any $x > 0$*

$$\mathbb{P}_i \left\{ \frac{\mathcal{N}(t)}{(\nu n)^{1/\nu}} W(t) < x \right\} \longrightarrow G(x) \quad \text{as } t \rightarrow \infty,$$

where the Laplace transform

$$\int_{\mathbb{R}_+} e^{-\theta x} dG(x) = \frac{1}{(1 + \theta^\nu)^{1+1/\nu}}.$$

Proof. Consider the Laplace transform

$$\phi(t; \theta) := \mathbb{E} e^{-\theta q(t)W(t)} = G_i(t; \theta(t)), \tag{68}$$

where $\theta(t) = \exp\{-\theta q(t)\}$ and

$$q(t) := R(t; 0) = \frac{\mathcal{N}(t)}{(\nu t)^{1/\nu}}.$$

It was shown in the proof of Theorem 17 that

$$G_i(t; s) \sim \frac{q(t)}{\nu t} \cdot \pi(s) \cdot \frac{1}{(1 + \mathcal{M}(s)/t)^{1+1/\nu}} \quad \text{as } t \rightarrow \infty, \tag{69}$$

where

$$\pi(s) = \frac{s}{(1 - s)^{1+\nu} \mathfrak{L}(1/(1 - s))} \tag{70}$$

and it follows from (31) and (34) that

$$\mathcal{M}(s) \sim \frac{1}{\nu(1 - s)^\nu \mathfrak{L}(1/(1 - s))} \quad \text{as } s \uparrow 1. \tag{71}$$

Put $s = \theta(t)$ in (69). It is clear $1 - \theta(t) \sim \theta q(t)$ and $R(t; s)/q(t) \rightarrow 1$ as $t \rightarrow \infty$. So by the property of SV-functions $\mathfrak{L}(1/R(t; \theta(t))) \sim \mathfrak{L}(1/(1 - \theta(t)))$. On account of all these, from (68)–(71) it is a matter of standard computation to verify that

$$\phi(t; \theta) \rightarrow \frac{1}{(1 + \theta^\nu)^{1+1/\nu}} \quad \text{as } t \rightarrow \infty.$$

The proof is completed. □

The Theorem 21 generalizes for all $0 < \nu \leq 1$ the known Harris theorem established under finite variance condition for the process with discrete time; see Athreya and Ney (1972, p.59). Really, in specific case $\nu = 1$, the Laplace transform specified in the theorem becomes $(1 + \theta)^{-2}$ that fits to the first order Erlang’s law

$$1 - e^{-x} - xe^{-x}.$$

5. Conclusion

We devote the paper to research of the asymptote of trajectory and limit structure of MBP $\{Z(t), t \in \mathcal{T}\}$. All our reasoning and results are based on the assertion of the Lemma 1 and Lemma 2. Local limit theorems proved in Section 2 and Section 3 improve same results from the paper of Imomov (2014a) excepting a finiteness of the second moment $f''(s \uparrow 1)$.

In the Section 4 we observe the structural and asymptotical properties of Markov Q-process (MQP). This process represents a stochastic population process with the trajectory never extinct. We see that MQP is subdivided two types of process depending on values of parameter $\beta = \exp\{f'(q)\}$. Classification and Ergodic properties of MQP are studied in this Section.

In our next researches we will improve the results for the critical MBP provided that the function $\mathfrak{L}(\cdot)$ is slowly varying with remainder, namely

$$\frac{\mathfrak{L}(\lambda x)}{\mathfrak{L}(x)} = 1 + o\left(\frac{\mathfrak{L}(x)}{x^\nu}\right) \quad \text{as } x \rightarrow \infty$$

for each $\lambda \in \mathbb{R}_+$; see Bingham *et al.* (1987, p.185). We already have some advancement at this assumption for a discrete case only. So if GF $F(s)$ of offspring law of discrete time branching process $\{Z(n), n \in \mathbb{N}\}$ has a representation

$$F(s) = s + (1 - s)^{1+\nu} \mathfrak{L}\left(\frac{1}{1-s}\right),$$

then

$$\frac{1}{\Lambda(1 - F(n; s))} - \frac{1}{\Lambda(1 - s)} \sim \nu n + \frac{1 + \nu}{2} \cdot \ln(1 + \nu n \Lambda(1 - s)) \quad (72)$$

as $n \rightarrow \infty$, where $F(n; s) = \mathbb{E}s^{Z(n)}$ and $\Lambda(y) = y^\nu \mathfrak{L}(1/y)$.

We are sure the statement (72) is fair and for MBP. Hence we can state convergence rates in limit theorems in the case $a = 0$ for MBP and in the case $\beta = 1$ for MQP.

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