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Beta Burr Type X with Application to Rainfall Data

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ABSTRACT

We introduce a new extension distribution for Burr type X with one parameter named the Beta Burr type X. The new distribution is extended from the Burr type X with one parameter. Several important properties of the new extension distribution are derived like the moment, and moment generating function. The maximum likelihood estimation is used to estimate the parameters involved. A rainfall data set from 1975 to 2005 for 35 stations in peninsular Malaysia is used for the application of this new model. It gives a better fit compared to several other distributions.

Keywords: Burr type X, Moment, Moment Generating Function, Maximum likelihood.

1. Introduction

Continuous univariate distributions are essential to statistical sciences and are the most powerful indispensable tool to applied statisticians. These distributions have been comprehensively used over the last years for fitting data set for several branches of research for instance biological studies, demography, medical and environmental sciences, agricultural, economics, engineering and finance. However, in many applied sciences and statistics it has become necessary for generate new forms of continuous univariate distributions. Burr (1942) introduced twelve cumulative distribution function (CDF) for modeling data . Among these twelve distributions functions, several distributions received much attention such as Burr type III (BII), Burr type X one parameter(BX1) and Burr type XII (BXII). The BX1 is used for modeling by many authors like Jaheen (1996, 1995) , Ahmad et al. (1997), Ahmad Sartawi and Abu-Salih (1991), Ali Mousa (2001), Surles and Padgett (1998). The CDF for the BX1 is defined as

$$F(y,\varphi) = \left(1 - e^{-y^2}\right)^{\varphi} = F_{\varphi}(y) \quad y > 0, \varphi > 0 \tag{1}$$

where φ denotes the shape parameter. The probability density function (PDF) corresponding to (1) is

$$f(y,\varphi) = 2\varphi\, y\, e^{-y^2}\left(1 - e^{-y^2}\right)^{\varphi-1} \tag{2}$$

The k^{th} central moments for the Burr type X one parameter distribution is $\mu^{(k)} = \varphi\Gamma(\frac{k}{2}+1)\sum_{j=0}^{\varphi-1}\binom{\varphi-1}{j}\frac{(-1)^j}{(j+1)^{\frac{k}{2}+1}}$.

Eugene et al. (2002) suggested and studied a new mode for building a different distribution from the CDF of baseline distribution and named it the beta class of generalized(BG) distribution. He added two extra shape parameters for the generator. The CDF of random variable (R.V) Y for beta class generalized distribution can be generated by utilizing the inverse CDF for beta distribution.

If G is the CDF of any R.V, the BG distribution is defined as:

$$G(y,\nu,\omega) = \frac{1}{B(\nu,\omega)}\int_0^{F(y)} z^{\nu-1}(1-z)^{\omega-1}dz \tag{3}$$

where the two extra shape parameters are ν and ω for the baseline distribution. The PDF of BG is denoted by equation (3) is

$$g(y,\nu,\omega) = \frac{f(y)}{B(\nu,\omega)}F(y)^{\nu-1}\Big[1 - F(y)\Big]^{\omega-1} \quad \nu > 0,\ \beta > 0, \tag{4}$$

where by

$$B(\nu,\omega) = \frac{\Gamma(\nu)\Gamma(\omega)}{\Gamma(\nu+\omega)}.$$

In this current study the extension of BX1 introduced by Burr (1942) is through Eugene et al. (2002) method and we named it the Beta Burr type X with one parameter distribution (BBX1). The aims for this work are firstly, to study and explore several properties for BBX1 distribution. Secondly, to establish heavy tailed distribution by adding two shape parameters. Finally, to verify systematically the fit of this new model compared to other models through the real data application.

Numerous authors follow the same idea of the paper by Eugene et al. (2002) with a different CDFs e.g. Akinsete et al. (2008), Lemonte (2012), Merovci and Sharma (2014), Nadarajah and Gupta (2004), Nadarajah and Kotz (2004, 2006), Pescim et al. (2010), Silva et al. (2010), Cordeiro et al. (2011), Paranaíba et al. (2011), Domma and Condino (2013), Cordeiro et al. (2013), Jafari et al. (2014), Merovci and Sharma (2014)and Merovci et al. (2016) among others. One significant advantage of this kind of class of distribution is its ability of fitting skewed data.

This paper is divided as follows: In Section 2, we introduce the PDF, CDF of BBX1 and figures of the PDF, CDF and Hazard function with chosen different parameters. In Section 3, we derive several important properties of BBX1. In Section 4, parameters are derived and estimated by using the method of maximum likelihood estimation. We illustrate the application of BBX1 by using a real data set in Section 5. Finally, section 6 ends with the conclusion.

2. Beta Burr Type X

We develop BBX1 distribution by using the Beta class of generalized distribution. Suppose G(y) is the CDF of BX1 distribution as given in equation(1). Inserting (1) and (2) into (4) we obtain the PDF for the BBX1 distribution as follows

$$\begin{aligned} g(y,\nu,\omega,\varphi) =& \frac{2\,\varphi\, y\, e^{-y^2}\left(1-e^{-y^2}\right)^{\varphi-1}}{B(\nu,\omega)}\left[\left(1-e^{-y^2}\right)^{\varphi}\right]^{\nu-1} \\ & * \left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\omega-1} \quad y>0,\nu>0,\omega>0,\varphi>0, \end{aligned} \tag{5}$$

that can be reduced to

$$g(y,\nu,\omega,\varphi)=\frac{2\,\varphi\, y\, e^{-y^2}\left(1-e^{-y^2}\right)^{\varphi\nu-1}}{B(\nu,\omega)}\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\omega-1}. \quad (6)$$

Suppose Y is a R.V with PDF (6) ,then $Y\ \sim$ BBX1(ν,ω,φ) .

The CDF for BBX1 distribution is

$$G(y,\nu,\omega,\varphi)=I_{\left(1-e^{-y^2}\right)^{\varphi}}(\nu,\beta)=\frac{1}{B(\nu,\omega)}\int_0^{\left(1-e^{-y^2}\right)^{\varphi}} z^{\nu-1}(1-z)^{\omega-1}dz \quad (7)$$

The hazard rate function (hf) of BBX1 can defined by

$$h_f(y,\nu,\omega,\varphi)=\frac{2\,\varphi\, y\, e^{-y^2}\left(1-e^{-y^2}\right)^{\varphi\,\nu-1}\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\omega-1}}{B(\nu,\omega)\left[1-I_{\left(1-e^{-y^2}\right)^{\varphi}}(\nu,\omega)\right]}\quad y>0. \quad (8)$$

Figure 1 and Figure 2 display a variety of possible shapes of the PDF and the CDF of BBX1 distribution for selected values of parameters ν,ω andφ. In Figure 3. the hazard function of the BBX1 distribution is shown and it can be seen that the hf is a monotonically increasing function for different values of parameters and that can be useful for modeling different types of data.

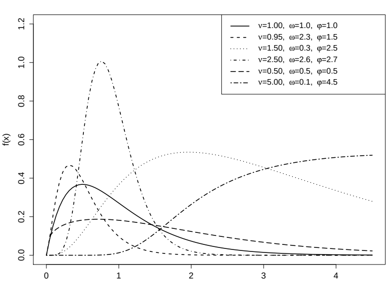

Figure 1: Plot of the BBX1 density function for several values.

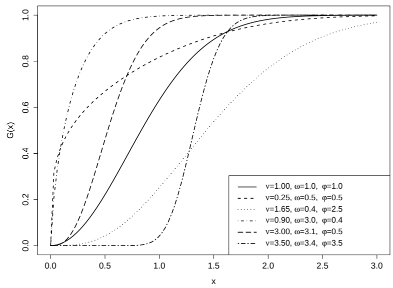

Figure 2: Plot of the BBX1 cumulative function for several values.

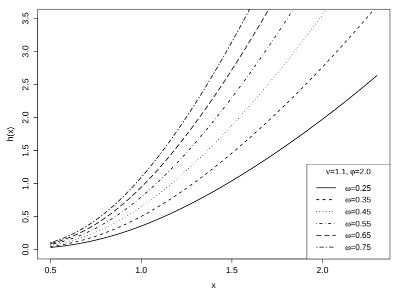

Figure 3: Plot of the BBX1 hazard function for several values.

3. Some Properties of the BBX1 Distribution

The followings are the properties of the distribution that have been established.

3.1 Quantile Function (QF), Bowley skewness (S_K) and Moors kurtosis(M_u)

The QF is defined as the inverse of CDF or $Q(q) = F^{-1}(q)$, it is easy to compute the QF by taking the inverse of (7)

$$y = Q(q) = G^{-1}(q) = \left(-\log\left\{1 - \left[I_q^{-1}(\nu, \omega)\right]^{\frac{1}{\varphi}}\right\}\right)^{\frac{1}{2}},$$

where $I_q^{-1}(\nu,\omega)$ denotes the inverse of the incomplete beta and can be defined from Wolfarm webste (http://functions.wolfram.com/06.23.06.0004.01)

$$\begin{aligned} I_q^{-1}(\nu,\omega) = w &+ \frac{(\omega-1)}{(\nu+1)}w^2 + \frac{(\omega-1)(\nu^2+3\nu\,\omega\ -\nu+5\omega-4)}{2(\nu+1)^2(\nu+1)}w^3 \\ &+ \frac{(\omega-1)[\nu^4+(6\omega-1)\nu^3+(\omega+2)(8\omega-5)\nu^2]}{3(\nu+2)(\nu+1)^3(\nu+1)}w^4 \\ &+ \frac{(\omega-1)[(33\omega^2-30\omega+4)\nu+\omega(31\nu-47)+18]}{3(\nu+1)(\nu+2)(\nu+1)^3}w^4 + O(P^{\frac{5}{\nu}}), \end{aligned}$$

where $w = [\nu\, q\, B(\nu,\omega)]^{1/\nu}$ for $\nu > 0$ and $0 < q < 1$.

The QF is important in finding several measures like theS_k and M_u. One of the earliest skewness measures is S_k and can be determine from equation(9), Kenney and Keeping (1954),

$$S_K = \frac{Q(\frac{75}{100}) + Q(\frac{25}{100}) - 2Q(\frac{50}{100})}{Q(\frac{75}{100}) - Q(\frac{25}{100})}. \tag{9}$$

The M_u Moors (1988) is

$$M_u = \frac{Q(\frac{10}{80}) + Q(\frac{30}{80}) + Q(\frac{70}{80}) - Q(\frac{50}{80})}{Q(\frac{75}{100}) - Q(\frac{25}{100})}. \tag{10}$$

S_k and M_u measures can be calculated without moment and are also less sensitive to outliers.

3.2 Moment and the Moment Generating Function

The moment about the origin ($\mu_r^{'}$) of continuous random variable Y has the CDF for $r \geq 1$ given by

$$E(Y^r) = \int_0^\infty y^r g(y,\nu,\omega,\varphi)dy.$$

From (6) we have

$$E(Y^r) = \frac{2\varphi}{B(\nu,\omega)}\int_0^\infty x^{r+1}e^{-y^2}\left[1-e^{-y^2}\right]^{\varphi\nu-1}\left[1-\left(1-e^{-y^2}\right)^{\varphi}\right]^{\omega-1}dy.$$

By using the binomial series expansion twice defined by

$$(1-\beta)^{w-1} = \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(w)}{\Gamma(w-j)j!}\beta^j,$$

the $E(X^r)$ is

$$\mu^{(r)} = E(X^r) = \frac{\varphi\Gamma(\frac{r}{2}+1)}{B(\nu,\omega)} \sum_{j,i=0}^{\infty} \frac{(-1)^{j+i}\Gamma(\omega)\Gamma(\varphi(\nu+j))}{\Gamma(\omega-j)\Gamma(\varphi(\nu+j)-i)i!j!(i+1)^{\frac{r}{2}+1}}. \quad (11)$$

Equation (11) can be written in another form for the r th central moment of the BBX1 distribution as follows

$$E(Y^r) = \frac{\varphi\Gamma(\frac{r}{2}+1)}{B(\nu,\omega)} \sum_{j=0}^{\infty} \binom{\omega-1}{j} \sum_{i=0}^{\infty} \binom{(\varphi(\nu+j)-1}{i} \frac{(-1)^{j+i}}{(i+1)^{\frac{r}{2}+1}}. \quad (12)$$

If $\nu = \omega = 1$ from equation (12)

$$E(Y^r) = \varphi\Gamma(\frac{r}{2}+1) \sum_{j=0}^{\theta-1} \binom{\varphi-1}{j} \frac{(-1)^j}{(i+1)^{\frac{r}{2}+1}}, \quad (13)$$

this is the r th moment for BX1 distribution.

The moment generating function (mgf) can be found by equation (14)

$$E(e^{x\,Y}) = M(x) = \int_0^{\infty} e^{x\,y} g(y,\nu,\omega,\varphi)dy. \quad (14)$$

By using the representation on page 29 Jeffrey and Zwillinger (2007),

$$e^{x\,y} = \sum_{k\geq 0} \frac{x^k\,y^k}{k!} \qquad .$$

We have a new form for mgf and we can write it as

$$M(x) = \int_0^{\infty} \sum_{k\geq 0} \frac{x^k}{k!} E(Y^k) g(y,\nu,\omega,\varphi)dy.$$

The $E(Y^k)$ has been obtained before as a power of r in equation (12), so the $E(e^{x\,Y})$ can be written as

$$E(e^{x\,Y}) = \frac{\varphi}{B(\nu,\omega)} \sum_{k,j,i\geq 0} \frac{x^k\,\Gamma(\frac{k}{2}+1)(-1)^{j+i}\Gamma(\omega)\Gamma(\varphi(\nu+j))}{\Gamma(\omega-j)\Gamma(\varphi(\nu+j)-i)k!\,j!\,i!(i+1)^{\frac{k}{2}+1}}. \quad (15)$$

4. Maximum Likelihood Estimation

There are several methods for estimating the parameter. The most widely used methods of statistical estimation is that of maximum likelihood estimation (MLE). We employ the MLE method to estimate the parameters.

Let $Y_1, Y_2, ..., Y_s$ be independent and identically distributed sample size s from the PDF of BBX1 distribution. Then the likelihood function (l) is given by

$$l(\nu,\omega,\varphi) = \frac{\varphi^s\, 2^s\, y^s \left[\Gamma(\nu+\omega)\right]^s}{\left[\Gamma(\nu)\right]^s \left[\Gamma(\omega)\right]^s} e^{-\sum_{i=1}^{s} y_i^2} \prod_{i=1}^{s}\left(1-e^{-y_i^2}\right)^{(\nu\,\varphi-1)} * \prod_{i=1}^{s}\left[1-\left(1-e^{-y_i^2}\right)^{\varphi}\right]^{\omega-1}. \tag{16}$$

Hence, the log-likelihood function($L = \log l$) for the parameters $\Theta = (\nu, \omega, \varphi)^T$ where Θ is a vector can be expressed as

$$\begin{aligned} L = s\Big[&\log\varphi + \log 2 + \log y + \log\Gamma(\nu+\omega) - \log\Gamma(\nu) - \log\Gamma(\omega)\Big] \\ &- \sum_{i=1}^{s} y_i^2 + (\nu\,\varphi - 1)\sum_{i=1}^{s}\log\left(1-e^{-y_i^2}\right) \\ &+ (\omega-1)\sum_{i=1}^{s}\log\left[1-\left(1-e^{-y_i^2}\right)^{\varphi}\right]. \end{aligned} \tag{17}$$

Taking partial derivatives of l in (17) with respect to ν, ω, and φ and equating to zero we have.

$$\begin{aligned} \frac{\partial l}{\partial \nu} &= s\Psi(\nu+\omega) - n\Psi(\nu) + \varphi\sum_{i=1}^{s}\log\left(1-e^{-y_i^2}\right) = 0, \\ \frac{\partial l}{\partial \omega} &= s\Psi(\nu+\omega) - n\Psi(\omega) + \sum_{i=1}^{s}\log\left[1-\left(1-e^{-y_i^2}\right)^{\varphi}\right] = 0, \\ \frac{\partial l}{\partial \varphi} &= \frac{s}{\varphi} + \varphi\sum_{i=1}^{s}\log\left(1-e^{-y_i^2}\right) \\ &\quad - (\omega-1)\sum_{i=1}^{s}\frac{\left(1-e^{-x_i^2}\right)^{\varphi}\log\left(1-e^{-y_i^2}\right)}{\left[1-\left(1-e^{-y_i^2}\right)^{\varphi}\right]} = 0. \end{aligned}$$

5. Application

The rainfall data set is used to illustrate that the BBX1 distribution is a better model than BX1, generalized logistic (GL) and Gamma (G) distributions. The available data is the annual series for the period of 30 years in

Peninsular Malaysia and the maximum daily rainfall data at 35 stations was used. These data consist of the mean of maximum daily rainfall for 30 years (1975 -2004) of 35 stations shown in Table 1, adapted from Zin et al. (2009) with some modifications. We estimated the parameters of the distribution by using MLE in R language, (package: Adequacy Model). To find the best fit of the data, we computed the MLE by using the L-BFGS-B. We used several measures of goodness -of- fit like A^* (Anderson-Darling), and W^* (Cram'er-von Mises) statistics defined by:

$$\begin{aligned} A^* &= a^2\Big(1+\frac{0.75}{m}+\frac{2.25}{m^2}\Big), \\ a^2 &= -m-\sum_{j=0}^{m}\frac{(2i-1)\ln u_j+(2m+1-2j)\ln(1-u_j)}{m} \end{aligned}$$

$$\begin{aligned} W^* &= w^2\Big(1+\frac{0.5}{n}\Big), \\ w^2 &= \sum_{i=0}^{n}\Big[u_i-\frac{(2i-1)}{2n}\Big] \end{aligned}$$

For more details of these statistics see Chen and Balakrishnan (1995). Moreover, the log-likelihood function (LL), AIC (Akaike information criterion), $AICC$ (corrected Akaike information criterion), and BIC (Bayesian information criterion) and $HQIC$ (Hannan-Quinn information criterion) were used in the comparison process for the data set, the statistics are defined by:

$$AIC = 2\,a - 2\,LL \ \ , BIC = a\ \log(m) - 2\,LL \ \ , AICc = AIC + \frac{2\,a\,(a+1)}{m-a-1}$$

$$HQIC = 2\,a\ \log\Big[\log(m)(a-2\,LL)\Big]$$

where m is the sample size, a is the number of parameters for any statistical model and LL is the maximized MLE value. The better distribution corresponds to smaller values of these statistics that indicate a better fit. The results are exhibited in Table 2. Figure 4 exhibits the estimated PDF of the BBX1, BX1, GL and G distributions and the histogram.

The values of statistics A^* and W^* show that BBX1 is better than BX1, GL and G distributions for the rainfall data set. Based on the criteria LL ,AIC ,$AICc$,BIC and $HQIC$, we found that BBX1 is the best fitted model for these data. With respect to BIC, the G distribution is better than BBX1, but overall, BBX1 is better than the other plots. The estimated PDF of the BBX1, BX1, GL and G distributions fitted to the rainfall data set are given in Figure 4. From this figure, BBX1 distribution looks superior compared to BX1, GL and G distributions in fitting this data set.

Table 1: Stations and the mean of rainfall for every station in decimeter (dcm) per day.

No.	Stations	mean
1	Temiang	0.933
2	Kg.Bahru	0.937
3	Kangar	0.989
4	Ampang Pedu	1.047
5	Bumbong Lima	1.187
6	Bukit Berapit	109.2
7	Bayan Lepas	1.249
8	Selama	1.110
9	Alor Pongsu	1.106
10	Bt. Kurau	1.003
11	Gua Musang	1.018
12	Ipoh	0.953
13	Ladang Boh	0.703
14	Sitiawan	0.955
15	Chui Chak	0.955
16	Paya Kangsar	0.991
17	Bagan Terap	0.842
18	Bkt. Ibam	1.298
19	Ampang	1.027
20	Subang	1.007
21	Petaling Jaya	1.068
22	Sg. Lui Halt	1.141
23	Sikamat	0.914
24	Port Dickson	1.164
25	Sungai Udang	1.234
26	Melaka	1.056
27	Tangkak	0.880
28	Kluang	1.163
29	Separap	0.835
30	Sembrong	1.077
31	Batu Pahat	1.048
32	Kota Tinggi	1.178
33	Senai	1.181
34	Tampok	1.196
35	Johor Bahru	1.134

Table 2: The ML estimates W^*, A^*, LL, AIC, BIC , AICc and HQIC for data set.

	Models			
Criterion	BBX1	BX1	GL	G
W^*	0.01869	0.023	0.0935	0.0332
A^*	0.130	0.205	0.697	0.282
P.Value	0.997	0.0000329	0.84	0
AIC	-38.8	9.23	-31.1	-38
BIC	-34.2	10.8	-28	-34.9
AICc	-38.1	9.35	-30.7	-37.6
HQIC	-37.2	9.77	-30	-36.9
LL	-22.4	3.501	-17.5	-21.2
MLEs	$\hat{\varphi}$=10.190	$\hat{\varphi}$=2.45	$\hat{\nu}$=1260.77	$\hat{\nu}$=62.44
(ST.Error)	(8.628)	(0.397)	(1002.414)	(8.62736)
	$\hat{\nu}$=27.893	-	$\hat{\lambda}$=7.27	$\hat{\omega}$=0.01677
	(42.514)	-	(0.854)	(0.00334)
	$\hat{\omega}$=0.786	-	-	-
	(0.957)	-	-	-

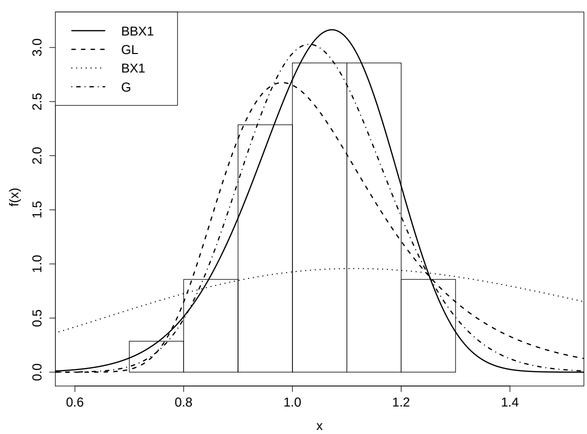

Figure 4: Plots of the estimated pdfs BBX1, BX1, GL and G distributions for the rainfall data.

6. Conclusion

We have introduced a new extension of BX1 distribution, named BBX1. We derived CDF, PDF and hazard function for BBX1. Furthermore, several statistical properties like QF, skewness, kurtosis, moments and mgf are also provided. We employ MLE to estimate the parameters. Finally, we applied the rainfall data and used statistical criteria to illustrate the goodness -of- fit of the rainfall data. This new distribution provides a better fit compare to BX1, GL and G distributions.

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