Malaysian Journal of Mathematical Sciences 12(1): 49–62 (2018)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

The Nonlocal Boundary Value Problem with Constant Coefficients for the Mixed Type of Equation of the First Kind in a Plane

Dzhamalov, S. Z.

Institute of Mathematics, National University of Uzbekistan, Uzbekistan

E-mail: siroj@63mail.ru

Received: 26 December 2016
Accepted: 20 September 2017

ABSTRACT

In the present work for the second order mixed type equation of the first kind we study unique solvability and smoothness of the generalized solution of nonlocal boundary value problem with constants coefficients.

Keywords: mixed type equation of the first kind, nonlocal problem, Sobolev spaces, unique solvability and smoothness of the generalized solution, "ε-regularization", a priory estimates and Galerkin's methods

1. Introduction and Formulation of the Problem

As it is known in Lavrentev and Shabat (1973), the problem of Dirichlet for the equation of the mixed type of the first kind is incorrect. Naturally there is a question: whether it is impossible to replace statements of the problem of Dirichlet with other conditions covering all border which provide a problem correctness?

The first time, such a condition have been offered and studied in the works in(Kalmenov (1978), Kalmenov (1990), Sabitov (2011) and Tsibikov (1986)).

As relatives on statement of studied problems we also note works (Berdyshev (1993),Dzhamalov (2014),Dzhamalov (016a),Dzhamalov (016b),Glazatov (1985),Karatopraklieva (1991),Pulkina and Savenkova (2016)).

Boundary value problems with nonlocal conditions have arisen for the first time in Frankl (1956) work at studying gas dynamics problems about a flow of profiles a stream of subsonic speed with the supersonic zone terminating in direct jump of consolidation.

In the present work, for the equation of the mixed type of the first kind in a plane the correctness of some nonlocal boundary value problem with constant coefficients is studied.

In the domain $Q = (-\alpha, \beta) \times (0, T)$ for the equation of the mixed type of the first kind:

$$Lu = K(x)u_{tt} - u_{xx} + a(x,t)u_t + c(x,t)u = f(x,t), \tag{1}$$

where, $x \cdot K(x) > 0$ at $x \neq 0$ and thus $-\alpha < x < \beta$; $\alpha > 0, \beta > 0$,

the following nonlocal boundary value problem is considered.

Problem. To find a generalized solution of the equation (1) from the Sobolev space $W_2^l(Q)$, $(1 \leq l$ is natural number), satisfying to nonlocal boundary conditions

$$\gamma D_t^p u|_{t=0} = D_t^p u|_{t=T}, \tag{2}$$

$$\eta D_x^p u|_{x=-\alpha} = D_x^p u|_{x=\beta}. \tag{3}$$

at $p = 0, 1$; here γ *and* $\eta - const \neq 0$, where $D_t^p u = \frac{\partial^p u}{\partial t^p}$, $D_t^0 u = u$.

Further, following definition and lemmas are necessary for us.

Through $W_2^l(Q)$ ($1 \leq l$ is natural number) we will designate Sobolev's space with a scalar product $(,)_l$ and norm $\|\cdot\|_l$, $W\,_2^0(Q) = L_2(Q)$ is the space of square sum able functions. Let $\nu = (\nu_t = \cos(\nu, t); \nu_x = \cos(\nu, x))$ be an unit vector of an interior normal to boundary ∂Q (Ladyzhenskaya (1973)). At deriving various a priori estimations we often use Young inequality. $\forall u, v > 0$, $\forall \sigma > 0$,$p > 1$

$$uv \leq \frac{\sigma^p u^{2p}}{2p} + \frac{v^{2q}}{2q\sigma^q}; \quad \frac{1}{p} + \frac{1}{q} = 1.$$

At $p = q = 1$, we will get Cauchy inequalities with σ (Ladyzhenskaya (1973)).

Definition 1.1. *We call the weak generalized solution of problem (1)-(3), function $u(x,t)$ from $W_2^1(Q)$, satisfying the following integral identity*

$$\Im(u,v) = \int\limits_Q e^{-\frac{(\lambda t + \mu x)}{2}} [K(x)u_t v_t - u_x v_x - (\frac{\lambda}{2}K(x)u_t - \frac{\mu}{2}u_x + \alpha u_t + cu)v]\, dxdt =$$

$$= -\int\limits_Q e^{-\frac{(\lambda t + \mu x)}{2}} f\, v\, dxdt$$

here λ and $\mu -$ constants, such that $\lambda > 0, \mu \geq 0$ for any periodic function $v(x,t)$ of variables x and of time t from $W\,_2^1(Q)$.

For simplicity at first we consider the case $l = 2$. We assume, that coefficients of equation (1) are smooth functions.

2. Uniqueness of the Solution for the Problem

Theorem 2.1. *Let aforementioned conditions to the coefficients of equation (1) are fulfilled, moreover let $2a(x,t) + \lambda K(x) \geq \delta_1 > 0$; $\lambda c - c_t \geq \delta_2 > 0$, where $\lambda = \frac{2}{T}\ln|\gamma|$, $|\gamma| > 1$, $|\eta| \geq 1$, $c(x,0) \leq c(x,T)$. Then for any function $f \in L_2(Q)$ there is a unique generalized solution of problem (1)-(3) in the space $W_2^2(Q)$.*

Proof. At first for any function $u \in W\,_2^2(Q)$, we will get the first basic inequality. In force the condition of theorem 1 and Cauchy inequalities with σ (Ladyzhenskaya (1973)), from problem (1)-(3) with integration, it is easy to obtain the following inequality

$$2\int\limits_Q Lu \exp(-\lambda t - \mu x) u_t dxdt \geq \int\limits_Q \exp(-\lambda t - \mu x) \times$$

$$\times \left\{(2a+\lambda K(x))u_t^2+\lambda u_x^2+(\lambda c-c_t)u^2\right\} dxdt+$$

$$+\int\limits_{\partial Q} \exp(-\lambda t-\mu x)\left\{K(x)u_t^2\nu_t-2u_xu_t\nu_x-u_x^2\nu_t+cu^2\nu_t\right\} ds-$$

$$-\sigma\cdot\|u_x\|_0^2-\mu^2\sigma^{-1}\cdot\|u_t\|_0^2, \qquad (4)$$

where $0\le\mu=\frac{2}{\alpha+\beta}\ln|\eta|$; σ, σ^{-1} are coefficients of the Cauchy inequality with σ (Ladyzhenskaya (1973)). Conditions of theorem 1 provide non- negativity of integral on the area Q and on the boundary ∂Q. Let $u\in W_2^2(Q)$ satisfying boundary conditions (2),(3) then

$$2\int\limits_{\partial Q} \exp(-\lambda t-\mu x)\left\{K(x)u_t^2\,\nu_t-2u_xu_t\nu_x-u_x^2\,\nu_t+cu^2\,\nu_t\right\} ds=$$

$$=[\exp(-\lambda T)\gamma^2-1]\int\limits_{-\alpha}^{\beta}\exp(-\mu x)K(x)(u_t^2(x,0)+u_x^2(x,0))dx+$$

$$+2[\exp(-\mu\beta)\eta^2-\exp(\mu\alpha)]\int\limits_0^T\exp(-\lambda t)u_x(-\alpha t)u_t(-\alpha t)dt+$$

$$+\int\limits_{-\alpha}^{\beta}\exp(-\mu x)[c(x,T)e^{-\lambda T}\gamma^2-c(x,0)]u^2(x,0)dx=\sum_{i=1}^{3}J_i,$$

J_i $(i=1,2,3)$ are integral along the boundary. Considering conditions of theorem 1, we get that boundary integrals $J_1=0, J_2=0$ and $J_3\ge 0$. Considering the aforesaid, from inequality (2), rejecting positive boundary integral, we obtain the following inequality

$$2\int\limits_Q Lu\exp(-\lambda t-\mu x)u_t dxdt\ge\int\limits_Q\left\{(2a+\lambda K(x))u_t^2+\lambda u_x^2+(\lambda c-c_t)u^2\right\}\times$$

$$\times\exp(-\lambda t-\mu x)dxdt-\sigma\cdot\|u_x\|_0^2-\mu^2\sigma^{-1}\cdot\|u_t\|_0^2, \qquad (5)$$

Choosing coefficients $\lambda-\sigma\ge\lambda_0>0$, $\delta_1-\mu^2\sigma^{-1}>\delta_0>0$, from inequality (2) we get the first a priori estimate

$$\|u\|_1\le m\|f\|_0,$$

from which uniqueness of the solution of problem (1)-(3) follows.

Here and further by m we designate positive constants, on exact values of which we are not interested. Theorem 2.1 is proved.

3. The Equation of the Third Order

Consider a nonlocal problem for the equation of the third order

$$L_\varepsilon u_\varepsilon = -\varepsilon \frac{\partial^3 u_\varepsilon}{\partial t^3} + L u_\varepsilon = f(x,t) \tag{6}$$

$$\gamma D_t^q u_\varepsilon|_{t=0} = D_t^q u_\varepsilon|_{t=T}\ , \quad q = 0,1,2\ . \tag{7}$$

$$\eta D_x^p u_\varepsilon|_{x=-\alpha} = D_x^p u_\varepsilon|_{x=\beta}, \ \ p = 0,1. \tag{8}$$

Here $D_t^p u = \frac{\partial^p u}{\partial t^p}$, $D_t^0 u = u$, ε-is a small positive number, $\eta,\ \gamma - const \neq 0$ such that $|\gamma| > 1$; $|\eta| \geq 1$. We use the equation of the third order (6) in quality "ε-regularization" the equations for equation (1)(refer Dzhamalov (2014), Kozhanov (1990),Tsibikov (1986)and Vragov (1983)).

In further by V we designate more low a class of the functions $u_\varepsilon(x,t) \in W_2^2(Q), \frac{\partial^3 u_\varepsilon}{\partial t^3} \in L_2(Q)$, satisfying corresponding conditions (7),(8).

Definition 3.1. *The regular solution of problem (6)-(8) we call function $u_\varepsilon(x,t) \in V$, satisfying to equation (6).*

Theorem 3.1. *Let aforementioned conditions to the coefficients of equation (1), be fulfilled, moreover let $2a + \lambda K(x) \geq \delta_1 > 0$; $\lambda c - c_t \geq \delta_2 > 0$, where $\lambda = \frac{2}{T}\ln|\gamma|$ and $|\gamma| > 1$; $|\eta| \geq 1$, $a(x,0) = a(x,T)$, $c(x,0) = c(x,T)$. Then for any function $f, f_t \in L_2(Q)$ such that $\gamma \cdot f(x,0) = f(x,T)$, there is a unique regular solution of problem (6)-(8) and the following inequalities are true:*

$$I) \qquad \varepsilon \left\| \frac{\partial^2 u_\varepsilon}{\partial t^2} u_\varepsilon \right\|_0^2 + \|u_\varepsilon\|_1^2 \leq m \|f\|_0^2,$$

$$II) \qquad \varepsilon \left\| \frac{\partial^3 u_\varepsilon}{\partial t^3} \right\|_0^2 + \|u_\varepsilon\|_2^2 \leq m \left[\|f\|_0^2 + \|f_t\|_0^2 \right] \ .$$

Proof. The proof is carried out stage by stage, using Galerkins method of with choice special bases-functions. The proof of an inequality I) is spent the same as also the proof of theorem 1 from which uniqueness of the regular solution of problem (6)-(8) follows (Kuzmin (1990),Ladyzhenskaya (1973)). Now we prove the first and the second a priori estimates. Let's consider the following spectral problems:

$$-X_j''(x) = \nu_j^2 X_j(x),$$

$$D_x^p X_j|_{x=-\alpha} = D_x^p X_j|_{x=\beta}, \quad p = 0, 1, \tag{9}$$

$$-T_j''(t) = \tau_j^2 T_j(t),$$

$$D_t^p T_j|_{t=0} = D_t^p T_j|_{t=T}, \quad p = 0, 1, \tag{10}$$

Through $\phi_j(x,t) = X_j(x)T_j(t)$ we define eigenfunctions, as the solution of problems (9),(10). From the general theory of the linear self-adjoint elliptic operators it is known that all $\{\phi_j(x,t)\}$ eigenfunctions of problems (9),(10) is a fundamental system in $W_2^2(Q)$ and it is orthogonal in the space $L_2(Q)$ (Berezinsky (1965),Ladyzhenskaya (1973)). Now by means of these sequences of functions we construct the solution of an auxiliary problem.

$$\ell\omega_j = e^{\frac{-(\lambda\cdot t+\mu\cdot x)}{2}} \frac{\partial \omega_j}{\partial t} = \phi_j, \tag{11}$$

$$\gamma \cdot \omega_j(x,0) = \omega_j(x,T), \tag{12}$$

where, $\gamma = const \quad \neq 0$, $\lambda = \frac{2}{T}\ln|\gamma|$, such that $|\gamma| > 1$, where $\eta = const \quad \neq 0$, $0 \leq \mu = \frac{2}{\alpha+\beta}\ln|\eta|$, $|\eta| \geq 1$. Obviously, the problem (11),(12) is uniquely solvable and its solution has the from

$$\ell^{-1}\phi_j = \omega_j = e^{\frac{\mu\cdot x}{2}} \cdot [\int_0^t \exp(\frac{\lambda\tau}{2})\phi_j d\tau + \frac{1}{\gamma-1}\int_0^T \exp(\frac{\lambda t}{2})\,\phi_j dt]. \tag{13}$$

It is clearly that functions $\omega_j(x,t)$ are linearly independent. Really, if $\sum_{j=1}^{N} c_j\omega_j = 0$ for some set of sequences of functions $\omega_1, \omega_2, ..., \omega_N$, then acting on this sum by the operator ℓ we have $\sum_{j=1}^{N} c_j \ell\omega_j = \sum_{j=1}^{N} c_j\phi_j = 0$, from this it follows that $c_j = 0$ for any $j = \overline{1,N}$. Note, from the construction of the function $\phi_j(x,t)$ the following conditions to the functions $\omega_j(x,t)$ follows:

$$\gamma\, D_t^q\, \omega_j|_{t=0} = D_t^q\, \omega_j|_{t=T}\,, \quad q = 0, 1, 2, \tag{14}$$

$$\eta\, D_x^p\, \omega_j|_{x=-\alpha} = D_x^p\, \omega_j|_{x=\beta}, \ \ p = 0, 1. \tag{15}$$

Now we search an approximate solution of problem (6)-(8) in the from $w = u_\varepsilon^N = \sum_{j=1}^{N} c_j\omega_j$, where the coefficient c_j for any $j = \overline{1,N}$ are defined as solutions of the linear algebraic system

$$\int_Q L_\varepsilon u_\varepsilon^N \cdot e^{\frac{-(\lambda\cdot t+\mu\cdot x)}{2}}\, \phi_j\, dxdt = \int_Q f \cdot e^{\frac{-(\lambda\cdot t+\mu\cdot x)}{2}}\, \phi_j\, dxdt. \tag{16}$$

Let's prove unequivocal resolvability of algebraic system (16). Multiplying every equation of (16) by $2c_j$ and summing up with respect j from 1 to N, considering problems (12),(13), from (16) we get the following identity

$$\int\limits_Q L_\varepsilon w \cdot e^{-(\lambda\cdot t+\mu\cdot x)} \cdot w_t dxdt = \int\limits_Q f \cdot e^{-(\lambda\cdot t+\mu\cdot x)} \cdot w_t dxdt \qquad (17)$$

From which, owing to a condition of theorem 2, integration of identity (17) we obtain for the approached solution of problem (6)-(8) estimations I), i.e.

$$\varepsilon \left\| \frac{\partial^2 u_\varepsilon^N}{\partial t^2} \right\|_0^2 + \left\| u_\varepsilon^N \right\|_1^2 \leq m \left\| f \right\|_0^2. \qquad (18)$$

This implies the solvability of (16). In particular, from the estimation (18) we obtain a weak solution of problem (6),(8). From here resolvability of algebraic system (16)(Ladyzhenskaya (1973)), follows due to the validness of uniqueness theorem. In particular, from an estimation (18) we will get existence of the weak generalized solution of problem (1)-(3)(Ladyzhenskaya (1973)). Really, due to an inequality (18), under the known theorem of weak compactness that from set of functions $\left\{ u_\varepsilon^N \right\}$, it is possible to extract poorly converging under sequence of functions in $W_2^1(Q)$ such that $\left\{ {u_{\varepsilon_j}}^{N_j} \right\} \to u$ at $N_j \to 0,\ \varepsilon_j \to 0$. On the foundation of it owing to uniqueness of a solution it is easy to show that, from identity (16) limiting function $u \in W_2^1(Q)$, satisfies to the integral identity in the sense of distributions

$$\Im(u,v) = -\int\limits_Q e^{\frac{-(\lambda t+\mu x)}{2}} f\, v\, dxdt$$

For any periodic function $v(x,t)$ of variable x and of time t from $W\,_2^1(Q)$ (Kozhanov (1990),Kuzmin (1990),Ladyzhenskaya (1973)).

Now we will prove the second a priori estimation II.

Differentiating equation (11) on a variable t two times and considering a statement of the problem (14)-(15), from identity (16) we get

$$-\frac{1}{\tau_j^2}\int\limits_Q L_\varepsilon w\, e^{\frac{-(\lambda\cdot t+\mu\cdot x)}{2}}\ \frac{\partial^2 \ell\omega_j}{\partial t^2} dxdt = -\frac{1}{\tau_j^2}\int\limits_Q f\, e^{\frac{-(\lambda\cdot t+\mu\cdot x)}{2}}\ \frac{\partial^2 \ell\omega_j}{\partial t^2} dxdt, \quad (19)$$

where, $\frac{\partial^2 \ell w}{\partial t^2} = \exp\ [\frac{-(\lambda t+\mu x)}{2}] \cdot (\frac{\partial^3 w}{\partial t^3} - \lambda w_{tt} + \frac{\lambda^2}{4} w_t)$;

Multiplying each equation of (19) by $2\tau_j^2 c_j$ and summing up j from 1 to N, considering a condition (14),(15), from (19) we have the following identity

$$-2\int\limits_Q L_\varepsilon w\, e^{\frac{-(\lambda\cdot t+\mu\cdot x)}{2}}\,\frac{\partial^2 \ell w}{\partial t^2}dxdt = -2\int\limits_Q f\, e^{\frac{-(\lambda\cdot t+\mu\cdot x)}{2}}\,\frac{\partial^2 \ell w}{\partial t^2}dxdt, \qquad (20)$$

Integrating (20) according to the conditions of Theorem 2 and the boundary conditions (14),(15), we obtain the following inequality

$$m\left[\,\|f_t\|_0^2+\|f\|_0^2\right]\geq \varepsilon\left\|\frac{\partial^3 w}{\partial t^3}\right\|_0^2+\int\limits_Q e^{-(\lambda\cdot t+\mu\cdot x)}\left\{(\,2a+\lambda K(x))\,w_{tt}^{\,2}+\lambda\, w_{tx}^{\,2}\right\}dxdt+$$

$$+\int\limits_{\partial Q} e^{-(\lambda\cdot t+\mu\cdot x)}[(K(x)\,w_{tt}^2+2a\,w_t\,w_{tt}-2w_{xx}\,w_{tt}+2c\,w\,w_{tt})\nu_t-2w_{tt}\,w_{xt}\,\nu_x]ds-$$

$$-\lambda_0\,(\,\|w_{xt}\|_0^2+\|w_{tt}\|_0^2\,)-c\,(\lambda_0)\,\|w\|_1^2=\sum_{i=1}^{2}J_i, \qquad (21)$$

where, J_1 is the integral along the domain, J_2 is the integral along the boundary.

Considering the condition of theorem 2 and boundary conditions (14),(15), we get, $J_1 > 0$ and $J_2 = 0$. Let $\delta_3 = \min\{\delta_1, \lambda\}$. Choosing $\delta_3 - \lambda_0 > \delta_0 > 0$, from an inequality (3) we obtain the second estimation

$$\varepsilon\left\|\frac{\partial^3 u_\varepsilon^N}{\partial t^3}\right\|_0^2+\left\|u_{\varepsilon,t\,t}^N\right\|_0^2+\left\|u_{\varepsilon,x\,t}^N\right\|_0^2\leq m\left[\|f\|_0^2+\|f_t\|_0^2\right]. \qquad (22)$$

Hence, the obtained estimations (18)-(22) allow to execute limiting transition on $N\to\infty$ and to conclude that a subsequence $\left\{u_\varepsilon^{N_k}\right\}$ converges in view of the uniqueness (Theorem 1) in $L_2(Q)$ together with a first -and second -order to the derivatives $u_{\varepsilon,t\,t}^{N_k}, u_{\varepsilon,x\,t}^{N_k}$ to the required regular solution $u_\varepsilon(x,t)$ of problem (6)-(8),the possessing the properties specified in Theorem 2. Owing to uniqueness (theorem 1) actually all sequence $\left\{u_\varepsilon^N\right\}$ converges to this solution. I.e. for function $\{u_\varepsilon\}$ owing to (22) the inequality is satisfaid

$$\varepsilon\left\|\frac{\partial^3 u_\varepsilon}{\partial t^3}\right\|_0^2+\left\|u_{\varepsilon,t\,t}\right\|_0^2+\left\|u_{\varepsilon,x\,t}\right\|_0^2\leq m\left[\|f\|_0^2+\|f_t\|_0^2\right]. \qquad (23)$$

Further, the function family $\{\,u_\varepsilon\}$, $(\varepsilon>0)$ satisfies to the elliptic equation with conditions (7),(8)

$$E\,u_\varepsilon=-u_{\varepsilon\,x\,x}\;=f+\varepsilon\,\frac{\partial^3 u_\varepsilon}{\partial\,t^3}-\;K(x)\;u_{\varepsilon\,t\,t}-a\,u_{\varepsilon\,t}-cu_\varepsilon=\quad F_\varepsilon, \quad (24)$$

From an aprioristic estimation (23) follows that the family of functions $\{F_\varepsilon\}, (\varepsilon > 0)$ is in regular intervals limited in norm of the space $L_2(Q)$, i.e. the inequality takes place

$$\|F_\varepsilon\|_0^2 \leq m \left[\|f\|_0^2 + \|f_t\|_0^2\right] \leq m \, \|f\|_1^2 \tag{25}$$

From here on the basis of aprioristic estimations for the elliptic equations (Dzhamalov (2014), Ladyzhenskaya (1973) and Vragov (1983)) and inequality (25) we get

$$\| u_\varepsilon\|_2^2 \leq m \, \|f\|_1^2 \, , i.e. \tag{26}$$

$$\varepsilon \left\| \frac{\partial^3 u_\varepsilon}{\partial t^3} \right\|_0^2 + \|u_\varepsilon\|_2^2 \leq m \, \left[\|f\|_0^2 + \|f_t\|_0^2 \right] . \tag{27}$$

Theorem 3.1 is proved.

4. Existence of a Solution for the Problem

Now by means of "ε-regularization" method we prove resolvability of a problem (1)-(3).

Theorem 4.1. *Let all conditions of theorem 3.1 be satisfied. Then the generalized solution of problem (1)-(3) of space $W_2^2(Q)$ exists and unique.*

Proof. Uniqueness of the generalized solution of problem (1)-(3) of $W_2^2(Q)$ is proved in Theorem 2.1. Now we prove existence of the generalized solution of problem (1)-(3) of $W_2^2(Q)$. For this purpose we consider in the area Q equation (6) with nonlocal boundary conditions (7),(8) at $\varepsilon > 0$. As all conditions of Theorem 3.1 there is a unique regular solution of problem (6)-(8) at are satisfied $\varepsilon > 0$ and for it are fair the first and second estimation. From here follows, under the known theorem of weak compactness that from set of functions $\{ u_\varepsilon\}, \quad \varepsilon > 0$ it is possible to take poorly-weakly converging under sequence of functions in V such that $\{ u_{\varepsilon_i} \} \to u$ at $\varepsilon_i \to 0$. we show that limiting function $u(x,t)$ satisfies to the equation $Lu = f(1)$. Really, since the sequence $\{ u_{\varepsilon_i}\}$ is converges weakly in $W_2^2(Q)$ and the sequence $\{ \frac{\partial^3 u_{\varepsilon_i}}{\partial t^3} \}, (\varepsilon > 0)$ is in regular intervals limited in $L_2(Q)$, and the operator L-is linear then we have

$$Lu - f = Lu - Lu_{\varepsilon_i} + \varepsilon_i \frac{\partial^3 u_{\varepsilon_i}}{\partial t^3} = L\,(u - u_{\varepsilon_i}) + \varepsilon_i \frac{\partial^3 u_{\varepsilon_i}}{\partial t^3} \tag{28}$$

From here, we get the following inequality

$$\|Lu-f\|_0=\|L(u-u_{\varepsilon_i})\|_0+\varepsilon_i\left\|\frac{\partial^3 u_{\,\varepsilon_i}}{\partial\, t^3}\right\|_0\leq m\,\|u-u_{\varepsilon_i}\|_2+\varepsilon_i\left\|\frac{\partial^3 u_{\,\varepsilon_i}}{\partial\, t^3}\right\|_0 \quad (29)$$

From (29) passing to the limit at $\varepsilon_i\to 0$, we get the unique solutions of problem (1)-(3)(Dzhamalov (2014), Ladyzhenskaya (1973) and Vragov (1983)).

Theorem 4.1 is proved.

5. Smoothness of the Solution for the Problem

Now we prove more general case,$l\geq 3$.Further we assume that coefficients of Eq.(1) are infinitely differentiable in a closed domain $\overline{Q}$.

Theorem 5.1. *Let conditions of theorem 4.1 are fulfilled, moreover let*

$D_t^{\,p}\,a|_{t=0}=D_t^p\,a|_{t=T}$; $D_t^{\,p}\,c|_{t=0}=D_t^{\,p}\,c|_{t=T}$. *Then for any function* $f(x,t)$, *such that* $f\in W_2^{\,p}(Q)$, $\quad D_t^{\,p+1}f\in L_2(Q)$, $\gamma\,D_t^{\,p}\,f|_{t=0}=D_t^{\,p}\,f|_{t=T}$, *there exists unique generalized solution of a problem (1)-(3) from the space* $W_2^{\,p+2}(Q)$,

noindent where $p=1,2,3,...$

Proof. From smoothness of the solution of the problem (9)-(12), the following conditions for the approximate solution of (6)-(8) follow:

$w=u_{\,\varepsilon}^{\,N}$;

$$\gamma D_t^q\,w|_{t=0}=D_t^q\,w|_{t=T}\ ,\qquad q=0,1,2,...$$
$$\eta D_x^p\,w|_{x=-\alpha}=D_x^p\,w|_{x=\beta},\quad p=0,1$$

Considering conditions of Theorem 3.1 at $\varepsilon>0$ and nonlocal conditions at $t=0,\ t=T$ from the equality

$$\left(e^{-\frac{\lambda t}{2}}L_\varepsilon u_\varepsilon\right)\Big|_{t=0}^{t=T}=\left(-\varepsilon e^{\frac{-\lambda t}{2}}\frac{\partial^3}{\partial t^3}u_\varepsilon+e^{\frac{-\lambda t}{2}}Lu_\varepsilon\right)\Big|_{t=0}^{t=T}=\left(e^{\frac{-\lambda t}{2}}f(x,t)\right)\Big|_{t=0}^{t=T}.$$

We get $\|\gamma u_{\varepsilon ttt}(x,0)-u_{\varepsilon ttt}(x,T)\|_0\leq const.$

Hence, follows that the function $v_\varepsilon(x,t) = u_{\varepsilon\, t}(x,t)$ belongs to V and satisfies the following equation

$$P_\varepsilon v_\varepsilon = L_\varepsilon v_\varepsilon = f_t - a_{\,t}\, u_{\varepsilon\, t} - c_t\, u_{\,\varepsilon} = F_\varepsilon. \tag{30}$$

From Theorem 3.1 follows that set of functions $\{F_\varepsilon\}$ is uniformly bounded in the space $L_2(Q)$, i.e.

$$\|F_\varepsilon\|_0 \leq m\, \left[\, \|f\|_0^2 + \|f_t\|_0^2\right].$$

Further from conditions of the Theorem 4.1, one can easily get that operators $P_\varepsilon(\varepsilon > 0)$ satisfy conditions of the Theorem 5.1, from here based on the estimates I), II), for a function $\{v_\varepsilon\}$ we obtain analogical estimates

$$\varepsilon\, \left\|\frac{\partial^2 v_\varepsilon}{\partial t^2}\right\|_0^2 + \|v_\varepsilon\|_1^2 \leq m\,(\|f\|_0^2 + \|f_t\|_0^2), \tag{31}$$

$$\varepsilon\, \left\|\frac{\partial^3 v_\varepsilon}{\partial t^3}\right\|_0^2 + \|v_\varepsilon\|_2^2 \leq m\, \left[\|f\|_1^2 + \|f_{tt}\|_0^2\right] \tag{32}$$

The function $\{\, u_\varepsilon\}$ satisfies parabolic equation with conditions (7),(8)

$$\Pi u_\varepsilon = u_{\varepsilon t} - u_{\varepsilon xx} = f + \varepsilon \frac{\partial^{\,3} u_\varepsilon}{\partial t^{\,3}} - K(x)u_{\varepsilon tt} - (a-1)u_{\varepsilon t} - cu_\varepsilon = \Phi_\varepsilon, \tag{33}$$

where $\Phi_\varepsilon \in L_2(Q)$, the set of functions $\{\Phi_\varepsilon\}$ is uniformly bounded in space $W_2^2(Q)$, i.e.

$$\|\Phi_\varepsilon\|_1^2 \leq m\left[\|f\|_1^2 + \|f_{tt}\|_0^2\right] \leq m\,\|f\|_2^2\,. \tag{34}$$

From here on the basis of aphoristic estimations for the parabolic equations (Dzhamalov (2014), Dzhamalov (016a), Ladyzhenskaya (1973)andVragov (1983)) and an inequality (34) we have

$$\|\, u_\varepsilon\|_3^2 \leq m\,\|f\|_2^2\,.$$

Further similarly one can prove that

$$\|u_\varepsilon\|_{p+2}^2 \leq m\,\|f\|_{p+1}^2\,,$$

where, $p = 2, 3, \ldots$. Theorem 5.1 is proved. □

Note. There is a question: How essential is a restriction on γ in formulation of problem? i.e. $|\gamma| > 1$

In this connection we consider the following example. Let the coefficients of equation (1) satisfy the following conditions.

$$K(x) = 0,\ \ a(x,t) = 1,\ \ c(x,t) = c_0 > 0,\ \ f(x,t) = 0,\ \ \eta = 1.$$

We note that the coefficients of equation (1) satisfy coefficients all conditions of theorem 2.1. We consider the following example.

Example. In the rectangle $Q = (-1,1) \times (0,T)$ we consider the following modeling problem

$$\Pi_1 u = u_t - u_{xx} + c_0 u = 0 \tag{35}$$

$$\gamma\, u\,(x,0) = u\,(x,T) \tag{36}$$

$$D_x^p u|_{x=-1} = D_x^p u|_{x=1},\ p = 0,1; \tag{37}$$

Further, let $\omega(x)$ be an eigenfunction of the Sturm-Lioville problem for the ordinary differential equation with periodic conditions.

$$\omega'' = -\lambda^2 \omega, \quad D_x^p \omega|_{x=-1} = D_x^p \omega|_{x=1},\ p = 0,1.$$

It is easy to check up that all conditions of the theorem 2.1 are fulfilled, but it despite nontrivial solutions of the nonlocal problem (35)-(37) will be functions $u(x,t) = Ae^{-(\lambda^2+c_0)t}\omega(x)$, ($A-$ is arbitrary constants)

Here $\gamma = e^{-(\lambda^2+c_0)T} \quad < 1$.

Thus, we have seen that the restrictions on γ,i.e $|\gamma| > 1$ in Theorem 2.1 are essential. If these conditions are not fulfilled, as shown, a uniqueness of the solution of the problem is violated.

6. Conclusion

In the present work, for the equation of the mixed type of the first kind in a plane the correctness of some nonlocal boundary value problem with constant coefficients is studied.

Acknowledgements

The author would like to sincerely thank the editor and referees for useful comments and suggestions on paper improvement.

References

Berdyshev, A. S. (1993). Nonlocal boundary value problems for equations of mixed type in a deviation from the characteristics. *Differ. Uravn.*, 29(12):2118–2125.

Berezinsky, Y. M. (1965). *Expansion in eigenfunctions of selfadjoint operators.* Naukova dumka, Kyev.

Dzhamalov, S. Z. (2014). On a nonlocal boundary value problem for a second-order equation of mixed type of the second kind. *Uzbek. Mat. Zh. (Russian)*, 2014(1):5–14.

Dzhamalov, S. Z. (2016a). On a nonlocal boundary value problem with constant coefficients for the equation of trikomi. *Uzbek. Mat. Zh. (Russian)*, 2016a(2):51–60.

Dzhamalov, S. Z. (2016b). On the correctness of a nonlocal problem for the second order mixed type equations of the second kind in a rectangle. *IIUM Engineering Journal*, 17(2):95–104.

Frankl, F. (1956). Flow of profiles a stream of subsonic speed with the supersonic zone terminating in direct jump objurgations. *Applied mathematics and mechanics*, 20(2):196–202.

Glazatov, S. N. (1985). Nonlocal boundary problems for mixed type equations in a rectangle. *Siberian Math. Journ.*, 26(6):162–164.

Kalmenov, T. S. (1978). About a floor to a periodic problem for the multidimensional equation mixed type. *Differ. Uravn.*, 14(3):546–548.

Kalmenov, T. S. (1990). On the dirichlet problem and nonlocal boundary problems for wave equation. *Differ. Uravn.*, 26(1):60–65.

Karatopraklieva, M. G. (1991). A nonlocal boundary-value problem for an equation of mixed type. *Differ. Uravn.*, 27(1):68–79.

Kozhanov, A. I. (1990). *Boundary problems for equations of mathematical physics of odd order.* Novosibirsk State University, Novosibirsk. (in Russian).

Kuzmin, A. (1990). *Non-classical mixed type equations and their applications to the gas dynamics.* Leningrad State University, Leningrad. (in Russian).

Ladyzhenskaya, O. A. (1973). *The Boundary Value Problems of Mathematical Physics.* Nauka, Moscow.

Lavrentev, M. A. and Shabat, B. V. (1973). *Hydrodynamics problems and their mathematical models.* Nauka, Moscow.

Pulkina, L. S. and Savenkova, A. E. (2016). A problem with non local condition on time for the many-dimensional hyperbolic equation. *Russian Math. (Izv. VUZ)*, 2016(10):41–52.

Sabitov, K. B. (2011). The dirichlet problem for a third-order equation of mixed type in a rectangular domain. *Differ. Uravn. (Russian)*, 47(5):705–713.

Tsibikov, B. (1986). About a correctness of a periodic problem for the multi-dimensional equation mixed type. *Non-classical equations of mathematical physics.*, 1986(1):201–206.

Vragov, V. (1983). *Boundary problems for the non-classical equations of mathematical physics.* NGU, Novosibirsk.