Malaysian Journal of Mathematical Sciences 12(2): 161–173 (2018)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

On Orthogonal Labelling for the Orthogonal Covering of the Circulant Graphs

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Received: 27 April 2016
Accepted: 16 May 2018

ABSTRACT

If we have two Abelian groups, then we can use the cartesian product of these two groups for labelling the circulants and this manages us to find the cyclic orthogonal double covers (CODCs) of these circulants by certain infinite graph classes, such as $K_{1,2m-2} \cup K_{1,2m(n-1)}$, $K_{1,4(n-1)} \cup K_{1,4(m-1)} \cup K_{1,4(m-1)(n-1)}$ with m and $n > 1$, and by other certain defined graphs in this paper.

Keywords: Circulant graph, Orthogonal double cover, Orthogonal labelling.

1. Introduction

Let $\mathcal{B}$ be Abelian group with identity 0 and S be a subset of $\mathcal{B}$ satisfying $0 \notin S$ and $S = -S$, hence $s \in S$ iff $-s \in S$. The *Cayley graph* $\mathrm{Cay}(\mathcal{B}; S)$ on $\mathcal{B}$ with *connection set* S is defined as follows (i) the vertices are the elements of $\mathcal{B}$ and (ii) there is an edge joining u and v if and only if $u = s + v$ for some $s \in S$. The circulant graphs are considered as Cayley graphs on cyclic groups.

The notation $\mathrm{Circ}(n; S)$ is used for the circulant graph of order n with connection set S. The circulant graph, $\mathrm{Circ}(7; \{2, 3, 4, 5\})$ is given in Figure 1 for more illustration.

The concept of the orthogonal double cover (ODC) of any graph J can be interpreted by supposing that J be a graph having m vertices and $\mathcal{J} = \{B_0, B_1, \ldots, B_{m-1}\}$ be a collection of m isomorphic subgraphs of J.

We consider $\mathcal{J}$ an ODC of J by B iff (i) All the edges of J are exactly repeated twice in $\mathcal{J}$ and (ii) If α and β are adjacent vertices in J, then B_α and B_β have one common edge. Our results in this paper are concerned with the cyclic orthogonal double covers (CODCs) of circulant graphs. For the CODCs definition, see Gronau et al. (1997). Many papers were introduced for the cyclic orthogonal double covers of circulant graphs, see Sampathkumar and Srinivassan (2011), El-Shanawany and Shabana (2014).

The *orthogonal labelling* notion was introduced by Gronau et al. (1997). For the graph $B = (V, E)$ having $m - 1$ edges, a one-to-one function $\Phi : V(B) \longrightarrow \mathbb{Z}_m$ is an orthogonal labelling of B if (i) B has two edges of length $d \in \{1, 2, \ldots, \lfloor \frac{(m-1)}{2} \rfloor\}$ exactly, and also one edge of length $m/2$ where m is even number, (ii) $\{r(d) : d \in \{1, \ldots, \lfloor \frac{(m-1)}{2} \rfloor\}\} = \{1, \ldots, \lfloor \frac{(m-1)}{2} \rfloor\}$, where $r(d)$ is the rotation-distance between two edges of the same length.

The following Theorem introduces the relation between the CODCs of the complete graphs and their orthogonal labellings and Theorem 1.1 was generalized by Theorem 1.2.

Theorem 1.1. *(Gronau et al. (1997)). The CODC of the complete graph by a graph B exists iff B has an orthogonal labelling.*

Theorem 1.2. *(Sampathkumar and Srinivassan (2011)). A CODC of $Circ(m; \{l_1, l_2, \ldots, l_k\})$ by a graph B exists iff B has an orthogonal $\{l_1, l_2, \ldots, l_k\}$-labellings.*

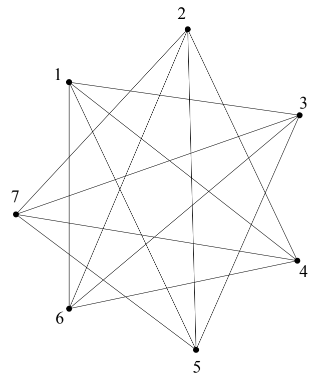

Figure 1: The circulant graph Circ(7; $\{2, 3, 4, 5\}$).

In the following Section, the notation $\star$ is used for referring to the normal multiplication, the notation $\times$ for the cartesian products and ab for $(a, b) \in \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$. In El-Shanawany et al. (2013), the new cartesian product notion for finding the ODCs of $K_{m,m}$ was introduced, and it is easy to notice that there is a relation between this method and the graph lift, for more illustration, see Shang (2012). Theorems 1.1, 1.2 are helpful tools for the following work. Since there is a bijective function $\Psi : \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \longrightarrow \mathbb{Z}_{m_1 \star m_2}$ defined by $\Psi(ab) = m_2 a + b$; $a \in \mathbb{Z}_{m_1}, b \in \mathbb{Z}_{m_2}$, $uv > wy$ if $u > w$ or if $u = w$ and $v > y$ where $uv,\ wy \in \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$ and $u \star v,\ w \star y \in \mathbb{Z}_{m_1 \star m_2}$. Let $\mathbb{Z}_{m_1} = \{0, 1, \ldots, m_1 - 1\}$ and $\mathbb{Z}_{m_2} = \{0, 1, \ldots, m_2 - 1\}$, then the circulant graph $Circ(m_1 \star m_2;\ Y)$ has a vertex set $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$; $Y \subset \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$. We say that the vertices uv and wy are adjacent iff $uv - wy = \pm(\gamma\delta)$; $\gamma\delta \in Y$, and $u,\ w$, and γ are calculated modulo m_1 and v, y, and δ are calculated modulo m_2. In $Circ(m_1 \star m_2;\ Y)$ the length of the edge $\{uv, wy\}$ is $min\{|uv - wy|, m_1 m_2 - |uv - wy|\}$. The rotation-distance $r(\gamma\delta)$ between E_1 and E_2 where $E_1 = \{xy, zt\}$ and $E_2 = \{op, qr\}$ are two edges having similar lengths, $\gamma\delta$ in $Circ(m_1 \star m_2;\ Y)$ is $r(\gamma\delta) = min\{ij, kl : \{xy + ij, zt + ij\} = E_2, \{op + kl, qr + kl\} = E_1\}$, where additions and differences for x, z, o, and q are calculated modulo m_1 and for y, t, p, and r are calculated modulo m_2. The rotation-distance for the two adjacent edges with the same length $\gamma\delta$ is $\gamma\delta$.

2. The cartesian product and the CODCs of circulant graphs

For the subgraph B of $Circ(n_1 \star n_2;\ Y)$, a one-to-one function $\Phi : V(B) \longrightarrow \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ is an orthogonal Y-labelling of B if the graph B verifies some conditions according to one of the subcases appeared through the proof of Theorem 2.1. In this paper we use only the subcases 2.1 and 4.1 of Theorem 3, so we introduce the proof of these subcases only as follows.

Theorem 2.1. *(El-Shanawany and El-Mesady (2014)). A CODC of* $Circ(n_1 \star n_2; Y)$ *by a graph* B *exists iff there is an orthogonal* Y*-labelling of* B.

Proof. Case 2. Let $n_1 > 1$ be odd and n_2 be even.
Subcase 2.1. For $n_2 > 2$, we find that:
(a) For every $\alpha\beta \in X_1$:

$$X_1 = \begin{cases} \alpha 0 & : \quad 1 \le \alpha \le \lfloor \frac{n_1}{2} \rfloor, \\ \alpha\beta & : \quad 0 \le \alpha \le \frac{1}{2}(n_1 - 1), 1 \le \beta \le \frac{n_2}{2} - 1, \\ n_1 n_2 - \alpha\beta & : \quad \frac{1}{2}(n_1 - 1) < \alpha < n_1, 1 \le \beta \le \frac{n_2}{2} - 1, \\ \alpha \frac{n_2}{2} & : \quad 1 \le \alpha \le \lfloor \frac{n_1}{2} \rfloor. \end{cases}$$

G contains exactly two edges of length $\alpha\beta$, and exactly one edge of length $X_2 = \{0\frac{n_2}{2}\}$,
(b) $\{\ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1,$ then $X = X_1 \cup X_2$.
Case 4. Let n_1 and n_2 be even.
Subcase 4.1. For $n_1 > 2$ and $n_2 > 2$, we find that:
(a) For every $\alpha\beta \in X_1$: $X_1 = \begin{cases} \alpha\beta & : \quad \alpha \in \{0, \frac{n_1}{2}\}, 1 \le \beta \le \frac{n_2}{2} - 1, \\ \alpha\beta & : \quad 1 \le \alpha \le \frac{n_1}{2} - 1, \beta \in \mathbb{Z}_{n_2}. \end{cases}$
G contains exactly two edges of length $\alpha\beta$, and exactly one edge of length $X_2 = \left\{\frac{n_1}{2}0, 0\frac{n_2}{2}, \frac{n_1}{2}\frac{n_2}{2}\right\}$,
(b) $\{\ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1$, then $X = X_1 \cup X_2$. □

Note: In Theorems 2.2 and 2.3, we have considered the circulant graphs as complete graphs.

Let m be a positive integer, $\gcd(m, 3) = 1$, $n \ge 2$, $k = l + m$, and $l \in \mathbb{Z}_m$. Then we consider the graph $H_1^{m,n}$ to be the graph with the edge set: $E(H_1^{m,n}) = \{(0l, 0l + ij) : ij \in Y_1 \backslash \{00\}\} \cup \{(0k, 0k + ij) : ij \in Y_2\} \cup \{(\delta\gamma, nw) : \delta \in \mathbb{Z}_{2n} \backslash \{0, n\}, \gamma \in \{2l, 2(l+m)\}, w \in \{l, (l+m)\}\}$, where $Y_1 = A_1 \times A_2$, $Y_2 = A_1 \times A_3$, $A_1 = \{0, n\}$, $A_2 = \{l, l + 2m\}$, and $A_3 = \{l + m, l + 3m\}$. It is easy to prove that, $|E(H_1^{m,n})| = 8mn - 1$ and $|V(H_1^{m,n})| = 2m(2n + 1)$.

Theorem 2.2. *Let* $n \ge 2$ *and* m *be positive integers and* $\gcd(m, 3) = 1$. *Then there is a CODC of* $Circ(8nm; X)$ *by* $H_1^{m,n}$ *w.r.t.* $\mathbb{Z}_{2n} \times \mathbb{Z}_{4m}$.

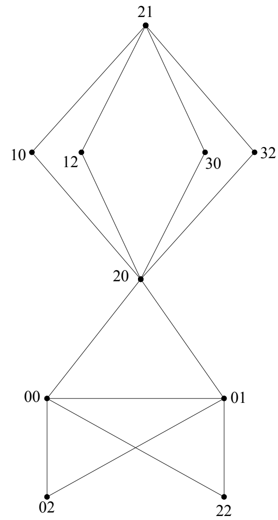

Figure 2: CODC generating graph of $Circ(16;\ X)$ by $H_1^{1,2}$ w.r.t. $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Proof. We define $\Phi : V(H_1^{m,n}) \longrightarrow \mathbb{Z}_{2n} \times \mathbb{Z}_{4m}$ by

$$\Phi(V_\alpha) = \begin{cases} 0\alpha & : \quad 0 \le \alpha \le 2m-1, \\ 0\gamma & : \quad 2m \le \alpha \le 3m-1, \gamma = 2(\alpha - m), \\ n\beta & : \quad 3m \le \alpha \le 5m-1, \beta = \alpha - 3m, \\ 2\delta & : \quad 5m \le \alpha \le 6m-1, \delta = 2(\alpha - 4m), \\ w\gamma & : \quad \alpha = i + (2w+4)m, 1 \le w \le n-1, \gamma = 2i, 0 \le i \le 2m-1, \\ xy & : \quad \alpha = i + (2x+4)m - 2m, n+1 \le x \le 2n-1, y = 2i, 0 \le i \le 2m-1. \end{cases}$$

It is clear that $H_1^{m,n}$, and from Subcase 4.1 of Theorem 2.1,
(i) If $\alpha\beta \in X_1$;

$$X_1 = \begin{cases} \alpha\beta & : \quad \alpha \in \{0, n\}, 1 \le \beta \le 2m-1, \\ \alpha\beta & : \quad 1 \le \alpha \le n-1, \beta \in \mathbb{Z}_{4m}. \end{cases}$$

We found that, the length $\alpha\beta$ is repeated twice in $H_1^{m,n}$, and there is only one edge of length $X_2 = \{n0, 0\gamma, n\gamma : \gamma = 2m\}$,
(ii) $\{\ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1$, then $X = X_1 \cup X_2$. $\square$

Theorem 2.2 can be illustrated by the following example, let $n = 2, m = 1$. Then there is a CODC of $Circ(16;\ X)$ by $H_1^{1,2}$ w.r.t. $\mathbb{Z}_4 \times \mathbb{Z}_4$, where $X = \{01, 02, 10, 11, 12, 13, 20, 21, 22\}$(see Figure 2).

Let $n \geq 2$ and $m > 1$ be positive integers and m be odd, then suppose that $H_2^{m,n}$ is a graph with the edge set:

$E\left(H_2^{m,n}\right) = \{(00, \gamma 0) : 1 \leq \gamma \leq m-1\} \cup \{(0n, \gamma 0) : 0 \leq \gamma \leq m-1\} \cup \{(01, \delta w) :$

$0 \leq \delta \leq m-1, 2 \leq w \leq n\} \cup \{(0k, \delta w) : k = n+1, 0 \leq \delta \leq m-1, 2 \leq w \leq n\}.$

It is easy to prove that,

$$|E\left(H_2^{m,n}\right)| = 1 + 2(2\lfloor \tfrac{m}{2} \rfloor + m(n-1)) \text{ and } |V\left(H_2^{m,n}\right)| = 2m + n.$$

Theorem 2.3. *Let $n \geq 2$ and $m > 1$ be positive integers and m be odd, then there is a CODC of $Circ(2mn; X)$ by $H_2^{m,n}$ w.r.t. $\mathbb{Z}_m \times \mathbb{Z}_{2n}$.*

Proof. We define $\Phi : V(H_2^{m,n}) \longrightarrow \mathbb{Z}_m \times \mathbb{Z}_{2n}$ by

$$\Phi(V_\alpha) = \begin{cases} 0\alpha & : \quad 0 \leq \alpha \leq n+1, \\ \beta 0 & : \quad \alpha = \beta n + 2, 1 \leq \beta \leq m-1, \\ \gamma\delta & : \quad \gamma n + 3 \leq \alpha \leq n(\gamma+1) + 1, \delta = \alpha - (\gamma n + 1), 1 \leq \gamma \leq m-1. \end{cases}$$

It is clear that $H_2^{m,n}$, and from Subcase 2.1 of Theorem 2.1,

(i) If $\alpha\beta \in X_1$; $X_1 = \begin{cases} \alpha\beta & : \quad 1 \leq \alpha \leq \lfloor \frac{m}{2} \rfloor, \beta = \{0, n\}, \\ \alpha\beta & : \quad 0 \leq \alpha \leq \frac{1}{2}(m-1), 1 \leq \beta \leq n-1, \\ m\gamma - \alpha\beta & : \quad \frac{1}{2}(m+1) \leq \alpha \leq m-1, 1 \leq \beta \leq n-1, \ \gamma = 2n. \end{cases}$

Then, we found that, the length $\alpha\beta$ is repeated twice in $H_2^{m,n}$, and there is only one edge of length $X_2 = \{0n\}$, (ii) $\{ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1$, then $X = X_1 \cup X_2$. □

Theorem 2.3 can be illustrated by the following example, let $m = 3, n = 2$, then there is a CODC of $Circ(12;\ X)$ by $H_2^{3,2} \cong K_{3,2} \cup^{02} K_{1,2,1}$ w.r.t. $\mathbb{Z}_3 \times \mathbb{Z}_4$,

where $X = \{01, 02, 10, 11, 12, 13\}$ and $K_{3,2} \cup^{02} K_{1,2,1}$ means that the two graphs $K_{3,2}$ and $K_{1,2,1}$ share the vertex 02(see Figure 3).

Note: In Theorems 2.4, 2.5, 2.6, and 2.7, we have constructed an ODC of circulant graphs, where $X_2 = \emptyset, X = X_1$, and this can be proved from the edge set in each Theorem.

Let m be a positive integer, $\gcd(m, 3) = 1$, $k = l + m$, and $l \in \mathbb{Z}_m$.

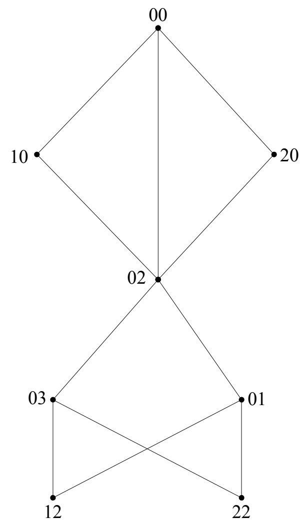

Figure 3: CODC generating graph of $Circ(12;\ X)$ by $H_2^{3,2} \cong K_{3,2} \cup^{02} K_{1,2,1}$ w.r.t. $\mathbb{Z}_3 \times \mathbb{Z}_4$.

Then H_3^m is the graph with the edge set:

$E\left(H_3^m\right) = \{(0l, 0l + ij) : ij \in Y_1 \backslash \{00, 0\gamma, 20, 2\gamma : \gamma = 2m\}\} \cup \{(0k, 0k + ij) : ij \in Y_2\} \cup \{(1l, 1l + ij) : ij \in Y_3\} \cup \{(1k, 1k + ij) : ij \in Y_4\}; Y_1 = A_1 \times A_2,\ Y_2 = A_1 \times A_4, Y_3 = A_3 \times A_2, Y_4 = A_3 \times A_4, A_1 = \{0, 2\},\ A_2 = \{l, l + 2m\},\ A_3 = \{1, 3\},\ A_4 = \{l + m, l + 3m\}$.

It is easy to prove that, $|E\left(H_3^m\right)| = 4(4m - 1)$ and $|V\left(H_3^m\right)| = 7m$.

Theorem 2.4. *Let m be a positive integer and $\gcd(m, 3) = 1$, then there is a CODC of $Circ(16m; X)$ by H_3^m w.r.t. $\mathbb{Z}_4 \times \mathbb{Z}_{4m}$.*

Proof. We define $\Phi : V(H_3^m) \longrightarrow \mathbb{Z}_4 \times \mathbb{Z}_{4m}$ by

$$\Phi(V_\alpha) = \begin{cases} 0\alpha & : \quad 0 \leq \alpha \leq 2m - 1, \\ 0\delta & : \quad 2m \leq \alpha \leq 3m - 1, \delta = 2(\alpha - m), \\ 1\gamma & : \quad 3m \leq \alpha \leq 5m - 1, \gamma = \alpha - 3m, \\ 2w & : \quad 5m \leq \alpha \leq 7m - 1, w = 2(\alpha - 5m). \end{cases}$$

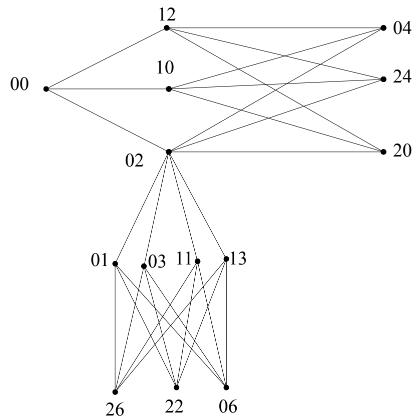

Figure 4: CODC generating graph of $Circ(32;\ X)$ by $H_3^2 \cong K_{3,4} \cup^{02} K_{4,4}$ w.r.t. $\mathbb{Z}_4 \times \mathbb{Z}_8$.

It is clear that H_3^m, and from Subcase 2.1 of Theorem 2.1,

(i) If $\alpha\beta \in X_1$;

$$X_1 = \begin{cases} \alpha\beta & : \quad \alpha \in \{0,2\}, 1 \leq \beta \leq 2m-1, \\ \alpha\beta & : \qquad \alpha = 1, \beta \in \mathbb{Z}_{4m}. \end{cases}$$

Then, we found that, the length $\alpha\beta$ is repeated twice in H_3^m and $X_2 = \emptyset$,
(ii) $\{\ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1$, then $X = X_1$. □

Theorem 2.4 can be illustrated by the following example, let $m = 2$, then there is a CODC of $Circ(32;\ X)$ by $H_3^2 \cong K_{3,4} \cup^{02} K_{4,4}$ w.r.t. $\mathbb{Z}_4 \times \mathbb{Z}_8$, where $X = \{01, 02, 03, 21, 22, 23, 10, 11, 12, 13, 14, 15, 16, 17\}$ and $K_{3,4} \cup^{02} K_{4,4}$ means that the two graphs $K_{3,4}$ and $K_{4,4}$ share the vertex 02(see Figure 4).

Let $n > 1$, $m > 1$ be positive integers and m be odd. Then $K_{1,2m-2} \cup K_{1,2m(n-1)}$ is the graph with the edge set: $E\left(K_{1,2m-2} \cup K_{1,2m(n-1)}\right)$ $= \{(0n, \delta\gamma) : \delta \in \mathbb{Z}_m, \gamma \in \mathbb{Z}_{2n} \setminus \{0, n\}\} \cup \{(00, \delta\gamma) : \delta \in \mathbb{Z}_m \setminus \{0\}, \gamma \in \{0, n\}\}$. It is easy to prove that, $\left|E\left(K_{1,2m-2} \cup K_{1,2m(n-1)}\right)\right| = 4\lfloor \frac{m}{2} \rfloor + 2m(n-1)$ and $\left|V\left(K_{1,2m-2} \cup K_{1,2m(n-1)}\right)\right| = 2mn$.

Theorem 2.5. *Let $n > 1$, $m > 1$ be positive integers and m be odd, then there is a CODC of $Circ(2mn; X)$ by $K_{1,2m-2} \cup K_{1,2m(n-1)}$ w.r.t. $\mathbb{Z}_m \times \mathbb{Z}_{2n}$.*

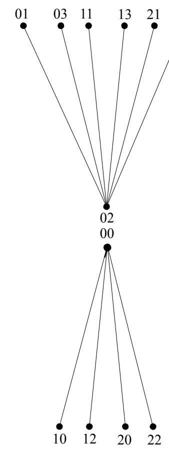

Figure 5: CODC generating graph of $Circ(12;\ X)$ by $K_{1,4} \cup K_{1,6}$ w.r.t. $\mathbb{Z}_3 \times \mathbb{Z}_4$.

Proof. We define $\Phi : V(K_{1,2m-2} \cup K_{1,2m(n-1)}) \longrightarrow \mathbb{Z}_m \times \mathbb{Z}_{2n}$ by

$$\Phi(V_\alpha) = \quad \beta i : \quad \alpha = i + 2\beta n, 0 \leq \beta \leq m-1, 0 \leq i \leq 2n-1.$$

It is clear that $K_{1,2m-2} \cup K_{1,2m(n-1)}$, and from Subcase 2.1 of Theorem 2.1, (i) If $\alpha\beta \in X_1$;

$$X_1 = \begin{cases} \alpha\beta & : \quad 1 \leq \alpha \leq \lfloor \frac{m}{2} \rfloor,\ \beta = \{0, n\} \\ \alpha\beta & : \quad 0 \leq \alpha \leq \frac{1}{2}(m-1), 1 \leq \beta \leq n-1, \\ m\gamma - \alpha\beta & : \quad \frac{1}{2}(m+1) \leq \alpha \leq m-1, 1 \leq \beta \leq n-1,\ \gamma = 2n. \end{cases}$$

Thus, the length $\alpha\beta$ is repeated twice in $K_{1,2m-2} \cup K_{1,2m(n-1)}$ and $X_2 = \emptyset$, (ii)$\{r(\alpha\beta) : \alpha\beta \in X_1\} = X_1$, then $X = X_1$. ☐

Theorem 2.5 can be illustrated from the following example, let $m = 3$, $n = 2$, then there is a CODC of $Circ(12; X)$ by $K_{1,4} \cup K_{1,6}$ w.r.t. $\mathbb{Z}_3 \times \mathbb{Z}_4$, where $X = \{01, 10, 11, 12, 13\}$ (see Figure 5).

Let $m > 1$ and $n > 1$ be positive integers. Then $K_{1,4(n-1)} \cup K_{1,4(m-1)} \cup K_{1,4(m-1)(n-1)}$ is the graph with the edge set: $E(K_{1,4(n-1)} \cup K_{1,4(m-1)} \cup K_{1,4(m-1)(n-1)}) = \{(0n, \delta\gamma) : \delta \in \{0, m\}, \gamma \in \mathbb{Z}_{2n} \backslash \{0, n\}\} \cup \{(m0, \delta\gamma) : \delta \in \mathbb{Z}_{2m} \backslash \{0, m\}, \gamma \in \{0, n\}\} \cup \{(mn, \delta\gamma) : \delta \in \mathbb{Z}_{2m} \backslash \{0, m\}, \gamma \in \mathbb{Z}_{2n} \backslash \{0, n\}\}$, it is easy to prove that,

$$\left| E(K_{1,4(n-1)} \cup K_{1,4(m-1)} \cup K_{1,4(m-1)(n-1)}) \right| = 4(mn-1)$$

and

$$\left|V(K_{1,4(n-1)} \cup K_{1,4(m-1)} \cup K_{1,4(m-1)(n-1)})\right| = 4mn - 1.$$

Theorem 2.6. *Let $m > 1$ and $n > 1$ be positive integers, then there is a CODC of $Circ(4mn; X)$ by $K_{1,4(n-1)} \cup K_{1,4(m-1)} \cup K_{1,4(m-1)(n-1)}$ w.r.t. $\mathbb{Z}_{2m} \times \mathbb{Z}_{2n}$.*

Proof. We define $\Phi : V(K_{1,4(n-1)} \cup K_{1,4(m-1)} \cup K_{1,4(m-1)(n-1)}) \longrightarrow \mathbb{Z}_{2m} \times \mathbb{Z}_{2n}$ by

$$\Phi(V_\alpha) = \begin{cases} 0w & : \quad 0 \le \alpha \le 2n-2, w = (\alpha+1), \\ \beta i & : \quad \alpha = i + 2\beta n - 1, 1 \le \beta \le 2m-1, 0 \le i \le 2n-1. \end{cases}$$

It is clear that $K_{1,4(n-1)} \cup K_{1,4(m-1)} \cup K_{1,4(m-1)(n-1)}$, and from Subcase 4.1 of Theorem 2.1,

(i) If $\alpha\beta \in X_1$;

$$X_1 = \begin{cases} \alpha\beta & : \quad \alpha \in \{0, m\}, 1 \le \beta \le n-1, \\ \alpha\beta & : \quad 1 \le \alpha \le m-1, \beta \in \mathbb{Z}_{2n}. \end{cases}$$

Then, we found that, the length $\alpha\beta$ is repeated twice in $K_{1,4(n-1)} \cup K_{1,4(m-1)} \cup K_{1,4(m-1)(n-1)}$ and $X_2 = \emptyset$,

(ii) $\{ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1$, then $X = X_1$. □

Theorem 2.6 can be illustrated by the following example, let $m = 3,\ n = 2$, then there is a CODC of $Circ(24;\ X)$ by $K_{1,4} \cup K_{1,8} \cup K_{1,8}$ w.r.t. $\mathbb{Z}_6 \times \mathbb{Z}_4$, where $X = \{01, 10, 11, 12, 13, 20, 21, 22, 23, 31\}$(see Figure 6).

Let m be a positive integer, $\gcd(m, 3) = 1,\ n \ge 2,\ k = l + m,$ and $l \in \mathbb{Z}_m$. Then $H_4^{m,n}$ is the graph with the edge set:

$E\,(H_4^{m,n}) = \{(0l, 0l + ij) : ij \in Y_1 \backslash \{00, n0, 0\sigma, n\sigma : \sigma = 2m\}\} \cup \{(0k, 0k + ij) : ij \in Y_2\} \cup \{(nw, \delta\gamma) : \delta \in \mathbb{Z}_{2n} \backslash \{0, n\}, \gamma \in \{2l, 2(l+m)\}, w \in \{l, (l+m)\}\}$, where $Y_1 = A_1 \times A_2,\ Y_2 = A_1 \times A_3, A_1 = \{0, n\},\ A_2 = \{l, l + 2m\},$ and $A_3 = \{l + m, l + 3m\}$.

It is easy to prove that, $|E\,(H_4^{m,n})| = 4(mn-1)$ and $|V\,(H_4^{m,n})| = 2m(2n+1)$.

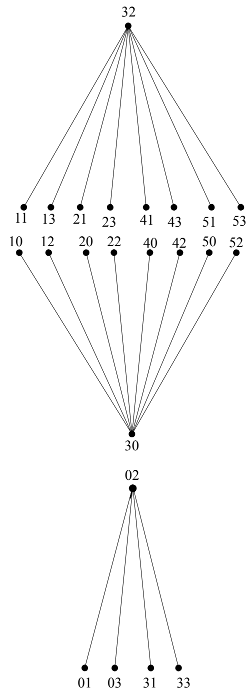

Figure 6: CODC generating graph of $Circ(24;\ X)$ by $K_{1,4} \cup K_{1,8} \cup K_{1,8}$ w.r.t. $\mathbb{Z}_6 \times \mathbb{Z}_4$.

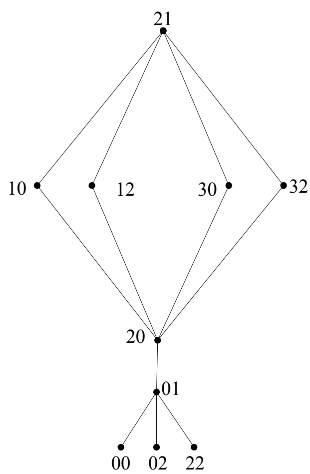

Figure 7: CODC generating graph of $Circ(16;\ X)$ by $H_4^{1,2} \cong K_{2,4} \cup^{20} K_{1,4}$ w.r.t. $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Theorem 2.7. *Let m and $n \geq 2$ be positive integers with* $\gcd(m, 3) = 1$. *Then there is a CODC of $Circ(8nm; X)$ by $H_4^{m,n}$ w.r.t. $\mathbb{Z}_{2n} \times \mathbb{Z}_{4m}$.*

Proof. We define $\Phi : V(H_4^{m,n}) \longrightarrow \mathbb{Z}_{2n} \times \mathbb{Z}_{4m}$ by

$$\Phi(V_\alpha) = \begin{cases} 0\alpha & : \quad 0 \leq \alpha \leq 2m-1, \\ 0\gamma & : \quad 2m \leq \alpha \leq 3m-1, \gamma = 2(\alpha - m), \\ n\beta & : \quad 3m \leq \alpha \leq 5m-1, \beta = \alpha - 3m, \\ 2\delta & : \quad 5m \leq \alpha \leq 6m-1, \delta = 2(\alpha - 4m), \\ w\gamma & : \quad \alpha = i + (2w+4)m, 1 \leq w \leq n-1, \gamma = 2i, 0 \leq i \leq 2m-1, \\ xy & : \quad \alpha = i + (2x+4)m - 2m, n+1 \leq x \leq 2n-1, y = 2i, 0 \leq i \leq 2m-1. \end{cases}$$

It is clear that $H_4^{m,n}$, and from Subcase 4.1 of theorem 2.1,

(i) If $\alpha\beta \in X_1; X_1 = \begin{cases} \alpha\beta & : \quad \alpha \in \{0, n\}, 1 \leq \beta \leq 2m-1, \\ \alpha\beta & : \quad 1 \leq \alpha \leq n-1, \beta \in \mathbb{Z}_{4m}. \end{cases}$

Then, we found that, the length $\alpha\beta$ is repeated twice in $H_4^{m,n}$ and $X_2 = \emptyset$,

(ii) $\{\ r(\alpha\beta) : \alpha\beta \in X_1\} = X_1$, then $X = X_1$. □

Theorem 2.7 can be illustrated by the following example, let $n = 2$ and $m = 1$, then there is a CODC of $Circ(16;\ X)$ by $H_4^{1,2} \cong K_{2,4} \cup^{20} K_{1,4}$ w.r.t. $\mathbb{Z}_4 \times \mathbb{Z}_4$, where $X = \{01, 21, 10, 11, 12, 13\}$, $K_{2,4} \cup^{20} K_{1,4}$ means that the two graphs $K_{2,4}$ and $K_{1,4}$ share the vertex 20(see Figure 7).

3. Conclusion

In conclusion, we got some new results for the CODCs by new graphs, where the helping tool is the cartesian product of the Abelian groups.

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