Malaysian Journal of Mathematical Sciences 12(2): 223–234 (2018)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

A Characterization of Non-Lightlike Curves With Respect to Parallel Transport Frame in Minkowski Space-Time

Büyükkütük, S. *[1], Kişi, I.[1], and Öztürk, G.[2]

[1] *Kocaeli University, Department of Mathematics, Turkey*
[2] *Izmir Democracy University, Department of Mathematics, Turkey*

E-mail: sezgin.buyukkutuk@kocaeli.edu.tr
* *Corresponding author*

Received: 24 May 2017
Accepted: 26 May 2018

ABSTRACT

In this paper, we characterize non-ligtlike curves ($x(s) = m_0(s)T(s) + m_1(s)M_1(s) + m_2(s)M_2(s) + m_3(s)M_3(s)$) in terms of their curvature functions $m_i(s)$ ($i = 0, 1, 2, 3$) with the help of parallel transport frame $\{T, M_1, M_2, M_3\}$. We obtain a result about spacelike constant ratio curves. We give the necessary and sufficient conditions for such curves to be T-constant and N-constant of first and second kind. Also, we give the necessary and sufficient conditions for T-constant curves to lie on a Lorentzian sphere.

Keywords: Parallel transport frame, position vector, constant ratio curves.

1. Introduction

Frenet frame is very famous in differential geometry and used to represent the geometric features of the curves, and so the kinematic features of objects that move through the differentiable curves. Thus, this frame is useful especially in mechanical designs and robotics also in medicine, computer graphics, and physics etc... Despite being useful, this frame cannot be constructed at the points where the curvature of the curve vanishes. In 1975, Bishop defined a new frame by the aid of parallel vector fields which is known as Bishop frame. This frame is well defined even if the second derivative of the curve vanishes in $3-$dimensional Euclidean space, see Bishop (1975), Hanson and Ma (1995). Afterwards, using the similar idea, authors give the alternative frames for curves in spaces $\mathbb{E}^4$ and $\mathbb{E}_1^4$ in Erdoğdu (2015), Gökçelik et al. (2014). In addition to many applications of the Bishop frame, some applications of Minkowski 3-space in modeling are also available. These applications are related with Minkowski Pythagorean hodograph curves and spacelike Bezier curves, see Jüttler and Mäurer (1999), Ugail et al. (2011).

The notion of constant ratio curves in Minkowski spaces is given by B. Y. Chen (2003a). In the same paper, the author gives the necessary and sufficient conditions, $x^T = 0$ or the ratio $\|x\| : \|x^T\|$ is constant, for curves to become constant ratio. Moreover, in Chen (2002), the author introduces *T-constant* and *N-constant* types of curves. If the norm of the tangential component (normal component) is constant, the curve is called as *T-constant* (*N-constant*). Also, if this norm is equal to zero, then the curve is a *T-constant* (*N-constant*) curve of first kind, otherwise second kind Gurpinar et al. (2015). Recently, the authors study with the constant ratio curves in Büyükkütük et al. (2016, 2017), Büyükkütük and Öztürk (2015a,b), Gurpinar et al. (2015), Kişi et al. (2017), Kişi and Öztürk (2015), Öztürk et al. (2017a,b,c).

In this paper, we deal with a non-lightlike curve in Minkowski space-time with respect to the its parallel transport frame $\{T, M_1, M_2, M_3\}$. Since $\{T, M_1, M_2, M_3\}$ is an orthonormal basis in $\mathbb{E}_1^4$, we write the position vector of the curve as

$$x(s) = m_0(s)T(s) + m_1(s)M_1(s) + m_2(s)M_2(s) + m_3(s)M_3(s) \qquad (1)$$

for some differentiable functions $m_i(s)$ $(i = 0, 1, 2, 3)$ called curvature functions of the curve x.

2. Basic Concepts

Minkowski 4-space is 4-dimensional pseudo-Euclidean space defined by the Lorentzian inner product

$$\langle v, w\rangle_{\mathbb{L}} = -v_1w_1 + v_2w_2 + v_3w_3 + v_4w_4, \tag{2}$$

where v_i, w_i, (i=1,2,3,4) are the components of the vectors v and w. Any arbitrary vector v is called timelike, lightlike or spacelike if the Lorentzian inner product $\langle v, v\rangle_{\mathbb{L}}$ is negative definite, zero or positive definite, respectively. Then, the length of the vector $v \in \mathbb{E}_1^4$ is calculated by

$$\|v\| = \sqrt{|\langle v, v\rangle_{\mathbb{L}}|}. \tag{3}$$

A curve $x = x(s) : I \to \mathbb{E}_1^4$ is timelike (lightlike (null), spacelike) if all tangent vectors $x'(s)$ are timelike (lightlike (null), spacelike). If $\|x'(s)\| = 1$, x is a unit speed curve, see O'Neill (1983). For a unit speed timelike curve $x(s)$, the parallel transport frame formulas are given as

$$\frac{d}{ds}\begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ k_1 & 0 & 0 & 0 \\ k_2 & 0 & 0 & 0 \\ k_3 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}, \tag{4}$$

where $\langle T, T\rangle_{\mathbb{L}} = -1$, $\langle M_i, M_i\rangle_{\mathbb{L}} = 1$ and k_i $(i = 1, 2, 3)$ are the principle curvature functions according to parallel transport frame of x.

Also, for a unit speed spacelike curve $x(s)$ in $\mathbb{E}_1^4$, the parallel transport formulas are given as

$$\frac{d}{ds}\begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 & k_3 \\ -\varepsilon_1 k_1 & 0 & 0 & 0 \\ -\varepsilon_2 k_2 & 0 & 0 & 0 \\ -\varepsilon_3 k_3 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} T \\ M_1 \\ M_2 \\ M_3 \end{bmatrix}, \tag{5}$$

where $\langle T, T\rangle_{\mathbb{L}} = 1$,$\langle M_i, M_i\rangle_{\mathbb{L}} = \varepsilon_i$ and k_i $(i = 1, 2, 3)$ are the principle curvature functions according to parallel transport frame. Here, for the timelike vector M_i, $\varepsilon_i = -1$; otherwise $\varepsilon_i = 1$ (see Erdoğdu (2015)).

3. A Characterization of Non-Lightlike Curves With Respect to Parallel Transport Frame in Minkowski Space-Time

In this section, we classify the non-lightlike curves in $\mathbb{E}_1^4$ with regards to their curvatures. Let $x = x(s) : I \to \mathbb{E}_1^4$ be a unit speed non-lightlike curve with the principle curvatures k_1, k_2 and k_3. The curve x satisfies the equation (1) for some differentiable functions $m_i(s)$ $(i = 0, 1, 2, 3)$. For the timelike curve x, differentiating (1) with regards to s and using the equalities (4), we get

$$\begin{aligned} x'(s) &= (m_0'(s) + k_1(s)m_1(s) + k_2(s)m_2(s) + k_3(s)m_3(s))T(s) \\ &\quad +(m_1'(s) + k_1(s)m_0(s))M_1(s) \\ &\quad +(m_2'(s) + k_2(s)m_0(s))M_2(s) \\ &\quad +(m_3'(s) + k_3(s)m_0(s))M_3(s). \end{aligned} \tag{6}$$

It follows that

$$\begin{aligned} m_0' + k_1m_1 + k_2m_2 + k_3m_3 &= 1, \\ m_1' + k_1m_0 &= 0, \\ m_2' + k_2m_0 &= 0, \\ m_3' + k_3m_0 &= 0. \end{aligned} \tag{7}$$

For the spacelike curve x, differentiating (1) with regards to s and using the parallel transport frame equations (5), we get

$$\begin{aligned} x'(s) &= (m_0'(s) - \varepsilon_1k_1(s)m_1(s) - \varepsilon_2k_2(s)m_2(s) - \varepsilon_3k_3(s)m_3(s))T(s) \\ &\quad +(m_1'(s) + k_1(s)m_0(s))M_1(s) \\ &\quad +(m_2'(s) + k_2(s)m_0(s))M_2(s) \\ &\quad +(m_3'(s) + k_3(s)m_0(s))M_3(s). \end{aligned} \tag{8}$$

It follows that

$$\begin{aligned} m_0' - \varepsilon_1k_1m_1 - \varepsilon_2k_2m_2 - \varepsilon_3k_3m_3 &= 1, \\ m_1' + k_1m_0 &= 0, \\ m_2' + k_2m_0 &= 0, \\ m_3' + k_3m_0 &= 0. \end{aligned} \tag{9}$$

3.1 Constant Ratio Curves

Definition 3.1. *Let $x : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a unit speed curve in $\mathbb{E}_1^4$. The position vector is separated into normal and tangential components:*

$$x = x^N + x^T.$$

If $\|x^T\| : \|x^N\|$ is constant at each point, then x is a constant ratio curve (see Chen (2003a)).

Let $\rho = \|x(s)\|$ is the distance function. Then the gradient is defined as

$$grad\rho = \frac{d\rho}{ds}T(s) = \frac{< x(s), T(s) >}{\|x(s)\|}T(s) \tag{10}$$

for the tangent vector field T.

Lemma 3.1. *Chen (2003b) Let $x : I \subset \mathbb{R} \to \mathbb{E}_t^n$ be a unit speed spacelike curve in $\mathbb{E}_t^n$ with index t. Then $\|grad\rho\| = c =$ constant if and only if we have $\|x(s)\| = cs$ with a convenient translation.*

The following proposition classifies constant-ratio spacelike curves with respect to parallel transport frame in $\mathbb{E}_1^4$.

Proposition 3.1. *Let $x : I \to \mathbb{E}_1^4$ be a unit speed spacelike curve in $\mathbb{E}_1^4$ with Bishop curvatures k_1, k_2 and k_3. Then, x is a constant ratio spacelike curve if and only if*

$$\sum_{i=1}^{3}\left(\varepsilon_i k_i(s)\int (c^2 s)k_i(s)ds\right) = 1 - c^2. \tag{11}$$

Proof. Assume that x is a constant ratio spacelike curve $\mathbb{E}_1^4$. Then, from Lemma 3.1 and for a real constant c, $\rho = \|x(s)\| = cs$. Thus, by the use of (10), we obtain

$$\|grad\rho\| = \frac{< x(s), x'(s) >}{\|x(s)\|} = c.$$

Also, using the the equality (1), we get $m_0 = c^2 s$. Hence, substituting this value into (9) for the spacelike curve x, we get

$$\begin{aligned} c^2-1 &= \varepsilon_1 k_1 m_1 + \varepsilon_2 k_2 m_2 + \varepsilon_3 k_3 m_3, \\ m_1 &= -\int (c^2 s) k_1(s) ds, \\ m_2 &= -\int (c^2 s) k_2(s) ds, \\ m_3 &= -\int (c^2 s) k_3(s) ds. \end{aligned} \tag{12}$$

Consequently, we yield the expected result. The converse statement is trivial. □

3.2 T-Constant Curves

Definition 3.2. *Let $x : I \to \mathbb{E}_1^4$ be a unit speed non-lightlike curve in $\mathbb{E}_t^n$. If $\|x^T\|$ is constant, then the curve x is a T-constant curve. Further, a T-constant curve x is called first kind if $\|x^T\| = 0$, otherwise second kind (see Gurpinar et al. (2015)).*

Theorem 3.1. *Erdoğdu (2015) Let $x : I \to \mathbb{E}_1^4$ be a unit speed timelike curve with nonzero principle curvatures $k_i(s)$ $(0 < i \leq 3)$ with respect to parallel transport frame. Then, the necessary and sufficient condition for x to lie on a Lorentzian sphere is*

$$ak_1(s) + bk_2(s) + ck_3(s) = 1 \tag{13}$$

for nonzero constants $a, b,$ and c.

Theorem 3.2. *Erdoğdu (2015) Let $x : I \to \mathbb{E}_1^4$ be a unit speed spacelike curve with nonzero principle curvatures $k_i(s)$ $(1, 2, 3)$ with respect to parallel transport frame. Then, the necessary and sufficient condition for x to lie on a Lorentzian sphere is*

$$a\varepsilon_1 k_1(s) + b\varepsilon_2 k_2(s) + c\varepsilon_3 k_3(s) = -1 \tag{14}$$

for nonzero constants $a, b,$ and c.

Corollary 3.1. *Let $x : I \to \mathbb{E}_1^4$ be a unit speed non-lightlike curve given with the parametrization (1). Then the necessary and sufficient condition for x to become a T-constant curve of first kind is to lie on a Lorentzian sphere.*

Proof. Assume that x is a T-constant non-lightlike curve of first kind. Then, using the second, the third and the fourth equalities in (7) and also in (9), we get

$m_i' = 0$, $i = 1, 2, 3$. Further, substituting $m_1 = a$, $m_2 = b$, $m_3 = c$ into the first equation, for the timelike and spacelike curve, we get $ak_1(s)+bk_2(s)+ck_3(s) = 1$ and $a\varepsilon_1 k_1(s) + b\varepsilon_2 k_2(s) + c\varepsilon_3 k_3(s) = -1$, respectively. From Theorem 3.1 and Theorem 3.2, we obtain the result. Conversely, let x lies on a Lorentzian sphere centered origin with radius r. Therefore, we can write

$$\langle x(s), x(s)\rangle = \pm r^2.$$

Differentiating this equation, we have

$$\langle T(s), x(s)\rangle = 0,$$

which means $m_0 = 0$. Thus, the curve x is a T-constant curve of first kind. □

Theorem 3.3. *Let $x : I \to \mathbb{E}_1^4$ be a unit speed non-lightlike curve in $\mathbb{E}_1^4$. Then,*

i) x is a T-constant timelike curve of second kind if and only if

$$k_1 \int k_1 ds + k_2 \int k_2 ds + k_3 \int k_3 ds = \frac{-1}{m_0}. \tag{15}$$

ii) x is a T-constant spacelike curve of second kind if and only if

$$\varepsilon_1 k_1 \int k_1 ds + \varepsilon_1 k_2 \int k_2 ds + \varepsilon_1 k_3 \int k_3 ds = \frac{1}{m_0}. \tag{16}$$

Proof. Let x be a non-lightlike curve in $\mathbb{E}_1^4$. If it is a T-constant timelike curve of second kind, from (7), we have

$$k_1 m_1 + k_2 m_2 + k_3 m_3 - 1 = 0. \tag{17}$$

If it is a T-constant spacelike curve of second kind, from (9), we get

$$\varepsilon_1 k_1 m_1 + \varepsilon_2 k_2 m_2 + \varepsilon_3 k_3 m_3 + 1 = 0. \tag{18}$$

Further, integrating the second, the third and the fourth equations in (12) and (9) also substituting them into (17) and (18), we get the result. The converse statement is trivial. □

Corollary 3.2. *Let $x : I \to \mathbb{E}_1^4$ be a unit speed non-lightlike curve in $\mathbb{E}_1^4$. Then*

i) If x is a T-constant timelike curve of second kind, for the curvature functions m_i,

$$-2m_0 s + c = m_1^2 + m_2^2 + m_3^2 \tag{19}$$

is satisfied.
ii) If x is a T-constant spacelike curve of second kind, for the curvature functions m_i,

$$2m_0 s + c = \varepsilon_1 m_1^2 + \varepsilon_2 m_2^2 + \varepsilon_3 m_3^2 \tag{20}$$

is satisfied for an integral constant c.

Proof. Assume that x is a T-constant curve of second kind. By the use of the second, the third and the fourth equations in (7) and (9), we have

$$k_1 = -\frac{m_1'}{m_0}, k_2 = -\frac{m_2'}{m_0}, k_3 = -\frac{m_3'}{m_0}.$$

Substituting these values into the first equation in (7) and (9), we obtain the differential equations

$$m_1 m_1' + m_2 m_2' + m_3 m_3' = -m_0$$

and

$$\varepsilon_1 m_1 m_1' + \varepsilon_2 m_2 m_2' + \varepsilon_3 m_3 m_3' = m_0$$

which have the solutions (19) and (20), respectively. □

Proposition 3.2. *Let $x : I \to \mathbb{E}_1^4$ be a T-constant non-lightlike curve of second kind. Then,*

$$\rho = \pm\sqrt{2\lambda s + c} \tag{21}$$

is hold. Here $\rho = \|x\|$ is the distance function, λ, c are real constants and for the timelike curve x, $\lambda = -m_0$, for the spacelike curve x, $\lambda = m_0$.

Proof. Differentiating $\rho^2 = \langle x(s), x(s)\rangle$ and by the use of (1), we obtain $\rho\rho' = -m_0$ and $\rho\rho' = m_0$ for the timelike and spacelike curve, respectively. Since, x is a T-constant curve of second kind, $m_0(s)$ is constant from the definition. Thus, we obtain the nontrivial solution (21). □

3.3 N-Constant Curves

Definition 3.3. *Let $x : I \to \mathbb{E}_1^4$ be a unit speed non-lightlike curve in $\mathbb{E}_t^n$. If $\|x^N\|$ is constant, then the curve x is a N-constant curve. Further, a N-constant curve x is called first kind if $\|x^N\| = 0$, otherwise second kind, see Gurpinar et al. (2015).*

We suppose that, x is a N-constant timelike (spacelike) curve in $\mathbb{E}_1^4$. Then, the length of the normal component satisfies the equations

$$\left\|x^N(s)\right\|^2 = m_1^2(s) + m_2^2(s) + m_3^2(s), \tag{22}$$

$$\left\|x^N(s)\right\|^2 = \varepsilon_1 m_1^2(s) + \varepsilon_2 m_2^2(s) + \varepsilon_3 m_3^2(s), \tag{23}$$

which are constant functions. Thus, by differentiating the above equations, we get

$$m_1 m_1^{'} + m_2 m_2^{'} + m_3 m_3^{'} = 0, \tag{24}$$

$$\varepsilon_1 m_1 m_1^{'} + \varepsilon_2 m_2 m_2^{'} + \varepsilon_3 m_3 m_3^{'} = 0. \tag{25}$$

Proposition 3.3. *Let $x : I \to \mathbb{E}_1^4$ be a unit speed non-lightlike curve in $\mathbb{E}_1^4$.*

i) For a N-constant timelike curve of first kind, x is congruent to a straight line.

ii) For a N-constant spacelike curve of first kind, the equation

$$\int m_0 \left(\varepsilon_1 k_1 + \varepsilon_2 k_2 + \varepsilon_3 k_3\right) ds = 0 \tag{26}$$

holds.

Proof. Assume that x is a N-constant timelike curve of first kind in $\mathbb{E}_1^4$, then the equation (22) is satisfied. Also, by the use of (22), $m_1 = m_2 = m_3 = 0$ which implies that $k_1 = k_2 = k_3 = 0$. Thus, x is congruent to a straight line. Further, assume that x is a N-constant spacelike curve of first kind in $\mathbb{E}_1^4$, $\left\|x^N\right\| = 0$, then

$$\left\|x^N(s)\right\|^2 = \varepsilon_1 m_1^2(s) + \varepsilon_2 m_2^2(s) + \varepsilon_3 m_3^2(s) = 0. \tag{27}$$

By the use of equation system (9), we get

$$\varepsilon_1 m_1 m_1' + \varepsilon_2 m_2 m_2' + \varepsilon_3 m_3 m_3' = -m_0(\varepsilon_1 k_1 + \varepsilon_2 k_2 + \varepsilon_3 k_3). \tag{28}$$

Integrating (28), we get

$$\frac{\varepsilon_1 m_1^2(s) + \varepsilon_2 m_2^2(s) + \varepsilon_3 m_3^2(s)}{2} = -\int m_0(\varepsilon_1 k_1 + \varepsilon_2 k_2 + \varepsilon_3 k_3).$$

From the equation (27) and (28), we complete the proof. □

Theorem 3.4. *Let $x : I \to \mathbb{E}_1^4$ be a unit speed non-lightlike curve in $\mathbb{E}_1^4$ with the arclength function s and the principle curvatures k_i $(i = 1, 2, 3)$. If x is a N-constant curve of second kind, then x is congruent to a T-constant curve which is given by*

$$x(s) = \lambda M_1(s) + \mu M_2(s) + \eta M_3(s), \tag{29}$$

where λ, μ, and η are real constants or the curve has the position vector as

$$\begin{aligned} x(s) &= (s+b)T(s) \\ &\quad - \left(\int k_1(s)(s+b)ds\right) M_1(s) \\ &\quad - \left(\int k_2(s)(s+b)ds\right) M_2(s) \\ &\quad - \left(\int k_3(s)(s+b)ds\right) M_3(s), \end{aligned} \tag{30}$$

where b is a real constant.

Proof. Let x be a N-constant non-lightlike curve of second kind in $\mathbb{E}_1^4$. Since the equality (24) or (25) is satisfied, we have $m_0(k_1m_1 + k_2m_2 + k_3m_3) = 0$ for the timelike curve. Also, we get $m_0(\varepsilon_1k_1m_1 + \varepsilon_2k_2m_2 + \varepsilon_3k_3m_3) = 0$ for the spacelike curve. Hence, there are two possible cases; $m_0 = 0$ or $k_1m_1 + k_2m_2 + k_3m_3 = 0$ $(\varepsilon_1k_1m_1 + \varepsilon_2k_2m_2 + \varepsilon_3k_3m_3 = 0)$. The first case with the equation (7) or (9) implies that $m_1 = \lambda = const$, $m_2 = \mu = const$, $m_3 = \eta = const$. Thus x is a T-constant curve of first kind given by (29). Furthermore, using (7) and (9), we get

$$\begin{aligned} m_0 &= s+b, \\ m_1 &= -\int k_1(s)(s+b)ds, \\ m_2 &= -\int k_2(s)(s+b)ds, \\ m_3 &= -\int k_3(s)(s+b)ds \end{aligned}$$

for the second case. □

Theorem 3.5. *Let $x : I \to \mathbb{E}_1^4$ be a N-constant non-lightlike curve of second kind. Then,*

$$\rho = \mp\sqrt{\lambda s^2 + 2\lambda bs + d} \tag{31}$$

holds where $\rho = \|x\|$ is the distance function, b, d are real constants and $\lambda \in \{-1, 0, 1\}$.

Proof. Differentiating $\rho^2 = \langle x(s), x(s)\rangle$ and by the use of (1), we obtain for the timelike curve

$$\rho\rho' = -m_0 \tag{32}$$

and for the spacelike curve

$$\rho\rho' = m_0. \tag{33}$$

Since x is a N-constant curve of second kind, the curve has the parametrization (29) or (30) with the curvature functions $m_0 = 0$ or $m_0(s) = s+b$, respectively. If $m_0(s) = s+b$, then by the use of (32) we obtain the solution (31) for $\lambda = -1$, or by the use of (33) we obtain the solution (31) for $\lambda = 1$. Similarly, if $m_0 = 0$ in (32) or (33), the solution (31) holds for $\lambda = 0$, which completes the proof. □

4. Conclusion

Constant ratio, T-constant, and N-constant curves are first defined by B.Y. Chen. In this paper, according to these definitions, we consider these types of curves with their parallel transport frames in Minkowski space-time and give some results about constant ratio, T-constant, and N-constant curves.

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