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On the Some Non Extandable Regular P_{-2} Sets

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ABSTRACT

Diophantine 3-tuples with property P_k, for$k$ an integer, are sets of n positive integers such that product of any two of them by adding$k$ is a square. In the present paper, we consider some regular P_k- triples and prove that they can not be extendible to Diophantine quadruple when $k = -2$ by using fundamental solution of Pell equations. Also, we determine several significant properties about such sets.

Keywords: P_k-Sets, Pell Equations, Fundamental Solutions, Residue Classes, Congruences, Legendre/Jacobi Symbol.

1. Introduction

The mathematician Diophantus started the problem of extendibility and characterization of P_k -sets. Many famous mathematicians obtained significant results on Diophantine m-tuples, but still some problems about Diophantine properties remain unsolved. A set of n distinct positive integers n $\{a_1, a_2, ..., a_n\}$ is called a P_k -set for any k integer if $a_i.a_j + k$ $(1 \leq i < j \leq n)$ is a perfect square when i is different from j.

Diophantine equations have central role in number theory and can be used in coding theory and cryprography. For real life applcations, Diophantine Equations are useful to solve problem of Business, network flow and so on. Firstly, $\{1, 3, 8, 120\}$ quadruple problem was considered by Fermat (1891) but Baker and Davenport (1969) proved that $\{1, 3, 8, 120\}$ quadruple is P_1 and can not be extended. Cenberci and Peke (2017) have given some P_2 triples sets which they can be nonextended. In the paper of Brown (1985), some unsecify results of Diophantine m-tuples were determined. Dujella and Jurasic (2011) gave the definition of regular triple, regular quadrule as well as other interesting problems in Diophantine m-tuples. Mohanty and Ramasamy (1984) and Kedlaya (1998) worked on P_{-1}- triples by using different methods. Tzanakis (2002) , considered elliptic curves method for solving Diophantine m-tuples problems.

The author Özer (2016a), Özer (2016b) and Özer (2017) worked on different types of Diophantine 3-tuples and got significant properties on such sets. Besides, some authors such as Gopalan et al. (2014), Grinstead Grinstead (1978), Kanagasabapathy and Ponnudurai (1975), Katayama (2000), Masser and Rickert (1996) considered the different methods for extendibility and characterization of simultaneous Diophantine equations. For further knowledge/information about Diophantine properties, we may refer to Dickson (2005), Mollin (2008) and Roberts (1992).

The aim of this paper is to prove that some regular P_{-2}- triples can not be extended P_{-2} - quadruples by using the fundamental solutions of $x^2 - dy^2 = +1$ or $x^2 - dy^2 = +4$ Pell Equations. Also, we demonstrate that P_{-2} - triples do not contain the primes satisfy $p \equiv 5(mod8)$ or $p \equiv 7(mod8)$ with other properties by considering quadratic reciprocity theorem and Legendre-Jacobi symbols. In the case k is equal -2, there does not exist any similar paper of us for Diophantine triples.

2. Preliminaries

Definition 2.1. [5] *A $D(n)$- triple$\{a,b,c\}$ is called regular if it satisfies the condition*

$$(c-b-a)^2 = 4(ab+n) \tag{1}$$

Equation (1) is symmetric under permutations of a,b,c.

Definition 2.2. [14] *If $n \in N$ and $\alpha \in Z$ with $gcd(\alpha,n)=1$, then α is to be a quadratic residue modulo n if there exists an integer x such that*

$$x^2 \equiv \alpha (modn) \tag{2}$$

and if equivalence has no such solution, then α is a quadratic nonresidue modulo n.

Definition 2.3. [14]) *If $a \in Z$ and $p > 2$ is prime, then*

$$\frac{a}{p} = \begin{cases} 0 & \textit{if } (p|a) \\ 1 & \textit{if } a \textit{ is quadratic residue mod } p \\ -1 & \textit{otherwise} \end{cases} \tag{3}$$

and $(\frac{a}{p})$ is called the Legendre Symbol of a with respect to p.

Theorem 2.1. [14] *If $p \neq q$ are odd primes, then*

$$(\frac{p}{q})(\frac{q}{p}) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}} \tag{4}$$

where $(\frac{p}{q})$ and $(\frac{q}{p})$ are Legendre symbol.

Theorem 2.2. [14] *If $u,v \in N$ are odd and relatively prime, then*

$$(\frac{u}{v})(\frac{v}{u}) = (-1)^{\frac{u-1}{2}\cdot\frac{v-1}{2}} \tag{5}$$

holds.

Theorem 2.3. [*14*] *For any odd prime p,*

$$(\frac{-1}{p}) = (-1)^{\frac{p-1}{2}}, \ (\frac{2}{p}) = (-1)^{\frac{(p^2-1)}{8}} \tag{6}$$

Definition 2.4. [*14*]) *If $a \in Z$ and $n = p_1^{s_1} p_2^{s_2} \dots p_m^{s_m} > 1$ is odd positive integer with $p_1, p_2, \dots, p_m$ primes, then*

$$(\frac{a}{n}) = (\frac{a}{p_1})^{s_1} (\frac{a}{p_2})^{s_2} \dots (\frac{a}{p_m})^{s_m} \tag{7}$$

3. Main Theorem and Results

Theorem 3.1. *A set $P_{-2} = \{2, 3, 9\}$ with size three is regular and can not be extended to further.*

Proof. By use of Definition 2.1, it is clear that $P_{-2} = \{2, 3, 9\}$ triple set is regular. Assume that $P_{-2} = \{2, 3, 9\}$ can be extended P_{-2} quadruple. Let consider the set $\{2, 3, 9, d\}$ as a P_{-2} set for any positive integer d. Then there exist x, y, z integers such that

$$2d - 2 = x^2 \tag{8}$$

$$3d - 2 = y^2 \tag{9}$$

$$9d - 2 = z^2 \tag{10}$$

By dropping d between (8) and (9), we get

$$2y^2 - 3x^2 = 2 \tag{11}$$

and from this, we obtain

$$2(y^2 - 1) = 3x^2 \tag{12}$$

It is clear that the left side of (12) is even integer. So, the right side of equation (12) must be even too. This means, there is a $x_1 \in Z$ such that $x = 2x_1$. If we put $x = 2x_1$ into the (12), we have

$$6x_1^2 + 1 = y^2 \tag{13}$$

(13) gives that y is odd integer and can be written as $y = 2y_1 + 1$ for $y_1 \in Z$. Then, (13) becomes

$$3x_1^2 = 2(y_1^2 + y_1) \tag{14}$$

this gives x_1 is even and written by $x_1 = 2x_2$ ($x_2 \in Z$).

If we consider $x = 2x_1$ and $x_1 = 2x_2$ for x_1 , $x_2 \in Z$, then we obtain $x = 4x_2$. If we write $x = 4x_2$ in the equation (12), then we have Pell equation as follows:

$$y^2 - 24x_2^2 = 1 \tag{15}$$

We determine fundamental solution of (15) demonstrated as $(y, x_2) = (5, 1)$ and other all positive solutions are generated by fundamental solution as $y_n + \sqrt{24}(x_2)_n = (5 + \sqrt{24})^n$. From the last equation, we obtain recurrence relation

$$y_n = 10y_{n-1} - y_{n-2} \tag{16}$$

for the values of (y_n) when $n \geq 3$. Considering (9) and (16), we get some values of d for any $n \in Z^+$. It is easily seen that any of these d values don't give any perfect square of integer for equation (10). i.e. There isn't any integer solution z satisfies (10).

So, $P_{-2} = \{2, 3, 9\}$ can not be extended. □

Theorem 3.2. *A $P_{-2} = \{3, 9, 22\}$ set is regular and can not be extended.*

Proof. If we consider Definition 2.1, it is easily seen that $P_{-2} = \{3, 9, 22\}$ set is regular and shares the property of P_{-2}. We will determine whether or not this set can be extendable. Let d be any other positive integer such that $\{3, 9, 22, d\}$. Then following equations hold for some x, y, z integers.

$$3d - 2 = x^2 \tag{17}$$

$$9d - 2 = y^2 \tag{18}$$

$$22d - 2 = z^2 \tag{19}$$

Eliminating d between (17) and (18), we have

$$y^2 - 3x^2 = 4 \tag{20}$$

and (20) is a Pell equation. Besides, fundamental solution of this (20) equation is found as $(y, x) = (4, 2)$. Some other solutions of (20) are given as follows:

Table 1: Some positive solutions of $y^2 - 3x^2 = 4$

Solutions	Solution 1	Solution 2	Solution 3	Solution 4	Solution ...
(y, x)	$(4, 2)$	$(14, 8)$	$(52, 30)$	$(194, 112)$	...

Using the solutions of $y^2 - 3x^2 = 4$ in the Table 1, we obtain general recurrence relation for solution of $y$as follows:

$$y_n = 4y_{n-1} - y_{n-2} \tag{21}$$

for $n > 2$. From (18) and (21), we have some values of d for any $n \in Z^+$We can easily see that such d values give no perfect square of integer for equation (19). It means that there is no integer solution z satisfies (19).

So, $P_{-2} = \{3, 9, 22\}$ is non-extendable. □

Theorem 3.3. *A $P_{-2} = \{18, 27, 89\}$ triple set is regular and non-extendible.*

Proof. Using (1) from Definition 2.1, we can easily see that $P_{-2} = \{18, 27, 89\}$ is regular triple set. Now, Suppose that $\{18, 27, 89, d\}$ is a P_{-2} set for any other positive integer d. Then, there are x, y, z integers satisfy following equations.

$$18d - 2 = x^2 \tag{22}$$
$$27d - 2 = y^2 \tag{23}$$
$$89d - 2 = z^2 \tag{24}$$

From (22) and (23), we have $2y^2 - 3x^2 = 2$ equation which is the same of (11). Using the direction of the Proof of Theorem 3.1 and following same steps from (11) to (15), we get $y^2 - 24x_2^2 = 1$ Pell Equation numbered as (15) above. As we mentioned above, we have $y_n = 10y_{n-1} - y_{n-2}$ (i.e.(16)) as the recurrence relation for the values of (y_n) and fundamental solution of (15) determine as $(y, x_2) = (5, 1)$.

Using (23) and (16), we obtain some values of d for any $n \in Z^+$. So, none of these d values give any perfect square of integer for equation (24) and this gives that there is no integer solution z satisfies (24).

That's why, a $P_{-2} = \{18, 27, 89\}$ can not be extended. □

Theorem 3.4. *A $P_{-2} = \{6, 11, 33\}$ is regular but it can not be extendable.*

Proof. It is clear that $P_{-2} = \{6, 11, 33\}$ set is regular triple set from Definition 2.1. We assume that $P_{-2} = \{6, 11, 33\}$ can be extended for any $d \in Z^+$. So, we can find x, y, z integers such that

$$6d - 2 = x^2 \tag{25}$$
$$11d - 2 = y^2 \tag{26}$$
$$33d - 2 = z^2 \tag{27}$$

Eliminating d between (26) and (27), we have Pell equation as follows:

$$z^2 - 3y^2 = 4 \tag{28}$$

The fundamental solution of (28) Pell Equation is $(z, y) = (4, 2)$ and other positive solutions generated by fundamental solution are as follows:

By use of the Table 2 and the fundamental solution of (28), we obtain general recurrence relation for (z_n) as following equation:

Table 2: Positive Solutions of $z^2 - 3y^2 = 4$

Solutions	Solution 1	Solution 2	Solution 3	Solution 4	Solution ...
z	4	14	52	194	...
y	2	8	30	112	...

$$z_n = 4z_{n-1} - z_{n-2}, (n \geq 3.) \quad (29)$$

Using (29), we have some values of d from (27). If we put these d in the (25), then any of these values don't give any perfect square of an integer x for the equation (25). This proves that a $P_{-2} = \{6, 11, 33\}$ can not be extended for any $d \in Z^+$. □

Theorem 3.5. *A $P_{-2} = \{11, 33, 82\}$ triple set is both regular and non-extendible.*

Proof. $P_{-2} = \{11, 33, 82\}$ set proves the condition of Definition 2.1. So, it is clear that $P_{-2} = \{11, 33, 82\}$ is regular set. Suppose that $\{11, 33, 82, d\}$ is a P_{-2} set. Then, x, y, z integers can be found as follows:

$$11d - 2 = x^2 \quad (30)$$

$$33d - 2 = y^2 \quad (31)$$

$$82d - 2 = z^2 \quad (32)$$

Eliminating d from (30) and (31), then

$$y^2 - 3x^2 = 4 \quad (33)$$

Pell equation is obtained. In a similar way of Proof of Theorem 3.3, we determine fundamental unit (33) as $(y, x) = (4, 2)$ and general recurrence relation for (y_n) as

$$y_n = 4y_{n-1} - y_{n-2}, (n \geq 3) \quad (34)$$

We get some values of d from (31) by use of (34). For these values of d, there isn't any perfect square of an integer z in (32). Therefore, a $P_{-2} = \{11, 33, 82\}$ is non-extendable for any $d \in Z^+$. □

Theorem 3.6. *There isn't any P_{-2} set contains primes provided $p \equiv 5 \pmod 8$.*

Proof. It is sufficient to prove this theorem for p primes such that $p \equiv 5 \pmod 8$. We assume that k is an element of set P_{-2} . If pk, $(k \in Z)$ is an element of set P_{-2}, then following equation

$$pk - 2 = L^2 \tag{35}$$

has to satisfy for some integer L. We obtain following equivalent

$$L^2 \equiv -2 \pmod p \tag{36}$$

if we deduce in $(modp)$. By evaluating the Legendre symbol and its properties, we obtain

$$(\frac{-2}{p}) = (\frac{-1}{p})(\frac{2}{p}) \tag{37}$$

From (6) in Theorem 2.3, we have following equivalents;

$$(\frac{-1}{p}) = (-1)^{\frac{p-1}{2}} \text{ and } (\frac{2}{p}) = (-1)^{\frac{1}{8}(p^2-1)}$$

If we consider and apply $p \equiv 5(mod8)$ in the (6) equivalents, we obtain

$$(\frac{-1}{p}) = (-1)^{\frac{p-1}{2}} = +1 \text{ and } (\frac{2}{p}) \equiv (-1)^{\frac{1}{8}(p^2-1)} = -1 \tag{38}$$

So, we get

$$(\frac{-2}{p}) = -1 \tag{39}$$

This means, the equation (35) isn't solvable. Hence, primes $p \equiv 5 \pmod 8$ can not be an element of P_{-2}. □

Remark 3.1. *There isn't any P_{-2} set includes n positive integers satisfy $n \equiv 5 \pmod 8$. Since it is easily seen from the Theorem 3.5 that n (positive integer satisfies $n \equiv 5 \pmod 8$) can not be an element of P_{-2}.*

Theorem 3.7. *There is no P_{-2} set includes primes ensured $q \equiv 7 \pmod 8$.*

Proof. Suppose that u is an element of set P_{-2} . If qu, is an element of set P_{-2} for any integer, then we obtain

$$qu - 2 = R^2 \tag{40}$$

for some integer R. Applying $(modq)$ on the both side of equation (40), we get

$$R^2 \equiv -2 \pmod q \tag{41}$$

By use of the Legendre symbol and its properties on the equivalent (41), followings are found.

$$(\frac{-2}{q}) = (\frac{-1}{q})(\frac{2}{q}) \tag{42}$$

Applying similar method of the proof of Theorem 3.5 (i.e. (6) in Theorem 2.3 and $q \equiv 7 \pmod 8$), then we obtain

$$(\frac{-2}{q}) = -1 \tag{43}$$

This is a contradiction and shows that the congruence (41) has no solution (From (3) in Definition 2.3). So, primes $q \equiv 7 \pmod 8$ can not be an element of P_{-2}. □

Remark 3.2. *There is no P_{-2} set includes m positive integers satisfied $m \equiv 7 \pmod 8$. In a similar way, one can easily proves that any m positive integer such that $m \equiv 7 \pmod 8$ can not be an element of P_{-2}, using the Definition 2.3, Definition 2.4, Theorem 2.1 and Theorem 2.2 as well as the proof of the Theorem 3.7.*

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