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Three States Hybrid Cellular Automata with Periodic Boundary Condition

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ABSTRACT

Even though cellular automata (CA) is a discrete model, the behaviors at many iterative times can be a close a continuous system. After modelling the CA structure, one of the important tasks is able to move forwards and backwards on CA to understand their behaviors. This happens if CA is to be a reversible one. In this paper, it is studied main theoretical views of 2D hybrid-linear periodic CA over the ternary field, i.e. $\mathbb{Z}_3$ or three states case. Here, we set up a relation between reversibility of CA and characterization of 2D hybrid linear CA with this special boundary conditions by using of the matrix algebra theory. For given special transition (information) rule matrices, it is proved that which hybrid 2D CAs are reversible or not. In other words, the reversibility problem of 2D hybrid, linear CA with periodic boundary is resolved completely over ternary field. It is believed that these type of CAs could be found many different real life applications in special case situation e.g. computability theory, mathematics, theoretical biology, image processing area, textile design, video processing, DNA research and microstructure modelling, etc. in near future.

Keywords: Cellular automata, hybrid rule, three states, periodic boundary, 2D linear hybrid CA.

1. Introduction

Cellular automata (CA) is a discrete dynamical system with many elegant properties. CA operate infinite space and time and its behaviors are entirely characterized by defined special some local given rules. Up to now researchers investigated and explored many aspects of CA with special interest in the study of different kinds of CA. These studies are mainly some special analytical ideas into the behavior of the iterative process. Cellular automata idea are firstly investigated by Von Neumann and Ulam VonNeumann (1966) around 1950's, were systematic ways investigated by Hedlund considering its just mathematics theoretical view. Neumann interested in the relation between biology and the investigation of CA. One dimensional (1D) CA are studied very extended mathematical ways and demonstrated for many various applications. However, not so much interest is presented to 2D uniform and hybrid CA. Neumann presented that CA can be universal considering theirs many elegant properties VonNeumann (1966). Considering theirs structured complexness, rules studied by Neumann never proceeded in computer programme language codes in that time. Around the eighties, S. Wolfram Wolfram (1983) has started to study some prominent properties of 1D uniform CA rules and investigated that many different simple rules can be used for showing very interesting and chaotic-complex behaviors. Uniform linear CA has been admitted striking interest in the past decades Akin et al. (2010), Uguz et al. (2018, 2013, 2017, 2014a,b, 2015). The papers Chou and Reggia (1997), Da et al. (2017), Dihidar and Choudhury (2004), Dogaru (2009), Gerlee and Anderson (2007, 2008), Penninger et al. (2010), Uguz et al. (2016, 2014a,b, 2015) investigate the mathematical analytic behavior and real life applications of the hybrid and uniform 2D CA with states values in ternary $\mathbb{Z}_3$ or any other fields.

Here we briefly summarize where to apply hybrid CA in the literature up to now. Hybrid CA for solid tumor growth is investigated by Gerlee and Anderson (2007). They propose CA model for solid tumour growth considering network structure. A hybrid CA system for clonal evolution model in cancer studies is also investigated by Gerlee and Anderson (2008). The paper Dogaru (2009) studies random number generator model by using hybrid CA. The hybrid CA algorithm was also studied in Penninger et al. (2010). Da et al. (2017) is presented a new method for the topological design for some interesting materials.

In the present paper it is concentrated on 2D finite linear-hybrid CA with periodic condition over the field $\mathbb{Z}_3$, i.e. 3-states case. The increased number of states (i.e. pixel values for image science) give us many number of possible rules. Therefore a reduced pixel values intensity representation is studied here, leading to a 3-states CA case is more practical for all computation. Here, we set up a specific relation between reversibility of hybrid cellular automata and characterization of 2D hybrid CA with periodic boundary condition. We study the determination of the characterization problem of this special cellular automaton by means of the matrix algebra theory. Due to CA nature is very simple to allow some important mathematical studies and to obtain very complicated and complex behaviors chaos in dynamical systems, it is believed that hybrid linear CA construction could be obtained many different kind of applications. Considering the linear transition rule matrices, the present results give further to the algebraic consequences of these 2D uniform and hybrid CA and relates some elegant applications found by the authors in the literature (i.e. Chou and Reggia (1997), Da et al. (2017), Dihidar and Choudhury (2004), Dogaru (2009), Gerlee and Anderson (2007, 2008), Penninger et al. (2010), Uguz et al. (2018, 2016, 2017, 2015)).

The present paper is constructed as below. In Section 2 we present the preliminaries of 2D uniform CA and the transition rule matrices of Rule 2460PB and Rule 2461PB. Transition rule matrices of 2D hybrid CA over the ternary field with periodic boundary is investigated in Section 3. The reversibility cases for these special 2D hybrid CA are studied in Section 4. Finally conclusions and discussions are summarized in Section 5.

2. Uniform Linear 2D CA (Rule 2460PB and Rule 2461PB)

We work CA established by a hybrid linear rule for the ternary field $\mathbb{Z}_3$ and we deal with the general case of the CA reversibility problem. The determining all 1D-2D uniform or hybrid CA is reversible or not is a challenging problem. Here, it is shortly emphasized the significance of the reversibility phenomena of 2D linear CA or understanding general case. Firstly it is found the rule matrices T_{rule} corresponding to finite 2D linear-hybrid CA, after then it is characterized the reversibility problem of 2D hybrid CA.

2.1 Preliminaries

One can present 2D CA over the ternary field Z_3 by considering some new local rules. First let us recall the definition of 2D CA. Let us consider

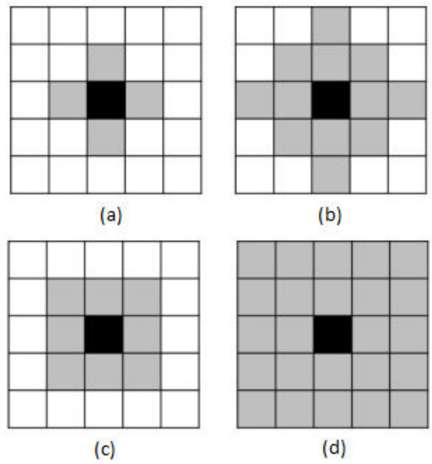

Figure 1: 2D CA with well-known neighborhoods in the literature: a) von Neumann neighborhood (VNN), b) VNN with radius 2, c) Moore neighborhood (MN) and d) MN with radius 2.

the 2D special integer Z^2 lattice and the model space $\Omega = \{0,1,2\}^{Z^2}$ and $\sigma : Z^2 \to \{0,1,2\}$ (see details in Uguz et al. (2016, 2014b, 2015)).

It is known that finite 2D CA ($m \times n$ CA cell values in m-n rows-columns respectively) cell values take the following spin values of 0, 1 or 2. When we consider 2D CA well-known nearest neighbor models (see Figure 1), one can see that nine nearest cells are arranged in a special 3×3 matrix. For the present work, we just investigate hybrid linear rules (briefly, Rule 2460PB and Rule 2461PB). Now it can be defined the $(t+1)^{th}$ state for $(i,j)^{th}$ cells as below.

$$x_{(i;j)}^{(t+1)} = ax_{(i-1;j)}^{(t)} + bx_{(i;j+1)}^{(t)} + cx_{(i+1;j)}^{(t)} + dx_{(i;j-1)}^{(t)} (mod\ 3), \quad (Rule\ 2460)$$

and

$$x_{(i,j)}^{(t+1)} = ax_{(i-1;j)}^{(t)} + bx_{(i;j+1)}^{(t)} + cx_{(i+1;j)}^{(t)} + dx_{(i;j-1)}^{(t)} + ex_{(i;j)}^{(t)} (mod\ 3), \ (Rule\ 2461)$$

where the constants $\{a,b,c,d,e\} \in Z_3^* = \{1,2\}$.

Here we just consider 2D finite linear CA constructed by specific hybrid rules with periodic boundary condition. Let $\Phi : M_{m\times n}(Z_3) \to Z_3^{mn}$. Φ takes the t^{th} state $[X_t]$ given by

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & \dots & x_{mn} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} & \dots & \dots & x_{mn} \end{pmatrix}^T.$$

Note that T given above is the matrix transpose operator. Let us consider the ternary valued information matrix $[X_t]_{m\times n}$ as below;

$$[X_t]_{m\times n} = \begin{pmatrix} x_{11}^{(t)} & \dots & \dots & x_{1n}^{(t)} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ x_{m1}^{(t)} & \dots & \dots & x_{mn}^{(t)} \end{pmatrix}.$$

This matrix can be considered as 2D finite CA configuration at time t. From these, one can define the following

$$(T_{Rule})_{mn\times mn}. \begin{pmatrix} x_{11}^{(t)} \\ . \\ . \\ x_{1n}^{(t)} \\ . \\ . \\ x_{m1}^{(t)} \\ . \\ . \\ x_{mn}^{(t)} \end{pmatrix}_{mn\times 1} = \begin{pmatrix} x_{11}^{(t+1)} \\ . \\ . \\ x_{1n}^{(t+1)} \\ . \\ . \\ x_{m1}^{(t+1)} \\ . \\ . \\ x_{mn}^{(t+1)} \end{pmatrix}_{mn\times 1}.$$

The matrix $(T_{Rule})_{mn\times mn}$ is known as a rule matrix for the 2D finite $CA_{m\times n}$.

Here we give some background of 2D CA boundary conditions. Considering the neighbourhood of the information cells, there are two well known studied boundary approaches (see Figure 2) in the literature.

- Null or fixed boundary: The boundary cells are zero spin values, or fixed 0-state.
- Periodic boundary: The boundary cells are contiguous to each border way.

An important note that, there are also different types of boundary conditions to study for future studies, i.e. reflexive, adiabatic etc.

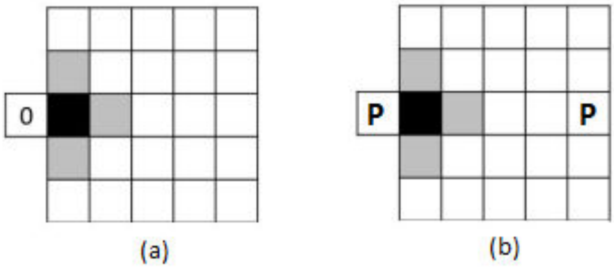

Figure 2: 2D CA with two boundary types in the literature: a) null or fixed-zero boundary and b) periodic boundary.

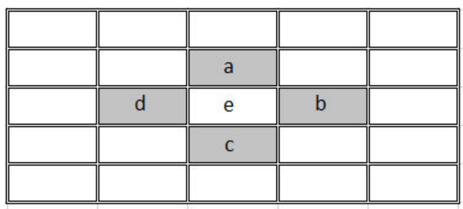

Figure 3: Non-zero coefficients of Rule 2460PB and Rule 2461PB.

2.2 Transition Rule Matrices of Rule 2460- Rule 2461 under periodic boundary

We determine the local rule matrix for 2D finite linear CA with periodic boundary over the ternary field Z_3 (see Figures 3 and 4). We also obtain the transformation Ψ such that $\Psi([X^{(t)}]_{m\times n}) = [X^{(t+1)}]_{m\times n}$.

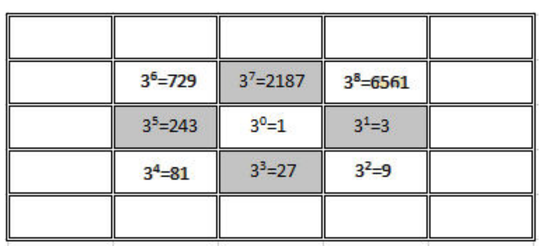

Figure 4: Linear rule convention of ternary spin values for 2D linear CA. Rule 2460=Rule 3 + Rule 27 + Rule 243 + Rule 2187. Similarly Rule 2461=Rule 1 + Rule 3 + Rule 27 + Rule 243 + Rule 2187. (See CA rule details in Dihidar and Choudhury (2004), Uguz et al. (2016)

Theorem (Rule 2460PB). Let us consider $a, b, c, d, e \in Z_3^*$, $m > 2$ and $n > 2$. Then, the transition rule matrix of $T_{Rule} = T_{2460PB}$ from $Z_3^{mn} \to Z_3^{mn}$ which takes the t^{th} state to the $(t+1)$-state is

$$(T_{2460PB})_{mn\times mn} = \begin{pmatrix} S(b,d) & cI & 0 & 0 & ... & 0 & aI \\ aI & S(b,d) & cI & 0 & ... & 0 & 0 \\ 0 & aI & S(b,d) & cI & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ cI & 0 & 0 & 0 & ... & aI & S(b,d) \end{pmatrix}$$

where partitioned matrix are given as follows.

$$S(b,d) = \begin{pmatrix} 0 & b & 0 & 0 & ... & 0 & d \\ d & 0 & b & 0 & ... & 0 & 0 \\ 0 & d & 0 & b & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & b \\ b & 0 & 0 & 0 & ... & d & 0 \end{pmatrix}_{n\times n}$$ and I is an $(n \times n)$ identity matrix.

Proof. For finding T_{Rule}, it should be determined the T_{Rule} action with the bases vectors. Firstly, let us consider the linear action Ψ from $m \times n$ matrix space elements to itself. Secondly, let us relate the action Ψ with rule matrix T_{Rule}. Denote e_{ij} for the $m \times n$ matrix where (i, i) position is equal to one and the rest is zero. In fact, these e_{ij} vectors are the standard basis for studied space (see Uguz et al. (2016, 2014b)). Given e_{ij}, the image of e_{ij} which is $\Psi(e_{ij})$ is combined to the transition rules neighbors. $\Psi(e_{ij}))$ is obtained as the linear combination for the following way

$$\Psi(e_{ij}) = a.e_{i+1,j} + b.e_{i,j-1} + c.e_{i-1,j} + d.e_{i,j+1}.$$

Hence the rule matrix is obtained as given in the theorem.

Theorem (Rule 2461PB). Let $a, b, c, d, e \in Z_3^*$, $m > 2$ and $n > 2$. Then, the rule matrix of $T_{Rule} = T_{2461PB}$ from $Z_3^{mn} \to Z_3^{mn}$ which takes the t^{th} state to the $(t+1)$-state is

$$T_{2461PB} = \begin{pmatrix} eI + S(b,d) & cI & 0 & ... & 0 & aI \\ aI & eI + S(b,d) & cI & ... & 0 & 0 \\ 0 & aI & eI + S(b,d) & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ cI & 0 & 0 & ... & aI & eI + S(b,d) \end{pmatrix}$$

where the partitioned matrix is given as below.

$$eI + S(b,d) = \begin{pmatrix} e & b & 0 & 0 & ... & 0 & d \\ d & e & b & 0 & ... & 0 & 0 \\ 0 & d & e & b & ... & 0 & 0 \\ : & : & : & : & : & : & b \\ b & 0 & 0 & 0 & ... & d & e \end{pmatrix}_{n\times n}.$$

Proof. For T_{Rule}, one can study the actions of T_{Rule} on base vector elements e_{ij}. the image of e_{ij} which is $\Psi(e_{ij})$ is connected to the transition rules. $\Psi(e_{ij}))$ is given by

$$\Psi(e_{ij}) = a.e_{i+1,j} + b.e_{i,j-1} + c.e_{i-1,j} + d.e_{i,j+1} + e.e_{i,j}.$$

Using these transition rules, it can be obtained the rule matrix as given in the theorem.

3. Transition Rule Matrices of 2D Hybrid CA Over The Ternary Field With Periodic Boundary

Here we study 2D hybrid linear CA over the ternary field with periodic boundary. Here the hybrid rule means that it is applied 2460PB and 2461PB respectively for each m-rows on $m \times n$ cellular automata (see details in Figure 5). The construction cases of the transition rule matrices will be presented by two specific Theorems called Theorem (Hybrid-Even) and Theorem (Hybrid-Odd). Thus, we study use of the matrix construction structure related to 2D hybrid CA and obtain the transition rule matrices for $m \times n$, where m is even and m is odd cases. So it is seen that there are two cases for these rule matrices. In this section, we obtain a general transition hybrid rule matrix case as follows.

- Case 1. When m is even, the matrix T^{even}_{hybrid} is given in the following theorem.

Theorem (Hybrid-Even). Let us consider $a, b, c, d, e \in Z_3^*$, $m > 2$ and $n > 2$. Then, T^{even}_{hybrid} from $Z_3^{mn} \to Z_3^{mn}$ which takes t^{th} state to $(t+1)$-state is

$$T^{even}_{hybrid} = \begin{pmatrix} S(b,d) & cI & 0 & ... & 0 & aI \\ aI & eI+S(b,d) & cI & ... & 0 & 0 \\ 0 & aI & S(b,d) & ... & 0 & 0 \\ : & : & : & : & : & : \\ cI & 0 & 0 & ... & aI & eI+S(b,d) \end{pmatrix}_{mn\times mn}$$

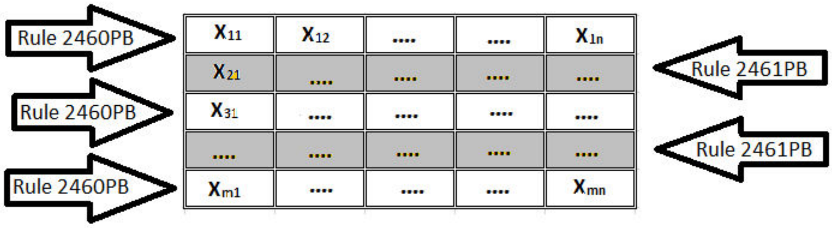

Figure 5: Hybrid rule means that it is applied 2460PB and 2461PB respectively for each m-rows on $m \times n$ cellular automata.

Proof. Using the idea similar used in the calculation of Theorems (Rule 2460PB) and (Rule 2461PB), we can find the desired result for this hybrid-even case.

- Case 2. If m is odd, T^{odd}_{hybrid} is constructed in the following theorem.

Theorem (Hybrid-Odd). Let us consider $a, b, c, d, e \in Z_3^*$, $m > 2$ and $n > 2$. Then, T^{odd}_{hybrid} from $Z_3^{mn} \rightarrow Z_3^{mn}$ which takes t^{th} state to $(t+1)$-state is

$$T^{odd}_{hybrid} = \begin{pmatrix} S(b,d) & cI & 0 & \ldots & 0 & aI \\ aI & eI + S(b,d) & cI & \ldots & 0 & 0 \\ 0 & aI & S(b,d) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ cI & 0 & 0 & \ldots & aI & S(b,d) \end{pmatrix}_{mn \times mn}$$

Proof. Combining results of the calculation of Theorems (Rule 2460PB) and (Rule 2461PB), then it is obtained the transition rule matrix for this hybrid-odd case.

To understand better of the situation, we present an example of 3×3 transition hybrid rule matrices in the following case.

Example: Let us consider $m = 3$ and $n = 3$, then we obtain the following the rule matrix T_{hybrid}.

$$
\begin{aligned}
(T_{hybrid})_{9\times 9} &= \left(\begin{array}{ccc|ccc|ccc}
0 & b & d & c & 0 & 0 & a & 0 & 0 \\
d & 0 & b & 0 & c & 0 & 0 & a & 0 \\
b & d & 0 & 0 & 0 & c & 0 & 0 & a \\ \hline
a & 0 & 0 & e & b & d & c & 0 & 0 \\
0 & a & 0 & d & e & b & 0 & c & 0 \\
0 & 0 & a & b & d & e & 0 & 0 & c \\ \hline
c & 0 & 0 & a & 0 & 0 & 0 & b & d \\
0 & c & 0 & 0 & a & 0 & d & 0 & b \\
0 & 0 & c & 0 & 0 & a & b & d & 0
\end{array}\right) \\
&= \left(\begin{array}{ccc}
S_3(b,d) & cI_3 & a_3 \\
aI_3 & eI_3 + S_3(b,d) & cI_3 \\
cI_3 & aI_3 & S_3(b,d)
\end{array}\right).
\end{aligned}
$$

4. Reversibility of 2D Hybrid CA

It can be stated the following iteration property among the column vectors $X^{(t)}$ with the rule matrix $T_{rules} : X^{(t+1)} = T_{rules}.X^{(t)}$ $(mod\ 3)$. If the transition rule matrix T_{rules} is a non-singular matrix, then we have

$$X^{(t)} = (T_{rules})^{-1}.X^{(t+1)}\ (mod\ 3).$$

Here a main problem is finding the rule matrix T_{rules} is invertible or non-invertible for any cases. If the transition matrices T_{rules} have full rank properties, then it is said to be a invertible CA, hence the 2D finite hybrid or uniform CA is a reversible one, on the other hand it is called a irreversible CA. The analysis on reversible or irreversible cases of the transition matrices found in the next subsection is given by a specific Theorem called Theorem (even). Thus, we study use of the following structure related to reversibility of CA in this section and detect the reversibility problem for $m \times n$, i.e m is even odd cases.

4.1 The rank of T_{hybrid} for m-even

Let T_i indicate the i'th row entry and $T_i[j]$ indicate the j'th position element of the i'th row of rule matrix T_{hybrid}, respectively. But we first transfer the first column of the rule matrix which m is even to the last column, we get

$$T_{hybrid}^{even} = \left(\begin{array}{ccccc|cc} cI & 0 & 0 & ... & 0 & aI & S(b,d) \\ eI+S(b,d) & cI & 0 & ... & 0 & 0 & aI \\ \hline aI & S(b,d) & cI & ... & ... & 0 & 0 \\ 0 & aI & eI+S(b,d) & ... & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & ... & aI & eI+S(b,d) & cI \end{array}\right)$$

Theorem (Even) Let the matrix

$$T_{hybrid}^{even} = \left(\begin{array}{ccccccc} cI & 0 & 0 & ... & 0 & aI & S(b,d) \\ eI+S(b,d) & cI & 0 & ... & 0 & 0 & aI \\ aI & S(b,d) & cI & ... & ... & 0 & 0 \\ 0 & aI & eI+S(b,d) & ... & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & ... & aI & eI+S(b,d) & cI \end{array}\right)$$

with $m - even, n \geq 3$ representing the 2D hybrid linear CA over the field $\mathbb{Z}_3$ under periodic boundary condition. Let

$T_1^1 = T_1,$

$T_1^{k+1} = -T_1^k[k].(T_{k+2}^1[k])^{-1}.T_{k+2}^1 + T_1^k,$

$T_2^1 = T_2,$

$T_2^{k+1} = -T_2^k[k].(T_{k+2}^1[k])^{-1}.T_{k+2}^1 + T_2^k$ for $1 \leq k \leq m-2$

Let us denote R by 2×2 block matrix, consisting of blocks of square matrices of $n \times n$ as below.

$$R = \begin{pmatrix} T_1^{m-1}[m-1] & T_1^{m-1}[m] \\ T_2^{m-1}[m-1] & T_2^{m-1}[m] \end{pmatrix}.$$

Then

$$rank(T_{hybrid}^{even}) = (m-2).n \; + \; rank(R).$$

Proof. By using induction (on m) and similar idea used in the calculation of the rank of T_{hybrid}, then we will find the linear algebraic relation of the rank of T_{hybrid}^{even} for any cases.

$$T^{even}_{hybrid} = \begin{pmatrix} cI & 0 & 0 & ... & 0 & aI & S(b,d) \\ eI+S(b,d) & cI & 0 & ... & 0 & 0 & aI \\ aI & S(b,d) & cI & ... & ... & 0 & 0 \\ 0 & aI & eI+S(b,d) & ... & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & ... & aI & eI+S(b,d) & cI \end{pmatrix}$$

Firstly, one can see above that the sub-matrix comprising of all rows except the first and second is an upper triangular shape and sub-matrix is of a full rank matrix that the sub-matrix rank is $(m-2).n$ for $a, b, c, d, e \in \mathbb{Z}_3^* = \{1, 2\}$. Now, if one can multiply the second row T_2 by $-T_1^1[1].(T_3^1[1])^{-1}.T_3^1$ and adding that product to T_1^1, so first-entry elements of the first new one row T_1^2 becomes zero. Hence, we change the the first row by $T_1^2 = -T_1^1[1](T_3^1[1])^{-1}.T_3^1 + T_1^1$. Now, we want to $T_1^2[2]$ is zero. In that case, we apply the same thing for T_1^2. So we have $T_1^3 = -T_1^2[2].(T_4^1[2])^{-1}.T_4^1 + T_1^2$. Thus the second entry elements of the new first row becomes zero. Repeatedly, we see that subsequently $m-2$ steps the only non-zero entries of T_1^{m-1} are $T_1^{m-1}[m-1]$ and $T_1^{m-1}[m]$.

$$\begin{aligned} T^{even}_{hybrid} &= \left(\begin{array}{cccccc|cc} 0 & 0 & 0 & ... & 0 & & T_1^{m-1}[m-1] & T_1^{m-1}[m] \\ 0 & 0 & 0 & ... & 0 & & T_2^{m-1}[m-1] & T_2^{m-1}[m] \\ \hline aI & S(b,d) & cI & ... & ... & & 0 & 0 \\ 0 & aI & eI+S(b,d) & ... & ... & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & ... & aI & & eI+S(b,d) & cI \end{array}\right) \\ &= \begin{pmatrix} 0 & R \\ P & Q \end{pmatrix} \end{aligned}$$

The left lower block matrix of P has full rank $(m-2).n$. Hence we can obtain that the rank equalities of the transition matrix is presented by $rank(T^{even}_{hybrid}) = (m-2).n + rank(R)$.

Example. Let us take $m = 4$, $n = 3$. Then and transition rule matrix is given as

$$(T^{even}_{hybrid})_{12\times 12} = \begin{pmatrix} cI & 0 & aI & S(b,d) \\ eI+S(b,d) & cI & 0 & aI \\ aI & S(b,d) & cI & 0 \\ 0 & aI & eI+S(b,d) & cI \end{pmatrix}.$$

Consider the special case $a, b, d = 2$ and $c, e = 1$. Then computation of the rank of T^{even}_{hybrid} gives 9. On the other part, if someone use Theorem(even), then it is proceeded in the next way:

$T^1_1 = T_1 = [cI \quad 0 \quad aI \quad S(b,d)]$

$T^2_1 = -T^1_1[1](T^1_3[1])^{-1}.T^1_3 + T^1_1 = [0 \quad -ca^{-1}S(b,d) \quad aI - c^2a^{-1}I \quad S(b,d)]$

$T^3_1 = -T^2_1[2](T^1_4[2])^{-1}.T^1_4 + T^2_1 = [0 \quad 0 \quad aI - c^2a^{-1}I + ca^{-1}S(b,d)a^{-1}.(eI + S(b,d)) \quad S(b,d) + ca^{-1}S(b,d)a^{-1}c]$

$T^1_2 = T_2 = [eI + S(b,d) \quad cI \quad 0 \quad aI]$

$T^2_2 = -T^1_2[1](T^1_3[1])^{-1}.T^1_3 + T^1_2 = [0 \quad cI - a^{-1}(eI + S(b,d)).S(b,d) \quad -(eI + S(b,d)).a^{-1}cI \quad aI]$.

$$\begin{aligned} T^3_2 \quad &= -T^2_2[2](T^1_4[2])^{-1}.T^1_4 + T^2_2 = [0\ 0\ -(cI - a^{-1}(eI + S(b,d)).S(b,d)) \\ &a^{-1}(eI + S(b,d))aI - (cI - a^{-1}(eI + S(b,d)).S(b,d))a^{-1}cI]. \end{aligned}$$

Thus,

$$R = \begin{pmatrix} T^3_1[3] & T^3_1[4] \\ T^3_2[3] & T^3_2[4] \end{pmatrix}.$$

If we consider $a, b, d = 2$ and $c, e = 1$, then we get

$$rank(R) = 3.$$

Therefore,

$$rank(T^{even}_{hybrid}) = (m-2).n + rank(R) = (4-2).3 + 3 = 9.$$

4.2 The rank of T_{hybrid} for m-odd

Theorem (Odd). Let us consider the last row elements in the first row in T^{odd}_{hybrid}. Then

$$(T^{odd}_{hybrid})_{mn \times mn} = \begin{pmatrix} cI & 0 & 0 & ... & aI & S(b,d) \\ S(b,d) & cI & 0 & ... & 0 & aI \\ aI & eI + S(b,d) & cI & ... & 0 & 0 \\ 0 & aI & S(b,d) & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & aI & S(b,d) & cI \end{pmatrix}$$

with $m - odd, n \geq 3$ representing the 2D hybrid linear CA for the ternary field $\mathbb{Z}_3$ over periodic boundary. Let

$$M_1^1 = M_1,$$

$$M_1^{k+1} = -M_1^k[k].(M_{k+2}^1[k])^{-1}.M_{k+2}^1 + M_1^k,$$

$$M_2^1 = M_2,$$

$$M_2^{k+1} = -M_2^k[k].(M_{k+2}^1[k])^{-1}.M_{k+2}^1 + M_2^k \text{ for } 1 \leq k \leq m-2.$$

Consider Y as 2×2 block matrix, consisting of blocks of square matrices of $n \times n$ as given below.

$$Y = \begin{pmatrix} M_1^{m-1}[m-1] & M_1^{m-1}[m] \\ M_2^{m-1}[m-1] & M_2^{m-1}[m] \end{pmatrix}.$$

Then

$$rank(T_{hybrid}^{odd}) = (m-2).n \ + \ rank(Y).$$

Proof. For the proof of m-odd case, similar approach is applied for the rank computations of m even T_{hybrid}^{even} as given in Theorem(even).

Remark. It is noted that the theory 2D hybrid periodic boundary CA of linear primary rules should be applied to real life application of image science area. It is seen that 2D linear uniform CA theory is applied successfully in self replica patterns of image science Khan et al. (1997, 1999), Redjepov et al. (2018), Sahin et al. (2013, 2015a,b), Uguz et al. (2018, 2013, 2016, 2014a,b, 2015). The some characterization and applications of any other 2D finite hybrid CA considering matrix algebra on ternary field should be investigated for the next studies. Von Neumann neighborhood corresponding to transition rule matrices of Rule 2460 for 2D integer lattices is really strange. It should be investigated Moore neighborhood cellular automata (Rule 9840 CA given Figure 1.c) and its reversibility cases for future studies. Using the matrix representation of 2D hybrid CA, it can be obtained any other reversibility algorithms for these CA defined by some other rules.

5. Conclusion

In this paper, it is studied the theoretical aspects of 2D hybrid-linear-periodic CA case over the ternary field $\mathbb{Z}_3$, i.e. three states case. It is introduced two main theorems for determining the reversibility of these CAs for a general case of hybrid linear transformation. Also after constructed the transition matrix representation of 2D hybrid-linear CA, one can find some real life applications for the 2D hybrid-linear CA, it is another goals of the next study. Von Neumann neighborhood corresponding to Rule 2460 for 2D integer lattices is really strange. Moore neighborhood cellular automata corresponding to Rule 9840 and its reversibility cases should be investigated for future studies. It is believed that CA hybrid theory could be applied successfully in especially image processing area Uguz et al. (2018, 2013, 2016, 2014a,b, 2015) and the other science branches in near future Sahin et al. (2013, 2015a,b, 2014), Siap and Uguz (2011). Some other interesting results and further connections on this direction wait to be explored in 2D hybrid CA Da et al. (2017), Dogaru (2009), Gerlee and Anderson (2007, 2008), Penninger et al. (2010).

References

Akin, H., Siap, I., and Uguz, S. (2010). Structure of 2-dimensional hexagonal cellular automata. *AIP Conference Proceedings*, 1309:16–26.

Chou, H. and Reggia, J. A. (1997). Emergence of self-replicating structures in a cellular automata space. *Physica D*, 110:252–276.

Da, D., Chen, J., Cui, X., and Li, G. (2017). Design of materials using hybrid cellular automata. *Structural and Multidisciplinary Optimization*, 56:131 – 137.

Dihidar, K. and Choudhury, P. P. (2004). Matrix algebraic formulae concerning some exceptional rules of two dimensional cellular automata. *Information Science*, 165:91–101.

Dogaru, R. (2009). Hybrid cellular automata as pseudo-random number generators with binary synchronization property. *Signals, Circuits and Systems ISSCS 2009*.

Gerlee, P. and Anderson, A. (2007). An evolutionary hybrid cellular automaton model of solid tumour growth. *Journal of Theoretical Biology*, 246:583 – 603.

Gerlee, P. and Anderson, A. (2008). A hybrid cellular automaton model of clonal evolution in cancer: the emergence of the glycolytic phenotype. *Journal of Theoretical Biology*, 250:705–722.

Khan, A., Choudhury, P., Dhider, K., Sarkar, P., and Verma, R. (1997). Vlsi architecture of a cellular automata machine. *Int. Journal Computers and Mathematics with Application*, 33:79–94.

Khan, A., Choudhury, P., Dhider, K., and Verma, R. (1999). Text compression using 2d cellular automata. *Int. Journal Computers and Mathematics with Application*, 37:115–127.

Penninger, C. L., Watson, T., Tovar, A., and Renaud, J. E. (2010). Convergence analysis of hybrid cellular automata for topology optimization. *Structural and Multidisciplinary Optimization*, 40:271 – 282.

Redjepov, S., Acar, E., and Uguz, S. (2018). Reversibility algorithm for 2d cellular automata with reflective condition. *Acta Physica Polonica A*, In press.

Sahin, U., Sahin, F., and Uguz, S. (2013). Hybridized fuzzy cellular automata thresholding algorithm for edge detection optimized by pso. *High Capacity Optical Networks and Enabling Technologies (HONET-CNS), 10th International Conference IEEE*, pages 228–232.

Sahin, U., Uguz, S., and Akin, H. (2015a). The transition rules of 2d linear cellular automata over ternary field and self-replicating patterns. *International Journal of Bifurcation and Chaos*, 25:1550011.

Sahin, U., Uguz, S., Akin, H., and Siap, I. (2015b). Three-state von neumann cellular automata and pattern generation. *Applied Mathematical Modeling*, 39:2003–2024.

Sahin, U., Uguz, S., and Sahin, F. (2014). Salt and pepper noise filtering with fuzzy-cellular automata. *Computers and Electrical Engineering*, 40:59–69.

Siap, I. an Akin, H. and Uguz, S. (2011). Structure and reversibility of 2d hexagonal cellular automata. *Computers Mathematics with Applications*, 62:4161–4169.

Uguz, S., Acar, E., and Redjepov, S. (2018). 2d triangular von neumann cellular automata with periodic boundary. *International Journal of Bifurcation and Chaos*, Accepted.

Uguz, S., Akin, H., and Siap, I. (2013). Reversibility algorithms for 3-state hexagonal cellular automata with periodic boundariess. *International Journal of Bifurcation and Chaos*, 23:1350101.

Uguz, S., Akin, H., Siap, I., and Sahin, U. (2016). On the irreversibility of moore cellular automata over the ternary field and image application. *Applied Mathematical Modeling*, 40:8017–8032.

Uguz, S., Redjepov, S., Acar, E., and Akin, H. (2017). Structure and reversibility of 2d von neumann cellular automata over triangular lattice. *International Journal of Bifurcation and Chaos*, 27:1750083.

Uguz, S., Sahin, U., Akin, H., and Siap, I. (2014a). 2d cellular automata with an image processing application. *Acta Physica Polonica A*, 125:435–438.

Uguz, S., Sahin, U., Akin, H., and Siap, I. (2014b). Self-replicating patterns in 2d linear cellular automata. *International Journal of Bifurcation and Chaos*, 24:1430002.

Uguz, S., Sahin, U., and Sahin, F. (2015). Edge detection with fuzzy cellular automata transition function optimized by pso. *Computers and Electrical Engineering*, 43:180 – 192.

VonNeumann, J. (1966). *The theory of self-reproducing automata.* University of Illinois Press, Urbana.

Wolfram, S. (1983). Statistical mechanics of cellular automata. *Reviews of Modern Physics*, 55(3):601–644.