Malaysian Journal of Mathematical Sciences 12(3): 383–400 (2018)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

Positive Solution of Pair Fully Fuzzy Matrix Equations

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Received: 29 March 2018
Accepted: 15 August 2018

ABSTRACT

A pair matrix equation is a matrix system that contains two matrix equations which is solved simultaneously to obtain its solution. In this study, an algorithm for obtaining the positive fuzzy solution of positive pair fully fuzzy matrix equation is proposed. The constructed algorithm utilizes fuzzy Kronecker product and fuzzy Vec-operator to transform pair fully fuzzy matrix equation into fully fuzzy linear system. Then, an associated linear system is used to reach the final solution. Necessary theorems, corollary and numerical example are presented to illustrate the proposed algorithm.

Keywords: Fully fuzzy pair matrix equation, Fully fuzzy linear system, Kronecker product, Vec-operator, Associated linear system

1. Introduction

In many applications, there exist situations where the crisp numbers are less adequate to represent the uncertainty, vagueness and ambiguity of information. In this case, fuzzy numbers plays a prominent role to model the fuzzy environment.

The past few decades have seen a growing trend towards the matrix equations in the fuzzy environment. There are fuzzy matrix equation (FME) of $A\tilde{X}_m = \tilde{B}_m$ (Guo and Gong, 2010), fully fuzzy matrix equation (FFME) of $\tilde{A}\tilde{X}_m = \tilde{B}_m$ (Otadi and Mosleh, 2012), fuzzy Sylvester matrix equation (FSE) of $A\tilde{X} + \tilde{X}B = \tilde{C}$ (Araghi and Hosseinzadeh, 2012, Guo, 2011, Guo and Bao, 2013, Guo and Shang, 2012, 2013, Salkuyeh, 2010) and also fully fuzzy Sylvester matrix equation of $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ (Malkawi et al., 2015, Shang et al., 2015) and $\tilde{A}\tilde{X} - \tilde{X}\tilde{B} = \tilde{C}$ (Daud et al., 2018b, Dookhitram et al., 2015). This considerable amount of literature have shown that, fuzzy set theory plays a significant role to model the matrix equations. It is undeniable that, the previous proposed methods demonstrated various significant contribution in solving the matrix equation in fuzzy environment. However, there are still many gaps that can be filled in this area.

This study aims to construct a new algorithm for solving a positive pair fully fuzzy matrix equation (PFFME). Basically, a pair matrix equation is a matrix system that contains two matrix equations which are solved simultaneously to obtain its solution. These equations are important in real application for example in control theory (Asari and Amirfakhrian, 2016). Previously, a study was carried out by Sadeghi et al. (2011), which proposed a significant knowledge in solving fuzzy pair matrix equation of $A_1\tilde{X} + \tilde{X}B_1 = \tilde{C}_1$ and $A_2\tilde{X}B_2 = \tilde{C}_2$, where A_1, B_1, A_2, B_2 are known crisp matrices, $\tilde{C}_1, \tilde{C}_2$ are known fuzzy matrices and $\tilde{X}$ is unknown fuzzy matrices.

Contrary to this study, two fully fuzzy matrix equations are solved simultaneously, which are fully fuzzy continuous-time Sylvester matrix equation of

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C} \tag{1}$$

and also fully fuzzy discrete-time Sylvester matrix equation of

$$\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C}. \tag{2}$$

Thus, a pair fully fuzzy matrix equation is given by

$$\begin{cases} \tilde{A}_1\tilde{X} + \tilde{X}\tilde{B}_1 = \tilde{C}_1 \\ \tilde{A}_2\tilde{X}\tilde{B}_2 - \tilde{X} = \tilde{C}_2 \end{cases} \tag{3}$$

where $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{B}_1$, $\tilde{B}_2$ represents $p \times p$ and $q \times q$ positive fuzzy matrices respectively, $\tilde{C}_1$ and $\tilde{C}_2$ are $p \times q$ arbitrary fuzzy matrices, and $\tilde{X}$ is a $p \times q$ positive fuzzy solution. This study utilizes fuzzy Kronecker product and fuzzy Vec-operator in converting the equation into a fully fuzzy linear system. In addition, the associated linear system based on (Malkawi et al., 2014) is adapted in obtaining the final solution. Overall, this study provides valuable contribution in finding the solution of PFFME, and, at the same time advances the understanding in theory of fuzzy sets and matrices.

The remaining part of the paper proceeds as follows. In Section 2, the fundamental concept of fuzzy set theory and Kronecker operation are provided. In Section 3, the algorithm for solving the PFFME is shown. Later on, a numerical example is illustrated in Section 4 followed by the conclusion in Section 5.

2. Preliminaries

In this section, some definitions and theorems used in this study are recalled.

Definition 2.1. *(Zadeh, 1965) A fuzzy number is a function such as $u : R \to [0, 1]$ satisfying the following properties:*

1. u is normal, that is, there exist an $x_0 \in R$ such that $u(x_0) = 1$;
2. u is fuzzy convex, that is $u(\lambda x + (1 - \lambda)y) \geq min\{u(x), u(y)\}$ for any $x, y \in R$, $\lambda \in [0, 1]$;
3. u is upper semicontinuous;
4. supp $u = \{x \in R | u(x) > 0\}$ is the support of u, and its closure cl(supp u) is compact.

Definition 2.2. *A fuzzy number $\tilde{M} = (m, \alpha, \beta)$ is said to be a triangular fuzzy number (TFN), if its membership function is given by:*

$$\mu_{\tilde{M}}(x) = \begin{cases} 1 - \frac{m-x}{\alpha}, & m - \alpha \leq x \leq m, \alpha > 0, \\ 1 - \frac{x-m}{\beta}, & m \leq x \leq m + \beta, \beta > 0, \\ 0, & otherwise. \end{cases} \tag{4}$$

In this case, m is the mean value of $\tilde{M}$, whereas α and β are right and left spreads, respectively.

Definition 2.3. *(Dubois and Prade, 1978) The arithmetic operations of two fuzzy numbers $\tilde{M} = (m, \alpha, \beta)$ and $\tilde{N} = (n, \gamma, \delta)$, are as follows:*

1. Addition:

$$\tilde{M} \oplus \tilde{N} = (m, \alpha, \beta) \oplus (n, \gamma, \delta) = (m + n, \alpha + \gamma, \beta + \delta) \tag{5}$$

2. Opposite:

$$-\tilde{M} = -(m, \alpha, \beta) = (-m, \beta, \alpha) \tag{6}$$

3. Subtraction:

$$\tilde{M} \ominus \tilde{N} = (m, \alpha, \beta) \ominus (n, \gamma, \delta) = (m - n, \alpha + \delta, \beta + \gamma) \tag{7}$$

4. Multiplication:

$$\tilde{M} \otimes \tilde{N} = (m, \alpha, \beta) \otimes (n, \gamma, \delta) \cong (mn, m\gamma + n\alpha, m\delta + n\beta) \tag{8}$$

Definition 2.4. *(Dehghan et al., 2006) An $n \times n$ fully fuzzy linear system (FFLS) is defined as follows.*

$$\begin{cases} \tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 + ... + \tilde{a}_{1n}\tilde{x}_n = \tilde{b}_1 \\ \tilde{a}_{21}\tilde{x}_1 + \tilde{a}_{22}\tilde{x}_2 + ... + \tilde{a}_{2n}\tilde{x}_n = \tilde{b}_2 \\ \qquad\qquad \vdots \\ \tilde{a}_{m1}\tilde{x}_1 + \tilde{a}_{m2}\tilde{x}_2 + ... + \tilde{a}_{mn}\tilde{x}_n = \tilde{b}_m \end{cases} \tag{9}$$

which can also be written in a matrix form of

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \dots & \tilde{a}_{1n} \\ \tilde{a}_{21} & \tilde{a}_{22} & \dots & \tilde{a}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1} & \tilde{a}_{m2} & \dots & \tilde{a}_{mn} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_m \end{pmatrix}, \tag{10}$$

and it is usually denoted in a form of

$$\tilde{A}\tilde{X} = \tilde{B}. \tag{11}$$

Definition 2.5. *(Dehghan et al., 2006) A positive fuzzy number $\tilde{X} = (x, y, z)$ where $x, y, z \geq 0$ be the solution of FFLS, $\tilde{A}\tilde{X} = \tilde{B}$, which $\tilde{A} = (A, M, N) \geq 0$ and $\tilde{B} = (b, h, g) \geq 0$ iff*

$$\begin{cases} Ax = b \\ Ay + Mx = h \\ Az + Nx = g. \end{cases} \tag{12}$$

Theorem 2.1. *(Malkawi et al., 2014) A system of linear equation*

$$SX = B, \tag{13}$$

is an associated linear system of the FFLS, $\tilde{A}\tilde{X} = \tilde{B}$, where

$$S = \begin{pmatrix} A & 0 & 0 \\ M & A & 0 \\ N & 0 & A \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} b \\ h \\ g \end{pmatrix}, \tag{14}$$

with A, M and N are square matrices in common size of n, whereas x, y, z, b, h and g are vectors of n components.

Theorem 2.2. *(Malkawi et al., 2014) The block matrix S in Eq.(14) is non-singular if and only if the matrix A is non-singular.*

Theorem 2.3. *(Malkawi et al., 2014) The unique solutions of $\tilde{A}\tilde{X} = \tilde{B}$ and $SX = B$ are equivalent.*

Definition 2.6. *(Malkawi et al., 2015) A matrix $(A)_{ij}$ is called as a positive matrix when all its elements are greater than zero, $A_{i,j} > 0$, $\forall i, j$.*

The following definitions and theorems elaborates the Kronecker properties and Vec-operator, (see Malkawi et al. (2015)).

Definition 2.7. *Let $\tilde{A} = (\tilde{a}_{ij})_{m\times m}$ and $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$ be fuzzy matrices. Fuzzy Kronecker product is represented as $\tilde{A} \otimes_k \tilde{B}$ with the operation*

$$\tilde{A} \otimes_k \tilde{B} = \begin{pmatrix} \tilde{a}_{11}\tilde{B} & \tilde{a}_{12}\tilde{B} & \dots & \tilde{a}_{1n}\tilde{B} \\ \tilde{a}_{21}\tilde{B} & \tilde{a}_{22}\tilde{B} & \dots & \tilde{a}_{2n}\tilde{B} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n1}\tilde{B} & \tilde{a}_{n2}\tilde{B} & \dots & \tilde{a}_{mn}\tilde{B} \end{pmatrix}. \tag{15}$$

Definition 2.8. *Vec-operator of a fuzzy matrix is a linear transformation that converts the fuzzy matrix of $\tilde{C} = (\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_n)$ into a column vector as*

$$Vec(\tilde{C}) = \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_n \end{pmatrix}. \tag{16}$$

Theorem 2.4. *If $\tilde{A} = (\tilde{a}_{ij})_{m\times m}$ be a fuzzy matrix, and $\tilde{U} = (\tilde{u}_{ij})_{p\times p}$ is a unitary fuzzy matrix defined as*

$$\tilde{U} = \begin{pmatrix} (1,0,0) & (0,0,0) & \dots & (0,0,0) \\ (0,0,0) & (1,0,0) & \dots & (0,0,0) \\ \vdots & \vdots & \ddots & \vdots \\ (0,0,0) & (0,0,0) & \dots & (1,0,0) \end{pmatrix}, \tag{17}$$

then

1. $\tilde{A}\tilde{U} = \tilde{U}\tilde{A} = \tilde{A}$
2. $\tilde{U}^T = \tilde{U}$.

Theorem 2.5. *Let $\tilde{A} = (\tilde{a}_{ij})_{m\times m}$, $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$ and $\tilde{X} = (\tilde{x}_{ij})_{m\times n}$, then by using Kronecker product and Vec-operator, the equation of $\tilde{A}\tilde{X} + \tilde{X}\tilde{B} = \tilde{C}$ can be rewritten as*

$$[(\tilde{U}_n \otimes_k \tilde{A}) + (\tilde{B}^T \otimes_k \tilde{U}_m)]Vec(\tilde{X}) = Vec(\tilde{C})$$

where $\tilde{U}_m$ and $\tilde{U}_n$ denotes the fuzzy identity matrices with order m and n, respectively.

Theorem 2.6. *(Daud et al., 2018a) Let $\tilde{A} = (\tilde{a}_{ij})_{m\times m}$, $\tilde{B} = (\tilde{b}_{ij})_{n\times n}$ and $\tilde{X} = (\tilde{x}_{ij})_{m\times n}$, then by using Kronecker product and Vec-operator, the equation of $\tilde{A}\tilde{X}\tilde{B} - \tilde{X} = \tilde{C}$ can be rewritten as*

$$[(\tilde{B}^T \otimes_k \tilde{A}) - \tilde{U}_{mn}]Vec(\tilde{X}) = Vec(\tilde{C}),$$

where $\tilde{U}_{mn}$ denotes the fuzzy identity matrix with order $m \times n$.

3. Theoretical Foundations

In this section, theoretical foundations are built which lead to the development of the algorithm in solving the PFFME.

Theorem 3.1. *The fuzzy coefficient $\tilde{A}_1$ and $\tilde{B}_1$ for the fully fuzzy matrix equation as shown in Eq.(1) must be square matrices.*

Proof. Let

$$(\tilde{A}_1)_{n\times n}\tilde{X}_{n\times p} + \tilde{X}_{n\times p}(\tilde{B}_1)_{p\times p} = \tilde{C}_{n\times p}$$
$$(\tilde{A}_1\tilde{X})_{n\times p} + (\tilde{X}\tilde{B}_1)_{n\times p} = \tilde{C}_{n\times p}$$

be the fully fuzzy matrix equation as shown in Eq.(1), where $\tilde{A}_1$ and $\tilde{B}_1$ are fuzzy coefficients and $\tilde{X}_{n\times p}$ is the fuzzy solution. If the fuzzy coefficents $\tilde{A}_1$ and $\tilde{B}_1$ are non-square with order $(\tilde{A}_1)_{m\times n}$ and $(\tilde{B}_1)_{p\times q}$, and the solution is $\tilde{X}_{n\times p}$, then

$$(\tilde{A}_1)_{m\times n}\tilde{X}_{n\times p} + \tilde{X}_{n\times p}(\tilde{B}_1)_{p\times q}$$
$$(\tilde{A}_1\tilde{X})_{m\times p} + (\tilde{X}\tilde{B}_1)_{n\times q}.$$

However, the addition of $(\tilde{A}_1\tilde{X})_{m\times p}$ and $(\tilde{X}\tilde{B}_1)_{n\times q}$ is invalid due to the difference in sizes. Thus, in all cases, $\tilde{A}_1$ and $\tilde{B}_1$ must be square matrices. □

Theorem 3.2. *The fuzzy coefficient $\tilde{A}_2$ and $\tilde{B}_2$ for the fully fuzzy matrix equation as shown in Eq.(2) must be square matrices.*

Proof. Let

$$(\tilde{A}_2)_{n\times n}\tilde{X}_{n\times p}(\tilde{B}_2)_{p\times p} - \tilde{X}_{n\times p} = \tilde{C}_{n\times p}$$
$$(\tilde{A}_2\tilde{X}\tilde{B}_2)_{n\times p} - \tilde{X}_{n\times p} = \tilde{C}_{n\times p}$$

be the fully fuzzy matrix equation as shown in Eq.(2), where $\tilde{A}_2$ and $\tilde{B}_2$ are fuzzy coefficients and $\tilde{X}_{n\times p}$ is the fuzzy solution. If the fuzzy coefficents $\tilde{A}_2$ and $\tilde{B}_2$ are non-square with order $(\tilde{A}_2)_{r\times n}$ and $(\tilde{B}_2)_{p\times s}$, and the solution is $\tilde{X}_{n\times p}$, then

$$(\tilde{A}_2)_{r\times n}\tilde{X}_{n\times p}(\tilde{B}_2)_{p\times s} - \tilde{X}_{n\times p}$$
$$(\tilde{A}_2\tilde{X}\tilde{B}_2)_{r\times s} - \tilde{X}_{n\times p}.$$

However, the subtraction of $(\tilde{A}_2\tilde{X}\tilde{B}_2)_{n\times s}$ and $\tilde{X}_{n\times p}$ is invalid due to the difference in sizes. Thus, in all cases, $\tilde{A}_2$ and $\tilde{B}_2$ must be square matrices. □

Corollary 3.1. *Let $\tilde{A}_1 = (\tilde{a}_{1ij})_{p\times p}$, $\tilde{A}_2 = (\tilde{a}_{2ij})_{p\times p}$ and $\tilde{B}_1 = (\tilde{b}_{1ij})_{q\times q}$, $\tilde{B}_2 = (\tilde{b}_{2ij})_{q\times q}$, and $\tilde{X} = (\tilde{x}_{ij})_{p\times q}$. A pair fully fuzzy matrix equation (PFFME) as in Eq.(3) can be expressed in the form*

$$\begin{cases}(\tilde{U}_q \otimes_k \tilde{A}_1) + (\tilde{B}_1^T \otimes_k \tilde{U}_p)\tilde{X}v = \tilde{C}v_1 \\ (\tilde{B}_2^T \otimes_k \tilde{A}_2) - \tilde{U}_{pq})\tilde{X}v = \tilde{C}v_2\end{cases} \tag{18}$$

where

$$\tilde{C}v_1 = Vec(\tilde{C}_1); \quad \tilde{C}v_2 = Vec(\tilde{C}_2); \quad \tilde{X}v = Vec(\tilde{X}).$$

Proof. The proof follows from Theorem 2.5 and 2.6. □

4. Solution for Positive PFFME

Now, the algorithm for finding the solution $\tilde{X} = (m^x, \alpha^x, \beta^x)$ of PFFME is outlined.

Step 1: Transforming the PFFME to pair FFLS.

Considering Corollary 3.1, let

$$\begin{cases} (\tilde{U}_q \otimes_k \tilde{A}_1) + (\tilde{B}_1^T \otimes_k \tilde{U}_p) = \tilde{L}_1 \\ (\tilde{B}_2^T \otimes_k \tilde{A}_2) - \tilde{U}_{pq} = \tilde{L}_2, \end{cases} \tag{19}$$

then an equation of pair FFLS is given by

$$\begin{cases} \tilde{L}_1 \tilde{X}v = \tilde{C}v_1 \\ \tilde{L}_2 \tilde{X}v = \tilde{C}v_2, \end{cases} \tag{20}$$

Step 2: Converting the equation of pair FFLS in a form of associated linear system.

The conversion of pair FFLS in Eq.(20) is done separately due to distraction of the fuzzy operators. The addition property fails on both fuzzy coefficients $\tilde{L}_1$ and $\tilde{L}_2$. Thus, considering the fuzzy numbers, $\tilde{L}_1 = (F_1, M_1, N_1)$, $\tilde{L}_2 = (F_2, M_2, N_2), \tilde{C}v_1 = (m^{c_1}, \alpha^{c_1}, \beta^{c_1}), \tilde{C}v_2 = (m^{c_2}, \alpha^{c_2}, \beta^{c_2})$ and $\tilde{X}v = (m^x, \alpha^x, \beta^x)$, the equation can be rewritten as

$$\begin{cases} (F_1, M_1, N_1)(m^x, \alpha^x, \beta^x) = (m^{c_1}, \alpha^{c_1}, \beta^{c_1}) \\ (F_2, M_2, N_2)(m^x, \alpha^x, \beta^x) = (m^{c_2}, \alpha^{c_2}, \beta^{c_2}). \end{cases}$$

By fuzzy arithmetic multiplication, we obtain

$$\begin{cases} F_1 m^x = m^{c_1} \\ F_1 \alpha^x + M_1 m^x = \alpha^{c_1} \\ F_1 \beta^x + N_1 m^x = \beta^{c_1} \\ F_2 m^x = m^{c_2} \\ F_2 \alpha^x + M_2 m^x = \alpha^{c_2} \\ F_2 \beta^x + N_2 m^x = \beta^{c_2}, \end{cases}$$

which can be rearranged as follows

$$\begin{cases} F_1 m^x = m^{c_1} \\ F_2 m^x = m^{c_2} \\ F_1 \alpha^x + M_1 m^x = \alpha^{c_1} \\ F_2 \alpha^x + M_2 m^x = \alpha^{c_2} \\ F_1 \beta^x + N_1 m^x = \beta^{c_1} \\ F_2 \beta^x + N_2 m^x = \beta^{c_2}. \end{cases}$$

Since $(F_1, M_1, N_1), (m^{c_1}, \alpha^{c_1}, \beta^{c_1})$ and $(F_2, M_2, N_2), (m^{c_2}, \alpha^{c_2}, \beta^{c_2})$ have similar common sizes respectively, it follows that

$$\begin{cases} (F_1 + F_2)m^x = (m^{c_1} + m^{c_2}) \\ (F_1 + F_2)\alpha^x + (M_1 + M_2)m^x = (\alpha^{c_1} + \alpha^{c_2}) \\ (F_1 + F_2)\beta^x + (N_1 + N_2)m^x = (\beta^{c_1} + \beta^{c_2}). \end{cases} \tag{21}$$

By assuming that the sum of two positive coefficient matrices are positive, we can let $F_1 + F_2 = F, M_1 + M_2 = M, N_1 + N_2 = N, m^{c_1} + m^{c_2} = m^c$, $\alpha^{c_1} + \alpha^{c_2} = \alpha^c$ and $\beta^{c_1} + \beta^{c_2} = \beta^c$. Therefore, Eq.(21) can be simplified into

$$\begin{cases} Fm^x = m^c \\ F\alpha^x + Mm^x = \alpha^c \\ F\beta^x + Nm^x = \beta^c. \end{cases} \tag{22}$$

Step 3: Composing the system in Eq.(22) to be an associated linear system of

$$\begin{pmatrix} F & 0 & 0 \\ M & F & 0 \\ N & 0 & F \end{pmatrix} \begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} m^c \\ \alpha^c \\ \beta^c \end{pmatrix} \tag{23}$$

where,

$$F = \begin{pmatrix} m_{11}^{F_1} + m_{11}^{F_2} & \dots & m_{1q}^{F_1} + m_{1q}^{F_2} \\ \vdots & \ddots & \vdots \\ m_{p1}^{F_1} + m_{p1}^{F_2} & \dots & m_{pq}^{F_1} + m_{pq}^{F_2} \end{pmatrix}, \quad M = \begin{pmatrix} \alpha_{11}^{F_1} + \alpha_{11}^{F_2} & \dots & \alpha_{1q}^{F_1} + \alpha_{1q}^{F_2} \\ \vdots & \ddots & \vdots \\ \alpha_{p1}^{F_1} + \alpha_{p1}^{F_2} & \dots & \alpha_{pq}^{F_1} + \alpha_{pq}^{F_2} \end{pmatrix},$$

$$N = \begin{pmatrix} \beta_{11}^{F_1} + \beta_{11}^{F_2} & \dots & \beta_{1q}^{F_1} + \beta_{1q}^{F_2} \\ \vdots & \ddots & \vdots \\ \beta_{p1}^{F_1} + \beta_{p1}^{F_2} & \dots & \beta_{pq}^{F_1} + \beta_{pq}^{F_2} \end{pmatrix},$$

$$m^c = \begin{pmatrix} m_{11}^{c_1} + m_{11}^{c_2} \\ \vdots \\ m_{pq}^{c_1} + m_{pq}^{c_2} \end{pmatrix}; \quad \alpha^c = \begin{pmatrix} \alpha_{11}^{c_1} + \alpha_{11}^{c_2} \\ \vdots \\ \alpha_{pq}^{c_1} + \alpha_{pq}^{c_2} \end{pmatrix}; \quad \beta^c = \begin{pmatrix} \beta_{11}^{c_1} + \beta_{11}^{c_2} \\ \vdots \\ \beta_{pq}^{c_1} + \beta_{pq}^{c_2}; \end{pmatrix}$$

$$m^x = \begin{pmatrix} m_{11}^x \\ \vdots \\ m_{pq}^x \end{pmatrix}; \quad \alpha^x = \begin{pmatrix} \alpha_{11}^x \\ \vdots \\ \alpha_{pq}^x \end{pmatrix}; \quad \beta^x \begin{pmatrix} \beta_{11}^x \\ \vdots \\ \beta_{pq}^x \end{pmatrix}.$$

Since PFFME can be converted to an associated linear system as in Eq.(23) and the solutions of the associated linear system is equivalent to the solutions of $\tilde{A}\tilde{X} = \tilde{C}$ (according to Theorem 2.3), it follows that the solutions of Eq.(23) is also equivalent to the solutions of PFFME. In order to obtain the solutions of Eq.(23), any classical linear algebra method such as inverse method, Cramers' rule, Gaussian elimination, LU decomposition or the iterative methods can be applied. In this paper, we implemented the inversion method.

Therefore,

$$\begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} F & 0 & 0 \\ M & F & 0 \\ N & 0 & F \end{pmatrix}^{-1} \begin{pmatrix} m^c \\ \alpha^c \\ \beta^c \end{pmatrix} \tag{24}$$

which is,

$$\begin{pmatrix} m^x \\ \alpha^x \\ \beta^x \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} m^x_{11} \\ \vdots \\ m^x_{pq} \end{pmatrix} \\ \begin{pmatrix} \alpha^x_{11} \\ \vdots \\ \alpha^x_{pq} \end{pmatrix} \\ \begin{pmatrix} \beta^x_{11} \\ \vdots \\ \beta^x_{pq} \end{pmatrix} \end{pmatrix}.$$

Thus, the solution is given by

$$\tilde{X} = \begin{pmatrix} (m^x_{11}, \alpha^x_{11}, \beta^x_{11}) & \dots & (m^x_{1q}, \alpha^x_{1q}, \beta^x_{1q}) \\ \vdots & \ddots & \vdots \\ (m^x_{p1}, \alpha^x_{p1}, \beta^x_{p1}) & \dots & (m^x_{pq}, \alpha^x_{pq}, \beta^x_{pq}) \end{pmatrix}.$$

Remark 4.1. *The fuzzy solution obtained is a unique fuzzy solution if and only if the matrix $F_1 + F_2$ in Eq.(21) is non-singular, which directly follows from Theorem 2.2.*

5. Numerical Example

Example 5.1. *Consider the pair fully fuzzy matrix equation,*

$$\begin{cases} \tilde{A}_1\tilde{X} + \tilde{X}\tilde{B}_1 = \tilde{C}_1 \\ \tilde{A}_2\tilde{X}\tilde{B}_2 - \tilde{X} = \tilde{C}_2 \end{cases}$$

where

$$\tilde{A}_1 = \begin{pmatrix} (8,4,7) & (5,4,5) \\ (9,6,1) & (7,2,7) \end{pmatrix}$$
$$\tilde{A}_2 = \begin{pmatrix} (5,1,1) & (6,1,2) \\ (7,1,0) & (4,0,1) \end{pmatrix}$$
$$\tilde{B}_1 = \begin{pmatrix} (6,6,7) & (5,4,3) & (7,2,3) \\ (9,3,2) & (7,1,7) & (5,3,1) \\ (8,5,3) & (5,2,3) & (1,1,7) \end{pmatrix}$$
$$\tilde{B}_2 = \begin{pmatrix} (6,2,3) & (5,1,3) & (7,1,3) \\ (9,3,2) & (7,1,7) & (6,3,1) \\ (8,1,3) & (9,2,3) & (10,1,7) \end{pmatrix}$$
$$\tilde{C}_1 = \begin{pmatrix} (259,316,349) & (195,214,253) & (193,237,286) \\ (241,307,386) & (180,230,228) & (208,269,274) \end{pmatrix}$$
$$\tilde{C}_2 = \begin{pmatrix} (1635,1880,2132) & (1537,1711,2251) & (1725,1884,2507) \\ (1713,1783,1780) & (1613,1628,1979) & (1787,1783,2179) \end{pmatrix}$$

and

$$\tilde{X} = \begin{pmatrix} (m^x_{11},\alpha^x_{11},\beta^x_{11}) & (m^x_{12},\alpha^x_{12},\beta^x_{12}) & (m^x_{13},\alpha^x_{13},\beta^x_{13}) \\ (m^x_{21},\alpha^x_{21},\beta^x_{21}) & (m^x_{22},\alpha^x_{22},\beta^x_{22}) & (m^x_{23},\alpha^x_{23},\beta^x_{23}) \end{pmatrix} \geq 0 \qquad (25)$$

Solution:

Step 1: The PFFME is transformed to FFLS according to its Kronecker properties.

$$(\tilde{U}_3 \otimes_k \tilde{A}_1) + (\tilde{B}_1^T \otimes_k \tilde{U}_2) =$$

$$\begin{pmatrix} (14,10,14) & (5,4,5) & (9,3,2) & (0,0,0) & (8,5,3) & (0,0,0) \\ (9,6,1) & (13,8,14) & (0,0,0) & (9,3,2) & (0,0,0) & (8,5,3) \\ (5,4,3) & (0,0,0) & (15,5,14) & (5,4,5) & (5,2,3) & (0,0,0) \\ (0,0,0) & (5,4,3) & (9,6,1) & (14,3,14) & (0,0,0) & (5,2,3) \\ (7,2,3) & (0,0,0) & (5,3,1) & (0,0,0) & (9,5,14) & (5,4,5) \\ (0,0,0) & (7,2,3) & (0,0,0) & (5,3,1) & (9,6,1) & (8,3,14) \end{pmatrix},$$

$$Vec(\tilde{C}_1) =$$

$$\begin{pmatrix} (259,316,349) \\ (241,307,386) \\ (195,214,253) \\ (180,230,228) \\ (193,237,286) \\ (208,269,274) \end{pmatrix}$$

and

$$(\tilde{B}_2^T \otimes_k \tilde{A}_2) - \tilde{U}_6 =$$

$$\begin{pmatrix} (29,16,21) & (36,18,30) & (45,24,19) & (54,27,30) & (40,13,23) & (48,14,34) \\ (42,20,21) & (23,8,18) & (63,30,14) & (36,12,17) & (56,15,21) & (32,4,20) \\ (25,10,20) & (30,11,28) & (34,12,42) & (42,13,56) & (45,19,24) & (54,21,36) \\ (35,12,21) & (20,4,17) & (49,14,49) & (27,4,35) & (63,23,21) & (36,8,21) \\ (35,12,22) & (42,13,32) & (30,21,11) & (36,24,18) & (49,15,45) & (60,16,62) \\ (49,14,21) & (28,4,19) & (42,27,7) & (24,12,10) & (70,17,49) & (39,4,38) \end{pmatrix},$$

$$Vec(\tilde{C}_2) =$$

$$\begin{pmatrix} (1635,1880,2132) \\ (1713,1783,1780) \\ (1537,1711,2251) \\ (1613,1628,1979) \\ (1725,1884,2507) \\ (1787,1783,2179) \end{pmatrix}.$$

Step 2: In order to achieve an associated linear system, all matrices obtained in previous step need to be converted to crisp form of matrices. Then,

$(\tilde{U}_3 \otimes_k \tilde{A}_1) + (\tilde{B}_1^T \otimes_k \tilde{U}_2)$ yields

$$F_1 = \begin{pmatrix} 14 & 5 & 9 & 0 & 8 & 0 \\ 9 & 13 & 0 & 9 & 0 & 8 \\ 5 & 0 & 15 & 5 & 5 & 0 \\ 0 & 5 & 9 & 14 & 0 & 5 \\ 7 & 0 & 5 & 0 & 9 & 5 \\ 0 & 7 & 0 & 5 & 9 & 8 \end{pmatrix}, M_1 = \begin{pmatrix} 10 & 4 & 3 & 0 & 5 & 0 \\ 6 & 8 & 0 & 3 & 0 & 5 \\ 4 & 0 & 5 & 4 & 2 & 0 \\ 0 & 4 & 6 & 3 & 0 & 2 \\ 2 & 0 & 3 & 0 & 5 & 4 \\ 0 & 2 & 0 & 3 & 6 & 3 \end{pmatrix}, N_1 = \begin{pmatrix} 14 & 5 & 2 & 0 & 3 & 0 \\ 1 & 14 & 0 & 2 & 0 & 3 \\ 3 & 0 & 14 & 5 & 3 & 0 \\ 0 & 3 & 1 & 14 & 0 & 3 \\ 3 & 0 & 1 & 0 & 14 & 5 \\ 0 & 3 & 0 & 1 & 1 & 14 \end{pmatrix}$$

and $Vec(\tilde{C}_1)$ yields

$$m^{c_1} = \begin{pmatrix} 259 \\ 241 \\ 195 \\ 180 \\ 193 \\ 208 \end{pmatrix}, \quad \alpha^{c_1} = \begin{pmatrix} 316 \\ 307 \\ 214 \\ 230 \\ 237 \\ 269 \end{pmatrix}, \quad \beta^{c_1} = \begin{pmatrix} 349 \\ 386 \\ 253 \\ 228 \\ 286 \\ 274 \end{pmatrix}.$$

On the other hand, the crisp matrices extracted from $(\tilde{B}_2^T \otimes_k \tilde{A}_2) - \tilde{U}_6$ yields

$$F_2 = \begin{pmatrix} 29 & 36 & 45 & 54 & 40 & 48 \\ 42 & 23 & 63 & 36 & 56 & 32 \\ 25 & 30 & 34 & 42 & 45 & 54 \\ 35 & 20 & 49 & 27 & 63 & 36 \\ 35 & 42 & 30 & 36 & 49 & 60 \\ 49 & 28 & 42 & 34 & 70 & 39 \end{pmatrix}, M_2 = \begin{pmatrix} 16 & 18 & 24 & 27 & 13 & 14 \\ 20 & 8 & 30 & 12 & 15 & 4 \\ 10 & 11 & 12 & 13 & 19 & 21 \\ 12 & 4 & 14 & 4 & 23 & 8 \\ 12 & 13 & 21 & 24 & 15 & 16 \\ 14 & 4 & 27 & 12 & 17 & 4 \end{pmatrix},$$

$$N_2 = \begin{pmatrix} 21 & 30 & 19 & 30 & 23 & 34 \\ 21 & 18 & 14 & 17 & 21 & 20 \\ 20 & 28 & 42 & 56 & 24 & 36 \\ 21 & 17 & 49 & 35 & 21 & 21 \\ 22 & 32 & 11 & 18 & 45 & 62 \\ 21 & 19 & 7 & 10 & 49 & 38 \end{pmatrix},$$

and $Vec(\tilde{C}_2)$ yields

$$m^{c_2} = \begin{pmatrix} 1635 \\ 1713 \\ 1537 \\ 1613 \\ 1725 \\ 1787 \end{pmatrix}, \quad \alpha^{c_2} = \begin{pmatrix} 1880 \\ 1783 \\ 1711 \\ 1628 \\ 1884 \\ 1783 \end{pmatrix}, \quad \beta^{c_2} = \begin{pmatrix} 2132 \\ 1780 \\ 2251 \\ 1979 \\ 2507 \\ 2179 \end{pmatrix}.$$

Thus, Eq.(22) can be written as follows

$$\begin{pmatrix} 43 & 41 & 54 & 54 & 48 & 48 \\ 51 & 36 & 63 & 45 & 56 & 40 \\ 30 & 30 & 49 & 47 & 50 & 54 \\ 35 & 25 & 58 & 41 & 63 & 41 \\ 42 & 42 & 35 & 36 & 58 & 65 \\ 49 & 35 & 42 & 29 & 79 & 47 \end{pmatrix} \begin{pmatrix} m^x_{11} \\ m^x_{21} \\ m^x_{12} \\ m^x_{22} \\ m^x_{13} \\ m^x_{23} \end{pmatrix} = \begin{pmatrix} 1894 \\ 1954 \\ 1732 \\ 1793 \\ 1918 \\ 1995 \end{pmatrix}$$

$$\begin{pmatrix} 26 & 22 & 27 & 27 & 18 & 14 \\ 26 & 16 & 30 & 15 & 15 & 9 \\ 14 & 11 & 17 & 17 & 21 & 21 \\ 12 & 8 & 20 & 7 & 23 & 10 \\ 14 & 13 & 24 & 24 & 20 & 20 \\ 14 & 6 & 27 & 15 & 23 & 7 \end{pmatrix} \begin{pmatrix} \alpha^x_{11} \\ \alpha^x_{21} \\ \alpha^x_{12} \\ \alpha^x_{22} \\ \alpha^x_{13} \\ \alpha^x_{23} \end{pmatrix} + \begin{pmatrix} 43 & 41 & 54 & 54 & 48 & 48 \\ 51 & 36 & 63 & 45 & 56 & 40 \\ 30 & 30 & 49 & 47 & 50 & 54 \\ 35 & 25 & 58 & 41 & 63 & 41 \\ 42 & 42 & 35 & 36 & 58 & 65 \\ 49 & 35 & 42 & 29 & 79 & 47 \end{pmatrix} \begin{pmatrix} m^x_{11} \\ m^x_{21} \\ m^x_{12} \\ m^x_{22} \\ m^x_{13} \\ m^x_{23} \end{pmatrix} = \begin{pmatrix} 2196 \\ 2090 \\ 1925 \\ 1858 \\ 2121 \\ 2052 \end{pmatrix}$$

$$\begin{pmatrix} 25 & 25 & 21 & 30 & 26 & 34 \\ 22 & 32 & 14 & 19 & 21 & 23 \\ 23 & 28 & 56 & 61 & 27 & 36 \\ 21 & 20 & 50 & 49 & 21 & 24 \\ 25 & 32 & 12 & 18 & 59 & 67 \\ 21 & 22 & 7 & 11 & 50 & 52 \end{pmatrix} \begin{pmatrix} \beta^x_{11} \\ \beta^x_{21} \\ \beta^x_{12} \\ \beta^x_{22} \\ \beta^x_{13} \\ \beta^x_{23} \end{pmatrix} + \begin{pmatrix} 43 & 41 & 54 & 54 & 48 & 48 \\ 51 & 36 & 63 & 45 & 56 & 40 \\ 30 & 30 & 49 & 47 & 50 & 54 \\ 35 & 25 & 58 & 41 & 63 & 41 \\ 42 & 42 & 35 & 36 & 58 & 65 \\ 49 & 35 & 42 & 29 & 79 & 47 \end{pmatrix} \begin{pmatrix} m^x_{11} \\ m^x_{21} \\ m^x_{12} \\ m^x_{22} \\ m^x_{13} \\ m^x_{23} \end{pmatrix} = \begin{pmatrix} 2481 \\ 2166 \\ 2504 \\ 2207 \\ 2793 \\ 2453 \end{pmatrix}.$$

Step 3: A crisp matrix equation is obtained from Eq.(23) as follows

$$\left(\begin{array}{cccccc|cccccc|cccccc} 43 & 41 & 54 & 54 & 48 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 51 & 36 & 63 & 45 & 56 & 40 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 30 & 30 & 49 & 47 & 50 & 54 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 35 & 25 & 58 & 41 & 63 & 41 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 42 & 42 & 35 & 36 & 58 & 65 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 49 & 35 & 42 & 29 & 79 & 47 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 26 & 22 & 27 & 27 & 18 & 14 & 43 & 41 & 54 & 54 & 48 & 48 & 0 & 0 & 0 & 0 & 0 & 0 \\ 26 & 16 & 30 & 15 & 15 & 9 & 51 & 36 & 63 & 45 & 56 & 40 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14 & 11 & 17 & 17 & 21 & 21 & 30 & 30 & 49 & 47 & 50 & 54 & 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 8 & 20 & 7 & 23 & 10 & 35 & 25 & 58 & 41 & 63 & 41 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14 & 13 & 24 & 24 & 20 & 20 & 42 & 42 & 35 & 36 & 58 & 65 & 0 & 0 & 0 & 0 & 0 & 0 \\ 14 & 6 & 27 & 15 & 23 & 7 & 49 & 35 & 42 & 29 & 79 & 47 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 25 & 25 & 21 & 30 & 26 & 34 & 0 & 0 & 0 & 0 & 0 & 0 & 43 & 41 & 54 & 54 & 48 & 48 \\ 22 & 32 & 14 & 19 & 21 & 23 & 0 & 0 & 0 & 0 & 0 & 0 & 51 & 36 & 63 & 45 & 56 & 40 \\ 23 & 28 & 56 & 61 & 27 & 36 & 0 & 0 & 0 & 0 & 0 & 0 & 30 & 30 & 49 & 47 & 50 & 54 \\ 21 & 20 & 50 & 49 & 21 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 35 & 25 & 58 & 41 & 63 & 41 \\ 25 & 32 & 12 & 18 & 59 & 67 & 0 & 0 & 0 & 0 & 0 & 0 & 42 & 42 & 35 & 36 & 58 & 65 \\ 21 & 22 & 7 & 11 & 50 & 52 & 0 & 0 & 0 & 0 & 0 & 0 & 49 & 35 & 42 & 29 & 79 & 47 \end{array}\right) \left(\begin{array}{c} m^x_{11} \\ m^x_{21} \\ m^x_{12} \\ m^x_{22} \\ m^x_{13} \\ m^x_{23} \\ \hline \alpha^x_{11} \\ \alpha^x_{21} \\ \alpha^x_{12} \\ \alpha^x_{22} \\ \alpha^x_{13} \\ \alpha^x_{23} \\ \hline \beta^x_{11} \\ \beta^x_{21} \\ \beta^x_{12} \\ \beta^x_{22} \\ \beta^x_{13} \\ \beta^x_{23} \end{array}\right) = \left(\begin{array}{c} 1894 \\ 1954 \\ 1732 \\ 1793 \\ 1918 \\ 1995 \\ \hline 2196 \\ 2090 \\ 1925 \\ 1858 \\ 2121 \\ 2052 \\ \hline 2481 \\ 2166 \\ 2504 \\ 2207 \\ 2793 \\ 2453 \end{array}\right).$$

Therefore,

$$X = \left(\begin{array}{cccccc|cccccc|cccccc}
43 & 41 & 54 & 54 & 48 & 48 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
51 & 36 & 63 & 45 & 56 & 40 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
30 & 30 & 49 & 47 & 50 & 54 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
35 & 25 & 58 & 41 & 63 & 41 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
42 & 42 & 35 & 36 & 58 & 65 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
49 & 35 & 42 & 29 & 79 & 47 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
26 & 22 & 27 & 27 & 18 & 14 & 43 & 41 & 54 & 54 & 48 & 48 & 0 & 0 & 0 & 0 & 0 & 0 \\
26 & 16 & 30 & 15 & 15 & 9 & 51 & 36 & 63 & 45 & 56 & 40 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 11 & 17 & 17 & 21 & 21 & 30 & 30 & 49 & 47 & 50 & 54 & 0 & 0 & 0 & 0 & 0 & 0 \\
12 & 8 & 20 & 7 & 23 & 10 & 35 & 25 & 58 & 41 & 63 & 41 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 13 & 24 & 24 & 20 & 20 & 42 & 42 & 35 & 36 & 58 & 65 & 0 & 0 & 0 & 0 & 0 & 0 \\
14 & 6 & 27 & 15 & 23 & 7 & 49 & 35 & 42 & 29 & 79 & 47 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
25 & 25 & 21 & 30 & 26 & 34 & 0 & 0 & 0 & 0 & 0 & 0 & 43 & 41 & 54 & 54 & 48 & 48 \\
22 & 32 & 14 & 19 & 21 & 23 & 0 & 0 & 0 & 0 & 0 & 0 & 51 & 36 & 63 & 45 & 56 & 40 \\
23 & 28 & 56 & 61 & 27 & 36 & 0 & 0 & 0 & 0 & 0 & 0 & 30 & 30 & 49 & 47 & 50 & 54 \\
21 & 20 & 50 & 49 & 21 & 24 & 0 & 0 & 0 & 0 & 0 & 0 & 35 & 25 & 58 & 41 & 63 & 41 \\
25 & 32 & 12 & 18 & 59 & 67 & 0 & 0 & 0 & 0 & 0 & 0 & 42 & 42 & 35 & 36 & 58 & 65 \\
21 & 22 & 7 & 11 & 50 & 52 & 0 & 0 & 0 & 0 & 0 & 0 & 49 & 35 & 42 & 29 & 79 & 47
\end{array}\right)^{-1}
\left(\begin{array}{c}
1894 \\ 1954 \\ 1732 \\ 1793 \\ 1918 \\ 1995 \\ \hline 2196 \\ 2090 \\ 1925 \\ 1858 \\ 2121 \\ 2052 \\ \hline 2481 \\ 2166 \\ 2504 \\ 2207 \\ 2793 \\ 2453
\end{array}\right)$$

which gives

$$X = \begin{pmatrix} 6 \\ 8 \\ 7 \\ 3 \\ 9 \\ 7 \\ 2 \\ 6 \\ 4 \\ 3 \\ 8 \\ 5 \\ 7 \\ 9 \\ 1 \\ 5 \\ 4 \\ 2 \end{pmatrix} \quad \text{or} \quad X = \left(\begin{array}{c} \begin{pmatrix} m^x_{11} \\ m^x_{21} \\ m^x_{12} \\ m^x_{22} \\ m^x_{13} \\ m^x_{23} \end{pmatrix} \\ \begin{pmatrix} \alpha^x_{11} \\ \alpha^x_{21} \\ \alpha^x_{12} \\ \alpha^x_{22} \\ \alpha^x_{13} \\ \alpha^x_{23} \end{pmatrix} \\ \begin{pmatrix} \beta^x_{11} \\ \beta^x_{21} \\ \beta^x_{12} \\ \beta^x_{22} \\ \beta^x_{13} \\ \beta^x_{23} \end{pmatrix} \end{array} \right) = \left(\begin{array}{c} \begin{pmatrix} 6 \\ 8 \\ 7 \\ 3 \\ 9 \\ 7 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 6 \\ 4 \\ 3 \\ 8 \\ 5 \end{pmatrix} \\ \begin{pmatrix} 7 \\ 9 \\ 1 \\ 5 \\ 4 \\ 2 \end{pmatrix} \end{array} \right)$$

Hence, the fuzzy solution is given by

$$\begin{aligned}\tilde{X} &= \begin{pmatrix} (m^x_{11}, \alpha^x_{11}, \beta^x_{11}) & (m^x_{12}, \alpha^x_{12}, \beta^x_{12}) & (m^x_{13}, \alpha^x_{13}, \beta^x_{13}) \\ (m^x_{21}, \alpha^x_{21}, \beta^x_{21}) & (m^x_{22}, \alpha^x_{22}, \beta^x_{22}) & (m^x_{23}, \alpha^x_{23}, \beta^x_{23}) \end{pmatrix} \\ &= \begin{pmatrix} (6,2,7) & (7,4,1) & (9,8,4) \\ (8,6,9) & (3,3,5) & (7,5,2) \end{pmatrix}.\end{aligned}$$

6. Conclusion

This study has constructed an algorithm for solving the PFFME, where the coefficients and the solution $\tilde{X}$ are positive fuzzy numbers. The proposed algorithm utilizes the fuzzy Kronecker product, fuzzy Vec-operator and also associated linear system approach in obtaining the solution, which is the main contribution of this study. The associated linear system transforms the fully fuzzy linear system to crisp linear system. The final solution is obtained by applying any method of crisp linear system. As a result, the proposed algorithm provides a new significant contribution in this particular area. In future, our proposed algorithm can be modified to solve negative, near-zero and mixed type of fuzzy coefficient, which is more representable for real applications.

Acknowledgements

The authors wishes to thank the Awang Had Salleh Graduate School (AHSGS), Universiti Utara Malaysia for funding this study under the Postgraduate Research Grant Scheme, S/O codes 15970. The authors would also like to express their gratitude to the anonymous referees for the constructive comments to improve this study.

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