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Commuting Differential Operators with Elliptic Coefficients

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ABSTRACT

Ordinary Differential Operators with elliptic coefficients have strong geometric properties and a long history which received new impetus in the context of integrable hierarchies. Partial Differential Operators with the same property, on the contrary, have hardly been investigated. We bring together results on the former, together with many open problems, and an extension to hyperelliptic coefficients; and propose and exemplify new methods on the latter. In both contexts, our emphasis is on maximal commuting algebras of operators over the complex numbers.

Keywords: Elliptic function, Maximal-commutative algebras in the ring of differential operators.

1. Introduction

The theory of Ordinary Differential Operators (ODOs) with elliptic coefficients has a long history. The Lamé operator:

$$\frac{d^2}{d\xi^2} + c\wp(\xi),$$

where $\wp(\xi)$ is Weierstrass' elliptic function for a curve

$$y^2 = (x - e_1)(x - e_2)(x - e_3)$$

and c is a complex number, has eigenfunctions called Lamé functions; when expanded in powers of $\wp(\xi)$ or $(\wp(\xi) - e_i)^{1/2}$ (for even, respectively odd n), the expansion terminates provided $c = n(n+1)$, with n a positive integer. Interest in this classical result due to E.L. Ince was revived in the 1980s in the context of the *finite-gap* solutions of certain integrable hierarchies of Partial Differential Equations (PDEs), such as Kortweg-de Vries (KdV) and Kadomtsev-Petviashvili (KP). An ODO $L = D^m + u_{m-2}(\xi)D^{m-2} + \cdots + u_s(\xi)$ $(D = d/d\xi)$ in normalized form can be deformed according to Sato's equation

$$\frac{\partial}{\partial t_i} L = [(L^{i/m})_+, L], \tag{1}$$

where the coefficients u_j now become functions of a sequence of variables, $t_1 = \xi, t_2, ...$, the algebraic operations take place in the ring Ψ of formal Pseudo Differential Operators (ΨDOs), and for an element $S = \sum_{j=-N}^{\infty} u_j(\xi)\partial^j \in \Psi$ (where N is a positive integer, u_j are formal power series and we replaced D by ∂), the notation $(S)_+$ means that the Laurent tail is truncated: $(S)_+ = \sum_{j\geq 0} u_j(\xi)\partial^j$. Equation (1) gives the'm reduction of the KP hierarchy, in particular the KdV hierarchy when $m = 2$. When the centralizer $\mathcal{C}(L)$ of L in the ring of ODOs is larger than the ring $\mathbb{C}[T]$, for an ODO T such that $L \in \mathbb{C}[T]$, and the greatest common divisor (gcd) of the orders of elements of $\mathcal{C}(L)$ is 1, the KP solutions are called *finite gap* and can be written explicitly in terms of theta functions for the *spectral curve* Spec $\mathcal{C}(L)$.

We call this the *Burchnall-Chaundy* case of an ODO, after the authors who asked the question and found a solution in the early 20th century in Burchnall and Chaundy (1928), but there are a great many unsettled issues, of which we review the most relevant to this paper in Section Rank One.

In particular, we highlight configurations of coverings of spectral curves that occur when the coefficients are elliptic, and give a new result, which excludes certain configurations and will be applied to the case of hyperelliptic coefficients, illustrated in the following Section. In Section Ultra-elliptic, we propose a generalization of the theory that has not been developed, namely, we construct operators whose coefficients are Abelian functions defined over a torus of dimension two, the Jacobian of a curve of genus two; classically such functions were called *ultra-elliptic*.

However, Spec $\mathcal{C}(L)$ can be an elliptic curve even for an L that does not have elliptic coefficients; this case is much more mysterious and we cover it in Section Rank Two.

Finally, analogous questions for PDOs are all but unexplored in Section Partial Differential Operators, we give the references that we are aware of, and propose a different method which would be more algebraic in nature. This calls for a refinement of the theory of differential resultants, and we describe a current project joint with Lewis (2010).

2. Rank One

In the subring $\mathcal{D} \subset \Psi$ of ODOs, $\mathcal{D} = \{\sum_0^s u_j \partial^j,\ s \geq 0\}$ we define the rank of a subset of $\mathcal{D}$ as the greatest common divisor of the orders of all the elements of $\mathcal{D}$.

Burchnall and Chaundy (1928) essentially proved that the centralizers of rank one are affine rings of algebraic curves; provided such centralizer is not a polynomial ring, the KP time deformations then take place on the Jacobian of the curve, a g-dimensional complex torus where g is the genus of the curve. Any non-constant L in the centralizer is called *finite-gap* because in the original Lamé (periodic) case, the Floquet spectrum is then finite gap. In Gesztesy and Weikard (1999) gives a review of their results) revived a theorem of Picard's and proved that when $L = \partial^2 + u(\xi)$, u is an elliptic finite-gap potential if and only if u is a Picard potential (i.e., if and only if for each $E \in \mathbb{C}$ every solution of $\psi''(\xi) + u(\xi)\psi(\xi) = E\psi(\xi)$ is meromorphic with respect to ξ).

However, even though in Gesztesy and Weikard (1999) offer an algorithm that computes the genus of the spectral curve, its explicit equation requires further work. In Eilbeck et al. (2007), using the eigenfuntion expansion, we give a computational procedure for the equation of the spectral curve, as well as a differential on it that reduces to the elliptic curve it covers, note that it is not even obvious *a priori* that when the genus is one, the spectral curve is isomorphic to the X where $u(\xi)$ is an elliptic function: the algorithm shows that it is. What we found surprising is that if X is equianharmonic, then the spectral curve has a basis of elliptic holomorphic differentials, and we were able to write them explicitly, for genus up to three. This is also what happens in the case of elliptic 3-rd-order L that we were able to deal with,

$$L = \partial^3 - \frac{4}{3}n^2\wp\partial + \frac{2n(n-3)(4n+3)}{27}\wp',$$

which we showed has centralizer larger than $\mathbb{C}[L]$; we asked the question since Halphen (1993), noticed that the equation $L = 0$ has solutions that can be expressed in terms of elliptic functions.

These geometric configurations, arising when the spectral curve covers one or more elliptic curve, have enormous importance in the computation of the KP solutions. In this section, we prove a new result on this case, after recalling the context.

Definition (Treibich and Verdier (1993), 2.2). Let Γ be a (projective,integral) curve of (arithmetic) genus $g > 0$, $p \in \Gamma$ a smooth point and $\pi\colon (\Gamma, p) \to (X, q)$ a finite pointed morphism to an elliptic curve. π is said to be a *tangential cover* if $\pi^*(\mathrm{JacX} \simeq X) \subset \mathrm{Jac}\Gamma$ is tangent to $A_\Gamma(\Gamma)$ at the origin of $\mathrm{Jac}\Gamma$, where A_Γ is the Abel map (based at the point p and defined on the smooth part of Γ). A tangential cover is said to be *minimal* if it does not factor through another tangential cover, except by isomorphism.

In Treibich (1989) and Treibich and Verdier (1990), three criteria equivalent to tangentiality are given: analytic (Γ is defined by a "tangential polynomial"); cohomological (there exists a surjective morphism $\pi^*(E) \to \mathcal{O}_\Gamma(p)$, where E is the nontrivial rank 2 bundle obtained extending $\mathcal{O}_X$ by $\mathcal{O}_X$); and geometric (Γ is mapped to the ruled surface $S = \mathrm{Proj}E$). It is on the surface S that tangential covers are classified.

Proposition.
(i) Any g-sheeted tangential cover $\pi : (\Gamma, p) \to (X, q)$ factors through a degree 1 morphism $\mu : \Gamma \to S$ by the natural projection $S \to X$.

(ii) π is minimal if and only if μ is an embedding of Γ in S (so π is the restriction of $S \to X$). Moreover any tangential cover dominates a minimal tangential cover, unique up an unique isomorphism.

(iii) The arithmetic genus of Γ is $\leq g$, with equality if and only if π is minimal.

(iv) The isomorphism classes of minimal tangential covers of degree g are in 1:1 correspondence with a $(g-1)$-dimensional affine variety.

(v) A dense Zariski-open subset of the minimal tangential covers lying on S are smooth.

Note. In particular, the moduli of genus-g tangential covers has dimension g, since to the count of $(g-1)$ in (iv) of the Proposition, we have to add the dimension of the moduli space of elliptic curves.

We need the following tangentiality criterion from Previato and Colombo (1994)(the proof is repeated here for convenience).

Theorem 2.1. *Let* $\pi\colon \Gamma \to X$ *be a covering of an elliptic curve* X *and let* $W \leq H^{1,0}(\mathrm{Jac}\Gamma)$ *be the 1-dimensional complex subspace that corresponds to* $H^{1,0}(X)$*. Then* π *is a tangential cover if and only if* $\exists p \in \Gamma$ *such that* $W^\perp = H^0(K_\Gamma(-p))$ *(where the inner product* $<,>$ *is induced on* $H^{1,0}(\mathrm{Jac}\Gamma) = H^0(K_\Gamma)$ *by the intersection pairing on the curve); if this is the case,* p *is the point of tangency.*

Proof. The condition for the covering $\pi\colon (\Gamma, p) \to (X, q)$ to be tangential translates via the diagram

$$\begin{array}{ccc} \Gamma & \stackrel{\pi}{\to} & X \\ A_\Gamma \searrow & & \nearrow \phi \\ & \mathrm{Jac}\Gamma & \end{array}$$

(where ϕ comes from the Albanese property of the Abel map A_Γ) into $T_0 A_\Gamma(\Gamma) = T_0\phi^* X$. Equivalently,

$$\begin{array}{ccc} \Gamma & \stackrel{\gamma}{\to} & \mathrm{Jac}\Gamma/\phi^* X \\ A_\Gamma \searrow & & \nearrow \\ & \mathrm{Jac}\Gamma & \end{array}$$

has the property $\mathrm{d}\gamma(T_p\Gamma) = 0$ in $T_{\gamma(p)}\,(\mathrm{Jac}\Gamma/\phi^* X)$. We let $W = \phi^*(H^0(K_X)) = T_p^*(\phi^* X)$ and consider the dual maps with respect to $<,>$; the condition becomes:

$$(d\gamma)^* \quad (T^*_{\gamma(p)}(\mathrm{Jac}\Gamma/\phi^* X) = W^\perp) = 0, \text{ namely } W^\perp = H^0(K_\Gamma(-p)).$$

□

Remark 2.1. *The equality* $W^\perp = H^0(K_\Gamma(-p))$ *means that the point* $\mathbb{P}(\phi^* H^0(K_X))$ *belongs to the canonical curve* $\phi_{|K_\Gamma|}(\Gamma)$*. In particular, when* $g = 3$ *and the curve* Γ *is canonically embedded, the image of the curve* X *becomes a point of* Γ*.*

As a consequence, we can see that **Corollary.** *Let* C *be a genus 2 curve which covers an elliptic curve* $C \stackrel{p_1}{\to} E_1$*, and let* $C \stackrel{p_2}{\to} E_2$ *be another covering so that there is an isogeny,* $\mathrm{Jac}C \sim E_1 \times E_2$*; then* p_i *is tangential exactly at the points where* p_j *is ramified,* $i \neq j$*.*

Proof. Let $p_1^*\omega_1, p_2^*\omega_2$ be a basis of holomorphic differentials on C; by the above Theorem, p_2 is tangential at q if and only if $p_1^*\omega_1(q) = 0$, and vice versa. □

We recall that by Poincaré's reducibility (see Lange and Birkenhake (1992)) when an abelian variety A contains a subabelian variety B, it is isogenous to a direct sum of abelian varietites, $B \oplus C$, say. Therefore, if we have a genus-3 tangential cover $(\Gamma, p) \to (X, q)$, we have up to isogeny a 2-dimensional abelian variety C such that $\text{Jac}\Gamma = X \oplus C$. It is natural to ask whether the curve Γ covers a curve Y of genus 2 so that $\text{Jac}\Gamma = X \oplus \text{Jac}Y$; we show that the answer is no.

Remark 2.2. *If a curve of genus 3 covers (non-trivially) a curve of genus 2, by the Riemann-Hurwitz formula the only possibility is that the degree of the cover be 2, with ramification number also 2, thus 2 distinct branchpoints – since a degree-2 cover can ramify at most with index 1.*

Proposition. *If a curve Γ of genus 3 covers curves X of genus 1 and Y of genus 2,*

$$\begin{array}{ccc} & \Gamma & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ X & & Y \end{array}$$

in such a way that Jac$\Gamma \sim X \oplus \text{Jac}Y$, then π_1 cannot be a tangential cover at any point.

Proof. By assumption, the pull-back of holomorphic differentials ω from X and η_1, η_2 from Y span the space of global sections of holomorphic differentials on Γ. The tangentiality criterion would require that both $\pi_2^*\eta_1$ and $\pi_2^*\eta_2$ vanish at some point of Γ, which is impossible since the ramification index is at most 1. □

Remark 2.3. *(1) By this Proposition we can see that the analog of the above Corollary does not hold in genus 3, namely, not all covers $\Gamma \to X$ are tangential at some point, for genus(Γ)=3 and genus(X)=1: a counterexample is given by any unramified double cover of a curve Y of genus 2, which has genus 3 by the Riemann-Hurwitz formula, and the projection $\pi : \Gamma \to X$ from the image of Γ in JacΓ to an elliptic curve X which splits (by Poincaré reducibility, up to isogeny) JacΓ over JacY.*

(2) The parameter count confirms this proposition: if the genus-3 Jacobian were isogenous to an elliptic curve times the Jacobian of a general genus-2

curve, then the moduli of genus-3 elliptic solitons would have dimension 1+3=4. The result should hold for every genus.

3. Ultra-elliptic

To devise a generalization of the elliptic solitons where instead the potential is ultra-elliptic, th classical name for an Abelian function on a 2-dimensional complex torus, in Enolski and Previato (2007) we introduce the following potentials,

$$u(t_1, t_2, \ldots) = 2\sum_{k=1}^{N} \wp_{22}(\boldsymbol{X} - \boldsymbol{X}_k), \;\; N \in \mathbb{N}$$

where $\wp_{22}$ is the Kleinian Abelian function defined on the Jacobi variety $\mathrm{Jac}(V_2)$ of the genus two curve, V_2

$$y^2 = 4x^5 - \lambda_3 x^3 - \lambda_2 x^2 - \lambda_1 x - \lambda_0$$

with moduli $\lambda_0, \ldots, \lambda_3$.

Remark 3.1. *The curve V_3 of genus three (Baker (1993))*

$$y^2 = \prod_{k=1}^{4}(x^2 - a_k^2) \equiv \Lambda_8 x^8 + \Lambda_7 x^7 + \ldots + \Lambda_0$$

admits the involution $(x, y) \to (-x, y)$ and covers a genus two hyperelliptic curve, $V_2 = (z, w)$, and an elliptic curve, $V_1 = (Z, W)$, given by equations

$$w^2 = 4z\prod_{k=1}^{4}(z - a_k^2), \quad W^2 = 4\prod_{k=1}^{4}(Z - a_k^2).$$

The elliptic cover is given by: $(x, y) \mapsto (Z = x^2, W = y)$, therefore it has four simple branch points.

In Enolski and Previato (2007), we are able to show that for the curve V_3 of genus three, the potentials $u(t_1, t_2, ...)$ give a solution to the KP equations; however, we have not constructed a completely integrable Hamiltonian system analogous to the Calogero-Moser-Krichever system Krichever (1980). This is a promising open question. In addition, the spectral curve is co-elliptic (namely it has a quotient which is an elliptic curve). We remark here that unfortunately it is not a *co-elliptic soliton*, as defined in Donagi and Previato (2001), which in this case would mean that the covering $V_3 \to V_1$ is simply ramified at two points and doubly ramified at one point.

Nevertheless, in Donagi and Previato (2001) we show that co-elliptic solitons form a completely integrable Hamiltonian system, the invariant tori are Prym varieties of the cover. To connect the two theories, we plan to modify Baker's curve and retest the Kleinian Abelian function for KP solution.

4. Rank Two

In the Weyl algebra, define $u = p^3 + q^2 + \alpha,\ v = \frac{1}{2}p,\ L = u^2 + 4v,\ B = u^3 + 3(uv - vu)$; then $\mathcal{C}(L) = \mathbb{C}[L, B]$ and $B^2 - L^3 = -\alpha$, as shown in **?**. By the assignment $p = \partial,\ q = x$ we obtain $L, B \in \mathcal{D}$ of order 6,9, but notice that the automorphism $\partial \mapsto -x,\ x \mapsto \partial$ will turn the orders into 4,6. Again, $\mathcal{C}_{\mathcal{D}}(L) = \mathbb{C}[L, B]$, the affine ring of the curve $\mu^2 = \lambda^3 - \alpha$; in particular, L has the Burchnall-Chaundy (BC) property $\mathcal{C}(L) \neq \mathbb{C}[T]$ for some operator T, and the rank of this algebra is three, two, respectively.

Since it can be shown that a change of variables transforms this rank-two centralizer into one with elliptic coefficients, these can be regarded as generalized Lamé operators. The theory for rank greater than two is unknown.

Since the algebra $\mathbb{C}[L, B]$ has no zero-divisors, it can be viewed as the affine ring $\mathbb{C}[X, Y]/(h)$ of a plane curve, with $h(X, Y)$ an irreducible polynomial. The BC curve $= \{(\lambda, \mu) \mid L, B,$ have a joint eigenfunction $Ly = \lambda y,\ By = \mu y\}$ is included in the curve Spec $\mathbb{C}[L, B]$ and since the latter is irreducible, they must coincide; this shows in particular that the BC polynomial, namely the differential resultant of $L - \lambda,\ B - \mu$, is some power of an irreducible polynomial h: $f(\lambda, \mu) = h^{r_1}$. In addition, each point of the spectral curve has a solution space: this gives a vector bundle over the curve. More precisely, let $r_2 =$ rank $\mathbb{C}[L, B]$, and $r_3 = \dim V_{(\lambda,\mu)}$ where $V_{(\lambda,\mu)}$ is the vector space of common eigenfunctions at any smooth point (λ, μ) of the BC curve. Then $r_1 = r_2 = r_3$. Moreover, this integer is the order of $G = gcd(L - \lambda, B - \mu)$, the operator (found by the Euclidean algorithm) of highest order for which a factorization holds, $B - \mu = T_1 G,\ L - \lambda = T_2 G$.

In theory, higher-rank algebras are classified by vector bundles over curves Mulase (1990), but there is no explicit dictionary between the vector bundles and the coefficients of the operators; a recent paper Burban and Zheglov (2018) completed, for singular spectral curves, the result in Previato and Wilson (1992), which analyzes the genus-one case of the spectral curve.

5. Partial Differential Operators

Lastly, it would be important to generalize Lamé operators to several variables, and this too is an open research area. The algebra of operators that commute with a given one is no longer necessarily commutative, the kernel of an operator is no longer a finite-dimensional space. Results by Chalykh et al. (2003) and Chalykh and Veselov (1990), use representation theory to construct examples. Their emphasis is on a Schrödinger operator $L = -\Delta + u(x)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$, with elliptic potential u of the following form

$$u(x) = \sum_{\alpha \in \mathcal{A}} c_\alpha \wp(\alpha(x)|\tau),$$

$\mathcal{A}$ a finite set of affine-linear functions on $\mathbb{C}^n$, such that the resulting potential $u(x)$ has the properties of periodicity with respect to a lattice and quasi-invariance (a technical condition).

A commutative ring of PDOs in n variables cannot have more than n algebraically independent elements (see Braverman et al. (1997)), however, for there to be $n+1$ commuting integrals instead of the usual n integrals in a commutative ring generated by n operators $L_1, ..., L_n$, Chalykh and Veselov (1990) make the assumption that the symbols of the L_i be algebraically independent, respectively, that the ring is not contained in another commutative ring generated by n operators (*complete*, resp., *supercomplete*), as well as that the ring be non-separable, namely do not contain a non-trivial operator in fewer than n variables.

In Kasman and Previato (2001), we posed the question whether a commutative ring generated by n operators is necessarily algebraic (namely, finitely generated, like the affine ring of an algebraic variety) and we showed by example that the answer is no.

In Kasman and Previato (2010), we focus on the goal of producing algorithmically algebraic equations satisfied by $n + 1$ commuting operators in n variables.

Our method, extending Burchnall and Chaundy (1928), was based on a possible definition of a differential resultant. Remaining open questions include properties of this resultant, especially analogs of the one-variable properties. For example, one might conjecture that in suitable coordinates, a supercomplete non-separable algebra (in the meaning above) would yield non-zero resultant.

The following example, worked out in Kasman and Previato (2010), is separable, has elliptic coefficients, and more importantly, yields a *spectral surface* (in the sense that each point of the surface represents eigenvalues of eigenfunctions common to the ring generated by these commuting PDOs) which is an elliptic fibration.

For a variable x and complex constants k_1 and k_2, let

$$L = L\,(x, \mathrm{d}x\,; k_1, k_2) = \mathrm{d}^2 x - 2\wp(x; k_1, k_2) \tag{2}$$

$$M = M\,(x, \mathrm{d}x\,; k_1, k_2) = \mathrm{d}^3 x - 3\wp(x; k_1, k_2)\,\mathrm{d}x - \frac{3}{2}\wp'(x; k_1, k_2) \tag{3}$$

where $f(z) = \wp(z; k_1, k_2)$ is the Weierstrass $\wp$-function associated to the (possibly singular) elliptic curve

$$\mathcal{E}_{k_1,k_2}(x, y) = y^2 - 4x^3 + k_1 x + k_2 = 0. \tag{4}$$

As can easily be derived from the fact that $f(x) = \wp(x; k_1, k_2)$ solves the differential equation

$$\mathcal{E}_{k_1,k_2}(f, f') = 0, \tag{5}$$

$[L, M] = 0$, and the operators identically satisfy the relationship

$$\mathcal{E}_{k_1,k_2}(L, 2M) = 0.$$

To put this in context, in soliton theory these operators serve as the Lax pair for a stationary (i.e. time independent) solution of the KdV equation, when the time deformations are defined as

$$\partial_i L = [(L^{i/2})_+, L],$$

in other words, this L satisfies $(L^{3/2})_+ = L^{3/2}$; as noted in Section I, the *wp* function in the coefficients of the operators is associated to the (elliptic) spectral curve.

From these we construct three partial differential operators in the variables x_1 and x_2:

$$A_1 = L\,(x_1, \mathrm{d}x_1\,; k_1, k_2) + L\,(x_2, \mathrm{d}x_2\,; a_1, a_2) \tag{6}$$

$$A_2 = M\,(x_1, \mathrm{d}x_1\,; k_1, k_2) \qquad A_3 = M\,(x_2, \mathrm{d}x_2\,; a_1, a_2)\,. \tag{7}$$

It is obvious that these three operators mutually commute and therefore satisfy an algebraic relationship. Using the method in Kasman and Previato (2001), we constructed a matrix whose minor determinants are polynomials in the variables μ_1, μ_2 and μ_3 that are equal to zero upon the substitution $\mu_i = A_i$.

One of these minor determinants is exactly $P(\mu_1, \mu_2, \mu_3) =$

$$
\begin{aligned}
&a_1^3\left(k_1\mu_1 + k_2 - 4\mu_1^3 + 4\mu_2^2\right)\\
&-a_2\left(-a_1^2\left(k_1 - 12\mu_1^2\right) + 2a_1\left(k_1^2 - 12k_1\mu_1^2 - 6\mu_1\left(k_2 + 8\mu_1^3 + 4\mu_2^2 - 8\mu_3^2\right)\right)\right.\\
&-k_1^3 + 12k_1^2\mu_1^2 - 12k_1\mu_1\left(k_2 + 4\left(-\mu_1^3 + \mu_2^2 + 4\mu_3^2\right)\right)\\
&\left.+12\left(k_2 2 + 4k_2\left(7\mu_1^3 + 2\left(\mu_2^2 + \mu_3^2\right)\right) + 16\left(\mu_1^6 + \left(7\mu_2^2 - 2\mu_3^2\right)\mu_1^3 + \left(\mu_2^2 + \mu_3^2\right)^2\right)\right)\right)\\
&-2a_1^2\left(k_1^2\mu_1 + k_1\left(k_2 + 2\mu_1^3 + 4\mu_2^2 - 2\mu_3^2\right) + 6\mu_1^2\left(k_2 + 4\left(-\mu_1^3 + \mu_2^2 + \mu_3^2\right)\right)\right)\\
&-12a_2^2\left(a_1\mu_1 - 2k_1\mu_1 + k_2 - 4\mu_1^3 + 4\mu_2^2 + 4\mu_3^2\right)\\
&+a_1\left(k_1^3\mu_1 + k_1^2\left(k_2 - 4\mu_1^3 + 4\mu_2^2 - 8\mu_3^2\right) + 24k_1\mu_1^2\left(k_2 + 2\mu_1^3 + 4\mu_2^2 + 4\mu_3^2\right)\right.\\
&+24\mu_1\left(k_2 2 + k_2\left(-2\mu_1^3 + 8\mu_2^2 + 2\mu_3^2\right)\right.\\
&\left.\left.-8\left(\mu_1^6 + \left(\mu_2^2 - 2\mu_3^2\right)\mu_1^3 - 2\mu_2^4 + \mu_3^4 - \mu_2^2\mu_3^2\right)\right)\right)\\
&-4a_2^3 - 4\left(k_1^3\left(\mu_1^3 - \mu_3^2\right) + 3k_2\left(k_1^2\mu_1^2 - 4k_1\left(2\mu_1^3 - 2\mu_2^2 + \mu_3^2\right)\mu_1\right.\right.\\
&\left.+16\left(\mu_1^6 + \left(7\mu_3^2 - 2\mu_2^2\right)\mu_1^3 + \left(\mu_2^2 + \mu_3^2\right)^2\right)\right)\\
&-12k_1^2\mu_1^2\left(\mu_1^3 - \mu_2^2 - \mu_3^2\right) + 3k_2 2\left(k_1\mu_1 + 4\left(-\mu_1^3 + \mu_2^2 + \mu_3^2\right)\right)\\
&+48k_1\mu_1\left(\mu_1^6 + \left(\mu_3^2 - 2\mu_2^2\right)\mu_1^3 + \mu_2^4 - 2\mu_3^4 - \mu_2^2\mu_3^2\right) + k_2 3\\
&\left.-64\left(\mu_1^9 - 3\left(\mu_2^2 + \mu_3^2\right)\mu_1^6 + 3\left(\mu_2^4 - 7\mu_3^2\mu_2^2 + \mu_3^4\right)\mu_1^3 - \left(\mu_2^2 + \mu_3^2\right)^3\right)\right)
\end{aligned}
$$

The connection between this polynomial and the polynomial $\mathcal{E}_{k_1,k_2}$ from (4) is as follows: Let (α, β) be the coordinates of a point on the elliptic curve $\mathcal{E}_{k_1,k_2}(x, y) = 0$ so that $\mathcal{E}_{k_1,k_2}(\alpha, \beta) = 0$. Then $P(x + \alpha, \beta/2, y/2) \in \mathbb{C}[x, y]$ is a polynomial (depending on the choice of parameters a_1, a_2, k_1, k_2 and the point (α, β)). It factors as

$$P(x + \alpha, \beta/2, y/2) = \mathcal{E}_{a_1,a_2}(x, y) \times (T_2(x, y)\mathcal{E}_{a_1,a_2}(x, y) + T_3(x, y))$$

where

$$T_2(x, y) = -4\left(3a_1\alpha + a_1x + a_2 - 36\alpha^2x - 9\alpha k_1 - 36\alpha x^2 - 6k_1x - 4x^3\right) - 4y^2$$

and

$$\begin{aligned} T_3(x,y) &= -12a_1^2\alpha^2 + a_1^2k_1 + 144a_1\alpha^4 + 432a_1\alpha^3x + 12a_1\alpha^2k_1 + 288a_1\alpha^2x^2 \\ &\quad -36a_1\alpha k_1x - 2a_1k_1^2 - 24a_1k_1x^2 - 1728\alpha^6 - 5184\alpha^5x + 432\alpha^4k_1 - 6912\alpha^4x^2 \\ &\quad +864\alpha^3k_1x - 5184\alpha^3x^3 - 36\alpha^2k_1^2 + 720\alpha^2k_1x^2 - 1728\alpha^2x^4 - 36\alpha k_1^2x \\ &\quad +432\alpha k_1x^3 + k_1^3 - 12k_1^2x^2 + 144k_1x^4 \end{aligned}$$

With greater computational resources, we would have been able to compute partial differential resultants of larger size, obtaining the equation of the spectral variety of the larger ring $\mathbb{C}[L_i, M_i | i = 1, 2]$, where L_i, M_i are operators representing the $\wp$ function and its derivative on the two elliptic curves $(i = 1, 2)$. This will correspond to the product of two spectral curves, producing the equation of $\mathcal{E}_{g_1,g_2} \times \mathcal{E}_{k_1,k_2}$ (here the elliptic curve parameters are fixed to be g_1, g_2 and k_1, k_2) in its Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ into $\mathbb{P}^8$ (see Mumford (1976)). The spectral problem will necessitate a suitable homogeneization of the coordinates, which is work in progress. If we define the coordinates in $\mathbb{P}^8$ lexicographically as customary $(X_0 = x_0y_0, X_1 = x_0y_1, ..., X_8 = x_2y_2)$, the six equations for the product of the two curves are:

$$X_0X_6^2 = 4X_3^3 - g_1X_0^2X_3 - g_2X_0^3, \quad X_1X_7^2 = 4X_4^3 - g_1X_1^2X_4 - g_2X_1^3,$$

$$X_2X_8^2 = 4X_5^3 - g_1X_2^2X_5 - g_2X_2^3, \quad X_0X_2^2 = 4X_1^3 - k_1X_0^2X_1 - k_2X_0^3,$$

$$X_3X_5^2 = 4X_4^3 - k_1X_3^2X_4 - k_2X_3^3, \quad X_6X_8^2 = 4X_7^3 - k_1X_6^2X_7 - k_2X_6^3,$$

while the linear functions in the Segre product will correspond to: $1 = x_0y_0, L_2 = x_0y_1, M_2 = x_0y_2, L_1 = x_1y_0, L_1L_2 = x_1y_1, L_1M_2 = x_1y_2, L_2 = x_2y_0, M_1L_2 = x_2y_1, M_1M_2 = x_2y_2$ and by eliminating from these equations algebraically we should find a product of elliptic curves which is the spectrum of the commutative ring of PDOs. Instead, the surfaces we produced in $\mathbb{P}^3$ seem of independent interest to us, in the spirit of using differential algebra to produce examples of algebro-geometric classes of commuting ODOs, because it isn't true that the ring $\mathbb{C}[L_i, M_i | i = 1, 2]$ contains, for example,

$$\frac{\partial^2}{\partial x_1^2} - 2\wp(x_1; k_1, k_2) - 2\wp(x_1; g_1, g_2).$$

The surface we gave above, the simplest possible we could compute at this point, does correspond to a subring.

6. Conclusion

In conclusion, we mention a new technique and current project: in order to speed the calculations of differential resultants and minors in several variables we plan, in collaboration with R.H. Lewis, to use the Dixon resultant in Lewis (2010), a different algorithm for choosing rows and columns. This would allows us to compute equations of spectral surfaces and check on examples several conjectures formulated (refer Chalykh et al. (2003)).

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