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Symmetry Analysis, Nonlinearly Self-adjoint and Conservation Laws of a Generalized (2+1)-dimensional Klein-Gordon Equation

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ABSTRACT

We study a generalization of Klein-Gordon equation (gKGe) in (2+1) dimensions which has an arbitrary element. Lie group classification is carried out on this equation. It is shown that gKGe admits a nine-dimensional Lie algebra of equivalence transformations and six-dimensional principal Lie algebra which has several possible extensions. The forms of the arbitrary element are linear, exponential, power law nonlinearity and others. Closed form solutions are obtained for some special cases of arbitrary element. Lastly, we derive conservation laws for nonlinearly self-adjoint subclass of the gKGe.

Keywords: Lie group classification; Nonlinearly self-adjoint; generalized (2+1)-dimensional Klein-Gordon equation; Lie symmetry analysis; Exact solutions.

1. Introduction

The sinh-Gordon (SG) equation

$$u_{tt} + \text{sinh} u = u_{xx} \tag{1}$$

is one of the equations which describes nonlinear wave motion. See for example Wazwaz (2005). The SG equation (1) has many technological applications in plasma physics, solid state physics, fluid mechanics, nonlinear optics and biology to name but a few. Recently, one as well as two soliton solutions of SG equation in (2+1) and (3+1) dimensions

$$u_{tt} + \text{sinh} u = u_{xx} + u_{yy}, \tag{2}$$

and

$$u_{tt} + \sinh u = u_{xx} + u_{yy} + u_{zz}, \tag{3}$$

respectively were investigated by Wazwaz (2012). SG equation is a special case of

$$u_{tt} + p(u) = u_{xx}, \tag{4}$$

where $p(u)$ represents various forms of many physical phenomena. The construction of different forms of these parameters is one of the most essential tasks in nonlinear science. Usually the various forms of such arbitrary element(s) are determined from experiments. Nevertheless, Lie symmetry method through its group classification (Baikov et al. (1997), Bluman and Kumei (1989), Ibragimov (1996), Lie (1881), Molati and Khalique (2012), Muatjetjeja and Khalique (2009, 2014), Olver (1993)) provides us systematically with different forms of these elements. Several researchers studied equation (4) for different forms of $p(u)$ (see for example Khalique and Magalakwe (2014), Kheiri and Jabbari (2010a,b), Tang and Huang (2007), Wazwaz (2006a,b,c, 2007a)). Kudra (1986) carried out group classification of (3+1)-dimensional nonlinear KG equation

$$u_{tt} + p(u) = u_{xx} + u_{yy} + u_{zz}.$$

Thereafter, Azad et al. (2010) performed group classification of (4) and constructed different forms of $p(u)$ that gave larger symmetry algebra.

The present study is motivated by the works in (Azad et al. (2010), Kudra (1986)). Here we study the extension of (4) to (2+1) dimensions, namely

$$u_{tt} + p(u) = u_{xx} - u_{yy}, \tag{5}$$

where $p(u)$ is an arbitrary function u.

We now give outline of this study. In Section 2 equivalence transformations of equation (5) are computed and principal Lie algebra is determined in Section 3. In Section 4 we accomplish Lie group classification of gKGe (5). Closed form solutions for various forms of $p(u)$ are presented in Section 5. Lastly, we present our investigations in Section 6.

2. Equivalence transformations (ETs)

The vector field

$$S = \tau\partial_t + \xi\partial_x + \psi\partial_y + \eta\partial_u + \mu\partial_p, \tag{6}$$

with τ, ξ, ψ, η depending on (t, x, y, u) and μ depending on (t, x, y, u, p), will generate equivalence group of gKGe (5) if and only if it admits the extended system

$$u_{tt} + p(u) = u_{xx} + u_{yy}, \tag{7a}$$

$$p_t = 0,\ p_x = 0,\ p_y = 0. \tag{7b}$$

The prolonged vector field for (7) is

$$\widetilde{S} = S^{[2]} + \omega_t\partial_{p_t} + \omega_x\partial_{p_x} + \omega_y\partial_{p_y} + \omega_u\partial_{p_u}, \tag{8}$$

where

$$\begin{aligned} S^{[2]} \quad = \quad & \tau\partial_t + \xi\partial_x + \psi\partial_y + \eta\partial_u + \mu\partial_p + \zeta_{tt}\partial_{u_{tt}} + \zeta_{xx}\partial_{u_{xx}} \\ & +\zeta_{yy}\partial_{u_{yy}} \end{aligned}$$

represents the second-prolongation of vector field (6). The coefficients ω's and ζ's are given by

$$\begin{aligned}
\omega_t &= \widetilde{D}_t(\mu) - p_t\widetilde{D}_t(\tau) - p_x\widetilde{D}_t(\xi) - p_y\widetilde{D}_t(\psi) - p_u\widetilde{D}_t(\eta),\\
\omega_x &= \widetilde{D}_x(\mu) - p_t\widetilde{D}_x(\tau) - p_x\widetilde{D}_x(\xi) - p_y\widetilde{D}_x(\psi) - p_u\widetilde{D}_x(\eta),\\
\omega_y &= \widetilde{D}_y(\mu) - p_t\widetilde{D}_y(\tau) - p_x\widetilde{D}_y(\xi) - p_y\widetilde{D}_y(\psi) - p_u\widetilde{D}_y(\eta),\\
\omega_u &= \widetilde{D}_u(\mu) - p_t\widetilde{D}_u(\tau) - p_x\widetilde{D}_u(\xi) - p_y\widetilde{D}_u(\psi) - p_u\widetilde{D}_u(\eta),
\end{aligned}$$

and

$$\begin{aligned}
\zeta_t &= D_t(\eta) - (u_t D_t(\tau) + u_x D_t(\xi) + u_y D_t(\psi)),\\
\zeta_x &= D_x(\eta) - (u_t D_x(\tau) + u_x D_x(\xi) + u_y D_x(\psi)),\\
\zeta_y &= D_y(\eta) - (u_t D_y(\tau) + u_x D_y(\xi) + u_y D_y(\psi)),\\
\zeta_{tt} &= D_t(\zeta_x) - (u_{tt} D_t(\tau) + u_{tx} D_t(\xi) + u_{ty} D_t(\psi)),\\
\zeta_{xx} &= D_x(\zeta_x) - (u_{tx} D_x(\tau) u_{xx} D_x(\xi) + u_{xy} D_x(\psi)),\\
\zeta_{yy} &= D_y(\zeta_y) - (u_{ty} D_y(\tau) + u_{xy} D_y(\xi) + u_{yy} D_y(\psi)),
\end{aligned}$$

respectively, with total derivatives

$$D_t = \partial_t + u_t\partial_u + \cdots, \quad D_x = \partial_x + u_x\partial_u + \cdots, \quad D_y = \partial_y + u_y\partial_u + \cdots$$

and total derivatives for extended system being

$$\begin{aligned}
\widetilde{D}_t &= \partial_t + p_t\partial_p + \cdots,\\
\widetilde{D}_x &= \partial_x + p_x\partial_p + \cdots,\\
\widetilde{D}_y &= \partial_y + p_y\partial_p + \cdots,\\
\widetilde{D}_u &= \partial_u + p_u\partial_p + \cdots.
\end{aligned}$$

Now utilizing (8) and invoking invariance conditions on (7) gives

$$\begin{aligned}
&S_1 = \partial_t,\ S_2 = \partial_x,\ S_3 = \partial_y,\ S_4 = \partial_u,\ S_5 = -y\partial_x + x\partial_y,\ S_6 = x\partial_t + t\partial_x,\\
&S_7 = y\partial_t + t\partial_y,\ S_8 = u\partial_u + p\partial_p,\ S_9 = x\partial_x + y\partial_y + t\partial_t - 2p\partial_p
\end{aligned}$$

as equivalent generators. Thus, equivalence group of nine-parameter is

$$\begin{aligned}
S_1 &: \bar{t} = t + a_1,\ \bar{x} = x,\ \bar{y} = y,\ \bar{u} = u,\ \bar{p} = p,\\
S_2 &: \bar{t} = t,\ \bar{x} = x + a_2,\ \bar{y} = y,\ \bar{u} = u,\ \bar{p} = p,\\
S_3 &: \bar{t} = t,\ \bar{x} = x,\ \bar{y} = y + a_3,\ \bar{u} = u,\ \bar{p} = p,\\
S_4 &: \bar{t} = t,\ \bar{x} = x,\ \bar{y} = y,\ \bar{u} = u + a_4,\ \bar{p} = p,\\
S_5 &: \bar{t} = t,\ \bar{x} = x - a_5 y,\ \bar{y} = y + a_5 x,\ \bar{u} = u,\ \bar{p} = p,\\
S_6 &: \bar{t} = t + a_6 x,\ \bar{x} = x + a_6 t,\ \bar{y} = y,\ \bar{u} = u +\ \bar{p} = p,\\
S_7 &: \bar{t} = t + a_7 y,\ \bar{x} = x,\ \bar{y} = y + a_7 t,\ \bar{u} = u,\ \bar{p} = p,\\
S_8 &: \bar{t} = t,\ \bar{x} = x,\ \bar{y} = y,\ \bar{u} = ue^{a_8},\ \bar{p} = pe^{a_8},\\
S_9 &: \bar{t} = te^{a_9},\ \bar{x} = xe^{a_9},\ \bar{y} = ye^{a_9},\ \bar{u} = u,\ \bar{p} = pe^{-2a_9},
\end{aligned}$$

whose composition gives

$$\begin{aligned}
\bar{t} &= (t + a_1 + a_6 x + a_7 y)e^{a_9}, \\
\bar{x} &= (x + a_2 - a_5 y + a_6 t)e^{a_9}, \\
\bar{y} &= (y + a_3 + a_5 x + a_7 t)e^{a_9}, \\
\bar{u} &= (u + a_4)e^{a_8}, \\
\bar{p} &= p e^{a_8 - 2a_9}.
\end{aligned}$$

3. The principal Lie algebra

Let

$$\Gamma = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \psi \frac{\partial}{\partial y} + \eta \frac{\partial}{\partial u} \tag{9}$$

be infinitesimal generator of symmetry group of gKGe (5). Application of $\Gamma^{[2]}$ to (5), expanding and splitting yields

$$\begin{aligned}
&\tau_u = 0,\ \psi_u = 0,\ \xi_u = 0,\ \eta_{uu} = 0,\ \tau_y = \psi_t,\ \xi_y + \psi_x = 0,\ \xi_t = \tau_x, \\
&\psi_y = \tau_t,\ \psi_y - \xi_x = 0,\ \tau_{tt} - \tau_{xx} - \tau_{yy} + 2\eta_{tu} = 0, \\
&\xi_{tt} - \xi_{xx} - \xi_{yy} + 2\eta_{xu} = 0,\ \psi_{tt} - \psi_{xx} - \psi_{yy} + 2\eta_{yu} = 0, \\
&p(u)\eta_u - 2p(u)\psi_y - p'(u)\eta - \eta_{tt} + \eta_{xx} + \eta_{xx} = 0.
\end{aligned} \tag{10}$$

The solution for arbitrary $p(u)$ of this system gives us six operators, viz.,

$$\begin{aligned}
&\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = \partial_y, \quad \Gamma_4 = y\partial_t + t\partial_y, \\
&\Gamma_5 = x\partial_t + t\partial_x, \quad \Gamma_6 = y\partial_x - x\partial_y,
\end{aligned}$$

which is principal Lie algebra of gKGe (5).

4. Lie group classification

The solution of (10) provides us with classifying relation (CR)

$$(u\beta + \gamma)p'(u) + \alpha p(u) + \lambda = 0$$

with constants β, γ, α and λ. This CR is invariant under ETs of Section 2 if

$$\bar{\beta} = \beta,\ \bar{\gamma} = a_4\beta + \gamma e^{-a_8},\ \bar{\alpha} = \alpha,\ \bar{\lambda} = \lambda e^{2a_9 - a_8}. \tag{11}$$

The above relation leads to the following five cases for the function $p(u)$.

Case 1 $p(u)$ arbitrary, but not as one of functions of Cases $2-5$ below.

This yields principal Lie algrebra, viz.,

$$\begin{aligned}&\Gamma_1=\partial_t,\quad \Gamma_2=\partial_x,\quad \Gamma_3=\partial_y,\quad \Gamma_4=y\partial_t+t\partial_y,\\&\Gamma_5=x\partial_t+t\partial_x,\quad \Gamma_6=y\partial_x-x\partial_y.\end{aligned}\tag{12}$$

Case 2 $p(u)=\sigma+\delta u$, where σ and δ are constants.

Here two subcases arise:

2.1 $\sigma,\delta\neq 0$.

The corresponding equation (5) gives an extension of principal Lie algebra by one operator

$$\Gamma_7=u\frac{\partial}{\partial u},\ \Gamma_8=H\left(t,x,y\right)\frac{\partial}{\partial u},$$

where H solves

$$H_{tt}-H_{xx}-H_{yy}+\delta H-C_1\sigma=0$$

and C_1 is a constant.

2.2 $\sigma\neq 0$, $\delta=0$.

This subcase extends the principal Lie algebra by six symmetries

$$\begin{aligned}&\Gamma_7=u\frac{\partial}{\partial u},\ \Gamma_8=(t^2+x^2+y^2)\frac{\partial}{\partial t}+t\left(2x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-u\frac{\partial}{\partial u}\right),\\&\Gamma_9=t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\ \Gamma_{10}=2y\left(t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}\right)+(t^2-x^2+y^2)\frac{\partial}{\partial y}-yu\frac{\partial}{\partial u},\\&\Gamma_{11}=2xt\frac{\partial}{\partial t}+(t^2+x^2-y^2)\frac{\partial}{\partial x}+x\left(2y\frac{\partial}{\partial y}-u\frac{\partial}{\partial u}\right),\ \Gamma_{12}=H\left(t,x,y\right)\frac{\partial}{\partial u},\end{aligned}$$

where $H(t,x,y)$ solves

$$2H_{tt}-2H_{xx}-2H_{yy}+10C_4\sigma t+5C_6\sigma x-10C_7\sigma y-2C_1\sigma+5C_{11}\sigma=0$$

and C_1,C_4,C_6,C_7,C_{11} are constants.

Case 3 $p(u)=\sigma+\delta u^n$, where σ is a constant, δ is non-zero constant and $n\neq 0,1$.

Three subcases arise. These are

3.1 $\sigma \neq 0$.

Here we have no additional Lie symmetry.

3.2 $\sigma = 0$, $n \neq 5$.

In this case we obtain one additional symmetry

$$\Gamma_7 = (n-1)\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - 2u\frac{\partial}{\partial u}.$$

3.3 $\sigma = 0$, $n = 5$.

In this subcase, four operators

$$\begin{aligned}
\Gamma_7 &= (t^2 + x^2 + y^2)\frac{\partial}{\partial t} + t\left(2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}\right),\\
\Gamma_8 &= 2y\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\right) + (t^2 - x^2 + y^2)\frac{\partial}{\partial y} - yu\frac{\partial}{\partial u},\\
\Gamma_9 &= 2tx\frac{\partial}{\partial t} + (t^2 + x^2 - y^2)\frac{\partial}{\partial x} + x\left(2y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}\right),\\
\Gamma_{10} &= 2\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - u\frac{\partial}{\partial u}
\end{aligned}$$

extend the principal Lie algebra.

Case 4 $p(u) = \sigma + \delta e^{nu}$, where σ is a constant, δ and n are non-zero constants.

Here two subcases arise.

4.1 $\sigma \neq 0$.

No additional symmetry is generated in this subcase.

4.2 $\sigma = 0$.

The extra Lie point symmetry is

$$\Gamma_7 = n\left(t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - 2\frac{\partial}{\partial u}.$$

Remark. In subcases 4.2 we retrieve two special equations, namely, the generalized Liouville equation in (2+1) dimensions (Wazwaz (2007b)) $u_{tt} - u_{xx} - u_{yy} + \delta e^{nu} = 0$ and the generalized (2+1)-dimensional combined sinh-cosh-Gordon (Fan et al. (2011)) $u_{tt} - u_{xx} - u_{yy} + \delta(\sinh(nu) + \cosh(nu)) = 0$.

Case 5 $p(u) = \sigma + \delta\,\mathrm{ln}u$, where σ is a constant and δ is nonzero constant.

This case reduces to Case 1.

5. Travelling wave solutions of two cases

Associated Lagrange's system

$$\frac{dt}{\tau} = \frac{dx}{\xi} = \frac{dy}{\psi} = \frac{du}{\eta}$$

is solved to generate exact solutions of gKGe (5). We consider two nonlinear cases, namely, Case 3.2 and Case 4.2.

5.1 Group-invariant solution of Case 3.2

In Case 3.2 gKGe (5) becomes

$$u_{tt} - u_{xx} - u_{yy} + \delta u^n = 0, \quad n \neq 0, 1. \tag{13}$$

We use the Lie point symmetry $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ to reduce equation (13) into a PDE with two new independent variables z, w and v as the new dependent variable. The symmetry Γ yields the invariants $u = v(z, w)$, $z = -t + x$ and $w = -t + y$ which transform (13) into the nonlinear second-order PDE

$$2v_{zw} + \delta v^n = 0. \tag{14}$$

Equation (14) admits the four symmetries

$$X_1 = \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial w}, \quad X_3 = (n-1)z\frac{\partial}{\partial z} + v\frac{\partial}{\partial v}, \quad X_4 = (n-1)w\frac{\partial}{\partial w} + v\frac{\partial}{\partial v}.$$

The symmetry $X_1 + cX_2$ provides $v = f(s)$ as group-invariant solution, with $s = w - cz$ (c a constant) and $f(s)$ satisfying the ODE

$$2cf^{''}(s) - \delta f(s)^n = 0. \tag{15}$$

Multiplying (15) by $f'(s)$ and integrating, we obtain

$$\frac{\delta f(s)^{n+1}}{n+1} - cf'^2(s) = C_1 \tag{16}$$

with C_1 a constant. Equation (16) is a variables separable equation, which on integration yields

$$-\frac{cf(s)\sqrt{\delta f(s)^{n+1} - C_1(n+1)}}{C_1\sqrt{c(n+1)}}\ {}_2F_1\left(1, \frac{1}{2} + \frac{1}{n+1}; 1 + \frac{1}{n+1}; \frac{\delta f(s)^{n+1}}{nC_1 + C_1}\right) = \pm s + C_2$$

with C_2 a constant and ${}_2F_1$ being the generalized hypergeometric function (Gradshteyn and Ryzhik (2000)). Thus in (t, x, y) variables the solution of gKGe (13) is

$$\begin{aligned} -\frac{c\,u\sqrt{\delta u^{n+1} - C_1(n+1)}}{C_1\sqrt{c(n+1)}}\ {}_2F_1\left(1, \frac{1}{2} + \frac{1}{n+1}; 1 + \frac{1}{n+1}; \frac{\delta u^{n+1}}{nC_1 + C_1}\right) \\ = \pm\{(c-1)t - cx + y\} + C_2. \end{aligned}$$

A special solution of (13) can be obtained by taking $C_1 = 0$ in (16). Then the integration of (16) with $C_1 = 0$ yields

$$u(t,x,y) = \left(\frac{2}{n-1}\right)^{\frac{2}{n-1}} \left[\sqrt{\frac{\delta}{c(n+1)}}\{(c-1)t - cx + y\} + C_2\right]^{\frac{2}{1-n}}, \quad n \neq -1, 1.$$

5.2 Group-invariant solution of Case 4.2

For the Case 4.2, the equation (5) becomes

$$u_{tt} - u_{xx} - u_{yy} + \delta e^{nu} = 0, \quad \delta, n \neq 0, \tag{17}$$

Again using the symmetry $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ and the invariants $u = v(z,w)$, $z = x - t$ and $w = y - t$, the equation (17) transforms into the nonlinear PDE

$$2v_{zw} + \delta e^{nv} = 0. \tag{18}$$

This equation admits the point symmetries

$$X_1 = \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial w}, \quad X_3 = f(z)\frac{\partial}{\partial v}, \quad X_4 = g(w)\frac{\partial}{\partial v}, \quad X_5 = w\frac{\partial}{\partial w} - z\frac{\partial}{\partial v}.$$

Invoking symmetry $X_1 + cX_2$ gives an invariant $s = w - cz$ (c a constant) with group-invariant solution $v = F(s)$, where F solves the ODE

$$2cF''(s) - \delta e^{nF(s)} = 0. \tag{19}$$

Integrate above equation twice and revert to (t, x, y) variables will give the solution of gKGe (17) that reads

$$u(t,x,y) = \frac{1}{n}\ln\left[\frac{cC_1}{\delta n}\left\{\tanh^2\left(\frac{1}{2}\sqrt{C_1}\left[C_2 + \{(c-1)t - cx + y\}\right]\right) - 1\right\}\right] \quad (20)$$

with C_1, C_2 being constants.

6. Subclass of nonlinearly self-adjoint equations and conservation Laws

We employ Ibragimov's theorem and obtain conservations laws for the nonlinearly self-adjoint (Gandarias et al. (2013), Ibragimov (2011a,b)) subclass of the generalized (2+1)-dimensional Klein-Gordon equation (5).

6.1 Preliminaries and Definitions

Consider rth-order PDE

$$E(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \quad (21)$$

where $x = (x^1, \ldots, x^n)$ and $u_{(1)} = \{u_i\}$, $u_{(2)} = \{u_{ij}\}, \ldots$ denote first, second, etc. orders partial derivatives of dependent variable u, that is, $u_i = \partial u/\partial x^i$, $u_{ij} = \partial^2 u/\partial x^i \partial x^j$. Adjoint equation (Ibragimov (2007)) of (21) is given as

$$E^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(r)}, v_{(r)}) = 0, \quad (22)$$

with

$$E^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(r)}, v_{(r)}) = \frac{\delta(v\,F)}{\delta u}, \quad (23)$$

where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{k=1}^{\infty}(-1)^k D_{i_1}\cdots D_{i_k}\frac{\partial}{\partial u_{i_1\cdots i_k}} \quad (24)$$

represents Euler-Lagrange operator. The variable v here is a new dependent variable, and D_i the usual total differentiations.

The following definitions are taken from (Gandarias et al. (2013), Ibragimov (2011a,b)).

Definition 6.1. *Equation (21) is self-adjoint provided equation obtained from (22) by letting $v = u$, viz.,*

$$E^*(x, u, u, u_{(1)}, u_{(1)}, \ldots, u_{(s)}, u_{(s)}) = 0,$$

is same as (21).

Definition 6.2. *Equation (21) is nonlinearly self-adjoint provided equation obtained from (22) by letting $v = h(x, t, u, u_{(1)}, \dots)$, with $h(x, t, u, u_{(1)}, \dots)$ such that $h(x, t, u, u_{(1)}, \dots) \neq 0$, is same as (21).*

6.2 Self-adjointness and nonlinearly self-adjointness

We derive nonlinearly self-adjointness equations from (5). Equation (23) gives

$$E^* = \frac{\delta}{\delta u}[v(u_{tt} - u_{xx} - u_{yy} + p(u))] = v_{tt} - v_{xx} - v_{yy} + p'(u)v. \qquad (25)$$

Putting $v = h(x, t, u)$ in (25) we obtain

$$\begin{aligned} &p'(u)h + h_{tt} + 2\,u_t h_{tu} + u_{tt}h_u + u_t^2 h_{uu} - h_{xx} - 2u_x h_{xu} \\ &-u_{xx}h_u - u_x^2 h_{uu} - h_{yy} - 2u_y h_{yu} - u_{yy}h_u - u_y^2 h_{uu}. \end{aligned}$$

We now assume that

$$E^* - \lambda(u_{tt} - u_{xx} - u_{yy} + p(u)) = 0, \qquad (26)$$

where λ is an undetermined coefficient. Condition (26) yields

$$\begin{aligned} &p'(u)h + h_{tt} + 2\,u_t h_{tu} + u_{tt}h_u + u_t^2 h_{uu} - h_{xx} - 2u_x h_{xu} - u_{xx}h_u - u_x^2 h_{uu} \\ &-h_{yy} - 2u_y h_{yu} - u_{yy}h_u - u_y^2 h_{uu} - \lambda u_{tt} + \lambda u_{xx} + \lambda u_{yy} - \lambda p(u), \end{aligned}$$

which gives

$$\begin{aligned} &h_u - \lambda = 0,\ h_{uu} = 0,\ h_{tu} = 0,\ h_{xu} = 0,\ h_{yu} = 0, \\ &p'(u)h + h_{tt} - h_{xx} - h_{yy} - p(u)h_u = 0. \end{aligned}$$

The solution of these PDEs yields

$$p(u) = c_2 u, \quad h = c_1 u + B(t, x, y),$$

with c_1, c_2 being constants and B solves

$$B_{tt} - B_{xx} - B_{yy} + c_2 B = 0. \qquad (27)$$

Thus we have theorem 6.1.

Theorem 6.1. *Equation (5) is nonlinearly self-adjoint when $p(u) = c_2 u$ with*

$$h = c_1 u + B(t, x, y)$$

for any function B satisfying condition (27).

6.3 Conservation laws

We recall Ibragimov's theorem (Ibragimov (2007)) and use it in conjunction with theorem 6.1 to compute conservation laws of nonlinearly self-adjoint equation.

Theorem 6.2. *Any symmetry (non-local or Lie-Bäcklund)*

$$G = \xi^i(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial x^i} + \eta(x, u, u_{(1)}, \ldots)\frac{\partial}{\partial u} \tag{28}$$

of (21) gives conservation law $D_i(T^i) = 0$ for (21)-(22), where its components are

$$\begin{aligned} T^i = \; & \xi^i L + W\left[\frac{\partial L}{\partial u_i} - D_j\left(\frac{\partial L}{\partial u_{ij}}\right) + D_j D_k\left(\frac{\partial L}{\partial u_{ijk}}\right) - \cdots\right] \\ & + D_j(W)\left[\frac{\partial L}{\partial u_{ij}} - D_k\left(\frac{\partial L}{\partial u_{ijk}}\right) + \cdots\right] + D_j D_k(W)\left[\frac{\partial L}{\partial u_{ijk}} - \cdots\right] + \cdots, \end{aligned} \tag{29}$$

with W and L being given by

$$W = \eta - \xi^j u_j, \quad L = v\,F\left(x, u, u_{(1)}, \ldots, u_{(s)}\right). \tag{30}$$

We invoke Theorem 6.2 and find conserved vectors for nonlinearly self-adjoint equation

$$u_{tt} - u_{xx} - u_{yy} + c_2 u = 0. \tag{31}$$

This equation has Lagrangian L given by

$$L = [c_1 u + B(t, x, y)]\,[u_{tt} - u_{xx} - u_{yy} + c_2 u] \tag{32}$$

and eight Lie symmetries

$$\begin{aligned} & X_1 = \partial_t,\, X_2 = \partial_x,\, X_3 = \partial_y,\, X_4 = x\partial_t + t\partial_x,\, X_5 = y\partial_t + t\partial_y, \\ & X_6 = -y\partial_x + x\partial_y,\, X_7 = u\partial_u \;\, X_8 = F\,(t, x, y)\,\partial_u, \end{aligned}$$

where F satisfies $F_{tt} - F_{xx} - F_{yy} + c_2 F = 0$.

Thus the conservation laws associated with eight symmetries have the following conserved vectors:

$$\begin{aligned} T_1^1 &= c_1 u(-u_{xx} - u_{yy} + c_2 u) + B(-u_{xx} - u_{yy} + c_2 u) + c_1 u_t^2 + u_t B_t, \\ T_1^2 &= -c_1 u_t u_x - u_t B_x + c_1 u u_{tx} + u_{tx} B, \\ T_1^3 &= -c_1 u_t u_y - u_t B_y + c_1 u u_{ty} + u_{ty} B; \end{aligned}$$

$$T_2^1 = c_1 u_t u_x + u_t B_x - c_1 u u_{tx} - u_{tx} B,$$
$$T_2^2 = c_1 u(u_{tt} - u_{yy} + c_2 u) + B(u_{tt} - u_{yy} + c_2 u) - c_1 u_x^2 - u_x B_x,$$
$$T_2^3 = -c_1 u_x u_y - u_x B_y + c_1 u u_{xy} + u_{xy} B;$$

$$T_3^1 = c_1 u_t u_y + u_y B_t - c_1 u u_{ty} - u_{ty} B,$$
$$T_3^2 = -c_1 u_x u_y - u_y B_x + c_1 u u_{xy} + u_{xy} B,$$
$$T_3^3 = c_1 u(u_{tt} - u_{xx} + c_2 u) + B(u_{tt} - u_{xx} + c_2 u) - c_1 u_y^2 - u_y B_y;$$

$$\begin{aligned}
T_4^1 &= c_1 x u(-u_{xx} - u_{yy} + c_2 u) + Bx(-u_{xx} - u_{yy} + c_2 u) + c_1 x u_t^2 + c_1 t u_t u_x + x u_t B_t \\
&\quad + t u_x B_t - c_1 u u_x - u_x B - c_1 t u u_{tx} - t u_{tx} B, \\
T_4^2 &= c_1 t u(u_{tt} - u_{yy} + c_2 u) + Bt(u_{tt} - u_{yy} + c_2 u) - c_1 x u_t u_x - c_1 t u_x^2 - x u_t B_x \\
&\quad - t u_x B_x + c_1 u u_t + u_t B + c_1 x u u_{tx} + x u_{tx} B, \\
T_4^3 &= -c_1 x u_t u_y - c_1 t u_x u_y - x u_t B_y - t u_x B_y + c_1 x u u_{ty} \\
&\quad + c_1 t u u_{xy} + x u_{ty} B + t u_{xy} B;
\end{aligned}$$

$$\begin{aligned}
T_5^1 &= c_1 y u(-u_{xx} - u_{yy} + c_2 u) + By(-u_{xx} - u_{yy} + c_2 u) + c_1 y u_t^2 + c_1 t u_t u_y + y u_t B_t \\
&\quad t u_y B_t - c_1 u u_y - u_y B - c_1 t u u_{yt} - t u_{yt} B, \\
T_5^2 &= -c_1 y u_t u_x - c_1 t u_x u_y - y u_t B_x - t u_y B_x + c_1 y u u_{tx} + y u_{tx} B + c_1 t u u_{xy} + t u_{xy} B \\
T_5^3 &= c_1 t u(u_{tt} - u_{xx} + c_2 u) + Bt(u_{tt} - u_{xx} + c_2 u) - c_1 y u_t u_y - c_1 t u_y^2 - y u_t B_y \\
&\quad - t u_y B_y + c_1 u u_t + u_t B + c_1 y u u_{ty} + y u_{ty} B;
\end{aligned}$$

$$\begin{aligned}
T_6^1 &= -c_1 y u_t u_x + c_1 x u_t u_y - y u_x B_t + x u_y B_t + c_1 y u u_{tx} - c_1 x u u_{ty} + y u_{xy} B - x u_{ty} B, \\
T_6^2 &= -c_1 y u(u_{tt} - u_{yy} + c_2 u) - By(u_{tt} - u_{yy} + c_2 u) + c_1 y u_x^2 - c_1 x u_x u_y + y u_x B_x \\
&\quad - x u_y B_x + c_1 u u_y + u_y B + c_1 x u u_{xy} + x u_{xy} B, \\
T_6^3 &= c_1 x u(u_{tt} - u_{xx} + c_2 u) + Bx(u_{tt} - u_{xx} + c_2 u) + c_1 y u_x u_y - c_1 x u_y^2 + y u_x B_y \\
&\quad - x u_y B_y - c_1 u u_x - u_x B - c_1 y u u_{xy} - y u_{xy} B;
\end{aligned}$$

$$\begin{aligned}
T_7^1 &= u_t B - u B_t, \\
T_7^2 &= u B_x - u_x B \\
T_7^3 &= u B_y - B u_y;
\end{aligned}$$

$$\begin{aligned}
T_8^1 &= c_1 u F_t + F_t B - F B_t - c_1 u_t F, \\
T_8^2 &= c_1 u_x F + F B_x - c_1 u F_x - F_x B \\
T_8^3 &= c_1 u_y F + F B_y - c_1 u F_y - F_y B,
\end{aligned}$$

respectively, where functions $B(t,x,y)$, $F(t,x,y)$ solves

$$\phi_{tt} - \phi_{xx} - \phi_{yy} + c_2 \phi = 0.$$

7. Conclusions

The Lie group classification was performed on the generalized (2+1)- dimensional Klein-Gordon equation (5). The arbitrary element $p(u)$ in (5) was found to be a linear function, power law function, exponential function and logarithmic function. From the classification we retrieved two special equations, namely, the generalized Liouville equation in (2+1) dimension and the (2+1)-dimensional generalized combined sinh-cosh-Gordon equation. In addition, group-invariant solutions of (1.5) were derived for power law and exponential function cases. We have also illustrated that the generalized (2+1)-dimensional Klein-Gordon equation is nonlinearly self-adjoint under the conditions given in Theorem 6.1. Lastly conservation laws for nonlinearly self-adjoint subclass were derived by invoking Ibragimov's theorem.

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