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Linear Lyapunov Functions of Infinite Dimensional Volterra Operators

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ABSTRACT

In the present paper, we investigate infinite dimensional Volterra quaratictic stochastic operators. We construct linear Lyapunov functions for such kind of operators, which allow to explore limiting set of associated with these kinds of operators.

Keywords: Quadratic Stochastic Operators, Volterra, Lyapunov functions, infinite dimensional, limit points, omega limiting set, trajectory,

1. Introduction

An original work on quadratic stochastic operators (in short QSOs) was done by Bernstein (1924) where such kind of operators appeared from the problems of population genetics (see also Lyubich et al. (1992)). These operator appear to have tremendous applications especially in modelings in many different fields such as biology Hofbauer and Sigmund (1998) (population and disease dynamics), physics Plank and Losert (1995), Takeuchi (1996)(non-equilibrium statistical mechanics) , economics and mathematics Lyubich et al. (1992), Takeuchi (1996) (replicator dynamics and games). A quadratic stochastic operator is usually used to present the time evolution of species in biology, which arises as follows. Consider evolution of species in biology as given in the following situation. Let $I = \{1, 2, \dots, n\}$ be the n type of species (or traits) in a population and we denote $x^{(0)} = (x_1^{(0)}, \dots, x_n^{(0)})$ be the probability distribution of the species in an early state of that population. By $P_{ij,k}$ we mean the probability of an individual in the i^{th} species and j^{th} species to cross-fertilize and produce an individual from k^{th} species (trait). Given $x^{(0)} = (x_1^{(0)}, \dots, x_n^{(0)})$, we can find the probability distribution of the first generation, $x^{(1)} = (x_1^{(1)}, \dots, x_n^{(1)})$ by using a total probability, i.e.,

$$x_k^{(1)} = \sum_{i,j=1}^{n} P_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k \in \{1, \dots, n\}.$$

This relation defines an operator which is denoted by V and it is called *quadratic stochastic operator (QSO)*. Each QSO can be reinterprate as evolutionary operator that describes the sequence of generations in terms of probabilistic distribution if the values of $P_{ij,k}$ and the distribution of the current generation are given. The most well-known class in the theory QSOs is Volterra one, namely

$$P_{ij,k} = 0 \text{ if } k \notin \{i, j\}. \quad (1)$$

The condition (1) biologically means that *each individual can inherit only the trait of the parents*. The dynamics of Volterra QSOs was somehow studied successfully by Ganikhodzhaev (1993). However, not all QSOs are of Volterra-type, therefore dynamics of non-Volterra QSOs remains open. This is not an easy task, hence many researcher are likely to introduce new classes of QSOs such as strictly non-Volterra QSOs, Centered QSOs, Bistochastic QSOs, $b-$bistochastic QSOs, F-QSOs, saparable QSOs, ℓ-Volterra and etc. Yet, all these classes do not cover the set of whole QSOs. Ganikhodzhaev et al. (2011), Mukhamedov and Ganikhodjaev (2015), it has given along self-contained exposition of the recent achievements and open problems in the theory of the QSOs. The main problem in the nonlinear operator theory is to study the behavior of nonlinear operators.

Presently, there are only a small number of studies on dynamical phenomena on higher dimensional systems, even though they are very important. Note that, most of the studies stated before were done on finite set of all probabilistic distributions. However, there are models where the probability distribution are given on countable set, which means that the corresponding QSOs are defined on infinite-dimensional space. The simplest case, the infinite-dimensional space should be the Banach space ℓ^1 of absolutely summable sequences. It is worth mentioning that Volterra QSOs, orthogonal preserving QSOs on infinite-dimensional space were studied by Mukhamedov et al. (2005), Mukhamedov and Embong (2017).

In the present paper, we are going to investigate infinite dimensional Volterra QSOs. Here we construct linear Lyapunov functions for such kind of operators, which allows investigation of limiting set.

2. Preliminaries

The space of ℓ^∞ is defined as follows:

$$\ell^\infty = \{\mathbf{x} = (x_1, x_2, \dots); \quad x_k \in \mathbb{R}, \sup |x_k| < \infty\}.$$

For a subset $\ell^1 \subset \ell^\infty$ denotes the space of all *absolutely summable sequences* i.e.,

$$\ell^1 = \left\{ \mathbf{x} = (x_k)_{k\in\mathbb{N}}; \|\mathbf{x}\|_1 = \sum_{k=1}^{\infty} |x_k| < \infty \right\}. \tag{2}$$

Denote

$$S = \{\mathbf{x} = (x_k)_{k\in\mathbb{N}} \in \ell_1 : \ x_k \geq 0 \text{ for all } k \in \mathbb{N}, \ \|\mathbf{x}\|_1 = 1\}. \tag{3}$$

It is known that $S = convh(ExtrS)$, where $Extr(S)$ is the extremal points of S and $convh(A)$ is the convex hall of a set A. Any extremal point of S has the following form:

$$\mathbf{e}_k = (0, \dots, 0, 1, 0, \dots), \quad k \in \mathbb{N},$$

where "1" stands at k^{th} position. That is, vertices of the simplex are extreme points of the simplex.

Now, let V be a mapping defined on the simplex by

$$V(\mathbf{x})_k = \sum_{i,j=1}^{\infty} P_{ij,k} x_i x_j \quad , k \in \mathbb{N}, \tag{4}$$

where, $P_{ij,k}$ are *hereditary coefficients* satisfying

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{\infty} P_{ij,k} = 1, \quad i,j,k \in \mathbb{N}. \tag{5}$$

One can check that V maps S into itself. The operator V is called *quadratic stochastic operator (QSO).*

We recall a QSO $V : S \to S$ is called Volterra if and only if

$$P_{ij,k} = 0 \text{ if } k \notin \{i,j\} \tag{6}$$

Taking into account (4), one easily can check that (6) is equivalent to the following canonical form of V

$$V(\mathbf{x})_k = x_k \left(1 + \sum_{i=1}^{\infty} a_{ki} x_i\right) \text{ for all } k \in \mathbb{N}, \tag{7}$$

where $a_{ki} = 1 - 2P_{ik,k}$. By means of (5) one finds

$$|a_{ki}| \leq 1 \text{ and } a_{ki} = -a_{ik}. \tag{8}$$

The symbols $\mathcal{V}$ and $\mathbb{A}$ denote the set of all Volterra QSOs and set of all matrices (a_{ki}) satisfying (8). The representation (7) establishes a bijective correspondence $\mathfrak{f} : \mathcal{V} \to \mathbb{A}$ by $\mathfrak{f}(V) = (a_{ki})$. Mukhamedov et al. (2005) investigated several properties of infinite dimensional Volterra operators but the dynamics were left out. So in this paper, we are going to investigate the dynamics of a class Volterra QSOs.

Here and henceforth, we use $V^{(n)}(\mathbf{x}_0)$ to represent the iterations of the given QSO V and $\mathbf{x}_0 \in S$ i.e.,

$$V^{(n+1)}(\mathbf{x}_0) = V(V^{(n)}(\mathbf{x}_0)), \; n \in \mathbb{N} \cup \{0\},$$

where $V^{(0)}(\mathbf{x}_0) = \mathbf{x}_0$.

Recall that, an ℓ^1-continuous function $\varphi : S \to \mathbb{R}$ is called a *Lyapunov function* for V if the limit $\lim\limits_{n\to\infty} \varphi(V^{(n)}(\mathbf{x}))$ exists for any initial point $\mathbf{x} \in S$.

3. The Lyapunov Functions

One of the main problems in mathematical biology is to study the asymptotic behavior of the trajectories. Ganikhodzhaev (1993, 1994), Ganikhodzhaev and Èshmamatova (2006) solved for the finite dimensional Volterra QSOs by using the theories of the Lyapunov function and tournaments. These Lyapunov functions used in Ganikhodzhaev (1993, 1994), Ganikhodzhaev and Èshmamatova (2006) were usually nonlinear. In particular, in Ganikhodzhaev et al. (2011) was stated an open problem: *to find new classes of Lyapunov function for Volterra QSOs*. Therefore in Jamilov (2012), the author motivated to construct new class of Lyapunov function namely, linear Lyapunov functions. So, we are interested to construct such Lypunov function for infinite dimensional Volterra QSOs.

Let $\{V^{(n)}(\mathbf{x}_0)\}_{n=1}^{\infty}$ be the trajectory of the point $\mathbf{x}_0 \in S$ under a QSO (6). By $\omega_V(\mathbf{x}_0)$ we denote the set of limit points (with respect to ℓ^1-norm) of the trajectory. Recall, for $\mathbf{x}^* \in \omega_V(\mathbf{x}_0)$ means that there exists a subsequence $\{n_k\}$ such that

$$\|V^{n_k}(\mathbf{x}_0) - \mathbf{x}^*\| \to 0, \quad n_k \to \infty.$$

Obviously, if $\omega_V(\mathbf{x})$ consists of a single point, then the trajectory converges, and $\mathbf{x} \in S$ is a fixed point of (6). However, looking ahead, we remark that convergence of the trajectories is not the typical case for the dynamical systems (6). Therefore, it is of particular interest to obtain an upper bound for $\mathbf{x}_0 \in S$, i.e., to determine a sufficiently "small" set containing $\mathbf{x}_0 \in S$. Denote

$$riS = \{x \in S; x_k > 0 \text{ for all } k \in \mathbb{N}\}, \quad \partial S = S \backslash riS.$$

Obviously, if $\lim\limits_{n\to\infty} \varphi(V^{(n)}(\mathbf{x}_0)) = c$, then $\omega(\mathbf{x}_0) \subset \varphi^{-1}(c)$.

In this section, we are going to construct Lypunov functions for three classes of infinite dimensional Volterra QSOs. For a given vector $\mathbf{b} \in \ell^{\infty}$, we define the following linear functional:

$$\varphi_{\mathbf{b}}(\mathbf{x}) = \sum_{k=1}^{\infty} b_k x_k, \qquad \mathbf{x} \in \ell^1. \tag{9}$$

One can see that the functional $\varphi_{\mathbf{b}}$ is well-defined on S (even on ℓ^1) i.e., $\varphi_{\mathbf{b}}(\mathbf{x}) < \infty$. Moreover it is ℓ^1-norm continuous.

Theorem 3.1. *Let $V \in \mathcal{V}$ and $\mathfrak{f}(V) = (a_{ij})$. Assume that $\boldsymbol{b} \in \ell^{\infty}$ such that for any pair $(k,i) \in \mathbb{N}^2$ one has $b_k a_{ki} \leq 0$ (resp. $b_k a_{ki} \geq 0$). Then the functional $\varphi_{\boldsymbol{b}}$ given by* (9) *on S is a Lyapunov function for V.*

Indeed, set of Volterra QSOs that satisfies condition given by Theorem 3.1 is non-empty. Consider the following example

Example 3.2. *Let us choose the following a skew-matrix associated with Volterra QSO V as follows:*

$$\begin{bmatrix}
0 & \cdots & a_{1m-1} & -1 & a_{1m+1} & \cdots & a_{1n-1} & 1 & a_{1n+1} & \cdots \\
a_{2,1} & \cdots & a_{2m-1} & -1 & a_{2m+1} & \cdots & a_{2n-1} & 1 & a_{2n+1} & \cdots \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{m-11} & \cdots & 0 & -1 & a_{m-1m+1} & \cdots & a_{m-1n-1} & 1 & a_{m-1n+1} & \\
1 & \cdots & 1 & 0 & 1 & \cdots & 1 & 1 & 1 & \cdots \\
a_{m+11} & \cdots & a_{m+1m-1} & -1 & 0 & \cdots & a_{m+1n-1} & 1 & a_{m+1n+1} & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n-11} & \cdots & a_{n-1m-1} & -1 & a_{n-1m+1} & \cdots & 0 & 1 & a_{n-1n+2} & \\
-1 & \cdots & -1 & -1 & -1 & \cdots & -1 & 0 & -1 & \cdots \\
a_{n+11} & \cdots & a_{n+1m-1} & -1 & a_{n+1m+1} & \cdots & a_{n+1n-1} & 1 & 0 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}$$

Choose $\mathbf{b} = (0, \dots, 0, -b_m, 0, \dots, 0, b_n, 0, \dots)$ such that $b_m, b_n > 0$. One sees that $b_k a_{ki} \leq 0$ for any $i, k \in \mathbb{N}$.

Now, we denote the following vector:

$$\begin{array}{ll}
\mathbf{b}_{\uparrow} = (b_1, \dots, b_n, \dots), & \text{such that } b_1 \leq \cdots \leq b_n \leq \cdots \\
\mathbf{b}_{\downarrow} = (b_1, \dots, b_n, \dots), & \text{such that } b_1 \geq \cdots \geq b_n \geq \cdots
\end{array}$$

Theorem 3.3. *Let $V \in \mathcal{V}$ such that $\mathfrak{f}(V) = (a_{ki})$ where $a_{ki} \geq 0$ for all $k < i$. If $\mathbf{b}_{\uparrow} \in \ell^{\infty}$, then a functional $\varphi_{\mathbf{b}_{\uparrow}}$ given by (9) on S is a Lyapunov function for V.*

Immediately form the last theorem, one can conclude the following result

Corollary 3.4. *Let $V \in \mathcal{V}$ such that $\mathfrak{f}(V) = (a_{ki})$ where $a_{ki} \leq 0$ for all $k < i$. If $\mathbf{b}_{\downarrow} \in \ell^{\infty}$, then a functional $\varphi_{\mathbf{b}_{\downarrow}}$ given by (9) on S is a Lyapunov function for V.*

Indeed, set of Volterra QSO that satisfies condition given by Theorem 3.3 is non-empty. Consider the following example:

Example 3.5. *Let us choose the following a skew-matrix associated with a Volterra QSO V as follows,*

$$\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & \cdots \\ -1 & 0 & 1 & \cdots & 1 & \cdots \\ -1 & -1 & 0 & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

4. Limiting Set

Finite dimensional simplex is compact but this is not true for infinite dimensional case. Hence, the dynamical investigation of QSOs on S may not applicable to employ well-known methods and techniques on a compact set. Moreover, the non-emptiness of limiting set ω_V of an infinite dimensional QSO is not obvious.

Fortunately, for a Volterra QSO V associated with skew-symmetric (a_{ki}) such that $a_{ki} \geq 0$ for $i > k$ (afterwards, the set of Volterra QSOs satisfy this condition is denoted by $\mathcal{V}^+$), the limiting set ω_V is non-empty.

Proposition 4.1. *Let $V \in \mathcal{V}^+$, then the limiting set $\omega_V \neq \emptyset$*

The following theorems are the main results in this paper which replicate finite dimensional case.

Theorem 4.2. *Let $V \in \mathcal{V}^+$, then for any initial point $\mathbf{x}_0 \in S \setminus Fix(V)$, we have $\omega_V(\mathbf{x}_0) \subset \partial S$.*

As a consequence of last theorem, we conclude the following corollary:

Corollary 4.3. *Let $V \in \mathcal{V}^+$ and $\mathfrak{f}(V) = (a_{ki})$. If there exists some $k_0, i_0 \in \mathbb{N}$ such that $a_{k_0 i_0} > 0$, then for $\mathbf{x}^* \in \omega_V(\mathbf{x}_0)$ we have*

$$\textit{either } x^*_{k_0} = 0 \textit{ or } x^*_{i_0} = 0$$

Theorem 4.2 reveals that the limiting set belongs to ∂S, but unfortunately we do not know about the exact location of $\omega_V(\mathbf{x})$ in ∂S. This problem is tricky for general case. Next two results partially answers to the mentioned question.

Proposition 4.1. *Let $V \in \mathcal{V}^+$ and $\mathfrak{f}(V) = (a_{ki})$. Assume there exists an integer $n_0 \geq 1$ satisfying $a_{n_0 i} > 0$ for all $i > n_0$. Then for any initial point $\mathbf{x} = (0, \dots, 0, x_{n_0}, x_{n_0+1}, \dots) \in S \setminus Fix(V)$ such that $x_{n_0} > 0$, we have*

$$\omega_V(\mathbf{x}) = \{\mathbf{e}_{n_0}\}$$

Theorem 4.4. *Let $V \in \mathcal{V}^+$ and $\mathfrak{f}(V) = (a_{ki})$. Assume that $a_{ki} > 0$ for all $i > k$. Then for any initial point $\mathbf{x} \in S \setminus Fix(V)$ we have*

$$\omega_V(\mathbf{x}) = \{\mathbf{e}_{\min\{supp(\mathbf{x})\}}\}.$$

Let us denote $\partial_\infty S$ be the infinite dimensional boundary of S (respectively, $\partial_f S$ be the finite dimensional boundary of S) i.e., for any $\mathbf{x} \in \partial_\infty S$ one has $|supp(\mathbf{x})| = \infty$ (respectively, for any $\mathbf{x} \in \partial_f S$ one has $|supp(\mathbf{x})| < \infty$). Next, we want to provide an example where the set ω_V lie on $ri\partial_\infty S$.

Example 4.5. *Let $V \in \mathcal{V}$ and $\mathfrak{f}(V) = (a_{ki})$ where $a_{2k-1,2k} > 0$ for all $k \in \mathbb{N}$ and the others be 0. One can see that the operator V belongs to $\mathcal{V}^+$ i.e., ω_V is non-empty (see Proposition 4.1). Moreover, it can be written in the following form:*

$$V(\mathbf{x}) = \begin{cases} V(\mathbf{x})_{2k-1} &= x_{2k-1}\left(1 + a_{2k-1,2k}x_{2k}\right), \\ V(\mathbf{x})_{2k} &= x_{2k}\left(1 + a_{2k,2k-1}x_{2k-1}\right), \end{cases}$$

for any $\mathbf{x} \in riS$ and $k \in \mathbb{N}$. From Corollary 4.3 we have

$$x^*_{2k-1} = 0 \text{ or } x^*_{2k} = 0 \text{ for any } k \in \mathbb{N} \tag{10}$$

where $\mathbf{x}^* \in \omega_V(\mathbf{x})$. Keeping in mind that $a_{2k-1,2k} > 0$, then for any coordinate $2k-1$ we have

$$\begin{aligned} V^{n+1}(\mathbf{x})_{2k-1} &= V^n(\mathbf{x})_{2k-1}\left(1 + a_{2k-12k}V^n(\mathbf{x})_{2k}\right) \\ &\geq V^n(\mathbf{x})_{2k-1} \end{aligned}$$

Due to $\mathbf{x} \in riS$, as n goes to infinity, we have

$$x^*_{2k-1} \geq \cdots \geq V^{n+1}(\mathbf{x})_{2k-1} \geq V^n(\mathbf{x})_{2k-1} \geq \cdots \geq V(\mathbf{x})_{2k-1} \geq \mathbf{x}_{2k-1} > 0$$

which means $x^*_{2k-1} > 0$ for every $k \in \mathbb{N}$. Using (10) one gets $x^*_{2k} = 0$ each of $k \in \mathbb{N}$. This gives us $\mathbf{x}^* \in ri\partial_\infty S$.

5. Conclusion

In this paper, we have considered a class infinite dimensional Volterra QSOs for which linear Lypunov functions have been constructed. This allowed us to describe the limiting (in the sense of norm convergence) set of the considered class of operators.

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