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Some Convergence Results Using K^* Iteration Process in Busemann Spaces

Ullah, K.[1], Khan, B. A.[1], Ozer, O. *[2], and Nisar, Z.[3]

[1] *Department of Mathematics, University of Science and Technology, Bannu, Pakistan*
[2] *Department of Mathematics, Kirklareli University, Turkey*
[3] *Department of Mathematics, International Islamic University, Islamabad, Pakistan*

E-mail: ozenozer39@gmail.com
* *Corresponding author*

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ABSTRACT

In this paper, some strong and Δ-convergence results for mapping satisfying condition (E) in the setting of uniformly convex Busemann spaces are proved. We are using newly introduced K^* iteration process for approximation of fixed point. We also give an example to show the efficiency of K^* iteration process. Our results are the extension, improvement and generalization of many known results in the literature of fixed point theory in Busemann spaces.

Keywords: Condition (E); Busemann spaces; K^* iterative process; Δ-convergence; strong convergence.

1. Introduction and preliminaries

Suppose that (X, d) is a metric space and $x, y \in X$. A geodesic path joining x to y is a mapping $\gamma : [a, b] \subseteq \mathbb{R} \to X$ such that $\gamma(a) = x, \gamma(b) = y$ and $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [a, b]$. In a particular, γ is an isometry and $d(x, y) = b - a$.

A geodesic segment joining x and y in X is the image of a geodesic path in X. The metric X is said to be a geodesic space, if every two points of X are joined by a geodesic. Moreover, X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$.

Let $\gamma : [a, b] \to X$ be a path in a metric space X. We say that γ is an affinely reparametrized geodesic if either γ is a constant path or there exists a geodesic path $\gamma' : [c, d] \to X$ such that $\gamma = \gamma' \circ \Psi$ where, $\Psi : [a, b] \to [c, d]$ is a unique affine homomorphism between the interval $[a, b]$ and $[c, d]$.

Suppose that X is a uniquely geodesic space and $\gamma([a, b])$ is a geodesic segment joining x and y, and $\lambda \in [0, 1]$. Then $z := \gamma((1 - \lambda)a + \lambda b)$ will be a unique point in $\gamma([a, b])$ satisfying $d(z, x) = \lambda d(x,\ y)$ and $d(z,\ y) = (1 - \lambda)d(x,\ y)$.

In the sequel, the notation $[x, y]$ is used for geodesic segment $\gamma([a, b])$ and z is denoted by $(1 - \lambda)x \oplus \lambda y$. A subset $K \subseteq X$ is said to be geodesically convex if K includes every geodesic segment joining any two of its points.

Let X be a geodesic metric space and $f : X \to \mathbb{R}$. We say that f is convex if for every geodesic path $\gamma : [a, b] \to X$, the map $f \circ \gamma : [a, b] \to \mathbb{R}$ is a convex. It is known that if $f : X \to \mathbb{R}$ is a convex function and $g : f(X) \to \mathbb{R}$ is an increasing convex function, then $g \circ f : X \to \mathbb{R}$ is convex.

Busemann (1948) developed a theory of non-positive curvature for path metric spaces, based on a simple axiom of convexity of the distance function.

Definition 1.1. *The geodesic metric space (X, d) is said to be Busemann space, if for any two affinely reparametrized geodesices $\gamma : [a, b] \to X$ and $\gamma' : [a', b'] \to X$, the map $D_{\gamma.\gamma'} : [a, b] \times [a', b'] \to \mathbb{R}$ defined by*

$$D_{\gamma.\gamma'}(t, t') = d(\gamma(h), \gamma'(t'))$$

is a convex; that is, the metric of Busemann space is convex.

The statement are equivalent en route for this meaning which are listed in (Papadopoulos, 2005, Proposition 8.1.2). Typical examples of Busemann space are $CAT(0)$ spaces, strictly convex Normed spaces, Minkowski spaces and simply connected Riemannian manifolds of nonpositive sectional curvature.

Let (X,d) be a Busemann space, and $x,y,z,w \in X$ and $\lambda, \lambda' \in [0,1]$. Then

(1) $d(z,(1-\lambda)x \oplus \lambda y) \leq (1-\lambda)d(z,x) + \lambda d(z,y)$,

(2) $d((1-\lambda)x \oplus \lambda y, (1-\lambda')x \oplus \lambda' y) = |\lambda - \lambda'|\, d(x,y)$,

(3) $(1-\lambda)x \oplus \lambda y = \lambda y \oplus (1-\lambda)x$,

(4) $d((1-\lambda)x \oplus \lambda z, ((1-\lambda)y \oplus \lambda w) \leq (1-\lambda)d(x,y) + \lambda d(z,w)$,

Hence Busemann spaces are also hyperbolic spaces, which were introduced by Kohlenbach (2005).

Definition 1.2. *The Busemann space X is called uniformly convex if for any $r > 0$ and $\varepsilon \in (0,2]$, there exists a map δ such that for every three points $a,x,y \in X$,*

$$\left.\begin{array}{r} d(x,a) \leq r \\ d(y,a) \leq r \\ d(x,y) \geq \varepsilon r \end{array}\right\} \Longrightarrow d(\frac{1}{2}x \oplus \frac{1}{2}y, a) \leq (1-\delta)r$$

for all $\varepsilon \in (0,2]$ and $\inf\{\delta : r > 0\} > 0$.

A mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$ for which $\eta(r,\varepsilon) := \delta$ for a given $r > 0$ and $\varepsilon \in (0,2]$ is called a modulus of uniform convexity.

From now on, modulus of uniform convexity with a decreasing modulus with respect to r (for a fixed ε) is called monotone modulus of uniform convexity. The following lemma shows a property of uniformly convex Busemann spaces which will be useful to obtain our main results.

Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is called (a) contraction if there exists $\theta \in (0,1)$ such that $d(Tx,Ty) \leq \theta d(x,y)$, for all $x,y \in K$ (b) nonexpansive if $d(Tx,Ty) \leq d(x,y)$, for all $x,y \in K$ and (c) quasi-nonexpansive if for all $x \in K$ and $q \in F(T)$ we have $d(Tx,q) \leq d(x,q)$, where $F(T)$ denotes the set $\{x \in X : T(x) = x\}$.

A mapping $T : K \to K$ is said to satisfy condition (C) if for all $x, y \in K$, we have

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq d(x, y).$$

A mapping $T : K \to K$ is called Suzuki generalized nonexpansive mappings if it satisfies condition (C). This class of mappings was introduced in Suzuki (2008).

Suzuki (2008) showed that the condition (C) is weaker than nonexpansiveness and stronger than quasi nonexpansiveness. Recently, Khan et al. (2017) established the existence and approximation results for SKC mappings in Busemann spaces.

Definition 1.3. *Let $\mu \geq 1$. A mapping $T : K \to K$ is said to satisfy condition (E_μ) if for all $x, y \in K$, we have*

$$d(x, Ty) \leq \mu d(x, Tx) + d(x, y).$$

We say that T satisfies condition (E), if T satisfies condition (E_μ) for some $\mu \geq 1$.

Following is the example of mapping satisfying condition (E), but not satisfying condition (C).

Example 1.1. *Define a mapping $T : K \longrightarrow K$ by*

$$T(x, y) = \begin{cases} (\frac{1+x}{3}, y) & if \;\; 0 \leq x \leq \frac{1}{3} \\ (0, y) & if \;\; \frac{1}{3} \leq x \leq 1, \end{cases}$$

where $K = [0, 1]^2 \subset X = (\mathbb{R}^2, d)$. Suppose that $x = (x_1, y_1)$ and $y = (x_2, y_2) \in K$.

If $x_1 = \frac{1}{3}$ and $x_2 = \frac{2}{3}$, then $x = (\frac{1}{3}, y_1)$ and $y = (\frac{2}{3}, y_2)$. Note that

$$\begin{aligned} d(x, T(x)) &= d((x_1, y_1), T(x_1, y_1)) \\ &= d((\frac{1}{3}, y_1), (\frac{1 + x_1}{3}, y_1)) \\ &= d((\frac{1}{3}, y_1), (\frac{1 + \frac{1}{3}}{3}, y_1)) \\ &= d((\frac{1}{3}, y_1), (\frac{4}{9}, y_1)) \\ &= \Big((\frac{1}{3} - \frac{4}{9})^2 + (y_1 - y_1)^2\Big)^{\frac{1}{2}} = \frac{1}{9}. \end{aligned}$$

That is, $\frac{1}{2}d(x,T(x)) = \frac{1}{18}$.

Also

$$\begin{aligned} d(x,y) &= d((x_1,y_1),(x_2,y_2)) \\ &= d\left((\frac{1}{3},y_1),(\frac{2}{3},y_2)\right) \\ &= \left((\frac{1}{3}-\frac{2}{3})^2+(y_1-y_2)^2\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{9}+(y_1-y_2)^2\right)^{\frac{1}{2}}. \end{aligned}$$

Hence

$$\frac{1}{2}d((x,T(x)) \leq d(x,y).$$

Now

$$\begin{aligned} d(T(x),T(y)) &= d(T(x_1,y_1),T(x_2,y_2)) \\ &= d\left((\frac{1+x_1}{3},y_1),(0,y_2)\right) \\ &= d\left((\frac{1+\frac{1}{3}}{3},y_1),(0,y_2)\right) \\ &= d\left((\frac{4}{9},y_1),(0,y_2)\right) \\ &= \left((\frac{4}{9}-0)^2+(y_1-y_2)^2\right)^{\frac{1}{2}} \\ &= \left(\frac{16}{81}+(y_1-y_2)^2\right)^{\frac{1}{2}} \\ &> \left(\frac{1}{9}+(y_1-y_2)^2\right)^{\frac{1}{2}} = d(x,y). \end{aligned}$$

Thus

$$\frac{1}{2}d(x,T(x)) \leq d(x,y) \nRightarrow d(T(x),T(y)) \leq d(x,y).$$

So T does not satisfy condition (C).

We now verify that T satisfies condition (E). Consider the following cases:

1. *$x_1 \leq \frac{1}{3}$ and $x_2 \leq \frac{1}{3}$ or $x_1 > \frac{1}{3}$ and $x_2 > \frac{1}{3}$.*

(a): If $x_1 = \frac{1}{3}$ *and* $x_2 = \frac{1}{4}$*, then* $x = (\frac{1}{3}, y_1)$ *and* $y = (\frac{1}{4}, y_2)$*. Note that*

$$\begin{aligned} d(x,y) &= d((x_1,y_1),(x_2,y_2)) \\ &= d((\frac{1}{3},y_1),(\frac{1}{4},y_2)) \\ &= ((\frac{1}{3}-\frac{1}{4})^2+(y_1-y_2)^2)^{\frac{1}{2}} \\ &= (\frac{1}{144}+(y_1-y_2)^2)^{\frac{1}{2}}. \end{aligned}$$

And

$$\begin{aligned} d(T(x),T(y)) &= d(T(x_1,y_1),T(x_2,y_2)) \\ &= d((\frac{1+x_1}{3},y_1),(\frac{1+x_2}{3},y_2)) \\ &= d((\frac{1+\frac{1}{3}}{3},y_1),(\frac{1+\frac{1}{4}}{3},y_1)) \\ &= d((\frac{4}{9},y_1),(\frac{5}{12},y_2)) \\ &= \left((\frac{4}{9}-\frac{5}{12})^2+(y_1-y_2)^2\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{1296}+(y_1-y_2)^2\right)^{\frac{1}{2}}. \end{aligned}$$

Thus $d(T(x),T(y)) \leq d(x,y)$ *implies that*

$$\begin{aligned} d(x,T(y)) &\leq d(x,T(x))+d(T(x),T(y)) \\ &\leq d(x,T(x))+d(x,y). \end{aligned}$$

(b) *: If* $x_1 = \frac{2}{3}$ *and* $x_2 = \frac{1}{2}$*, then* $x = (\frac{2}{3}, y_1)$ *and* $y = (\frac{1}{2}, y_2)$*. Clearly,*

$$\begin{aligned} d(x,y) &= d((x_1,y_1),(x_2,y_2)) \\ &= d((\frac{2}{3},y_1),(\frac{1}{2},y_2)) \\ &= \left((\frac{2}{3}-\frac{1}{2})^2+(y_1-y_2)^2\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{36}+(y_1-y_2)^2\right)^{\frac{1}{2}}, \end{aligned}$$

and

$$\begin{aligned} d(T(x),T(y)) &= d(T(x_1,y_1),T(x_{2,}y_2)) \\ &= d((0,y_1),(0,y_2)) \\ &= ((0-0)^2+(y_1-y_2)^2)^{\frac{1}{2}}. \end{aligned}$$

Thus $d(T(x),T(y)) \leq d(x,y)$ *implies that*

$$\begin{aligned} d(x,T(y)) &\leq d(x,T(x)) + d(T(x),T(y)) \\ &\leq d(x,T(x)) + d(x,y). \end{aligned}$$

2. In case, $(x_1 \leq \frac{1}{3}$ *and* $x_2 > \frac{1}{3})$ *or* $(x_1 > \frac{1}{3}$ *and* $x_2 \leq \frac{1}{3})$
(a) : *If* $x_1 = \frac{1}{3}$ *and* $x_2 = \frac{1}{2}$ *then* $x = (\frac{1}{3}, y_1)$ *and* $y = (\frac{1}{2}, y_2)$*, we have*

$$\begin{aligned} d(y,T(y)) &= d((x_2,y_2),T(x_{2,}y_2)) \\ &= d((\frac{1}{2},y_2),(0,y_2)) \\ &= \left((\frac{1}{2}-0)^2 + (y_2-y_2)^2\right)^{\frac{1}{2}} = \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} d(x,T(x)) &= d((\frac{1}{4},y_1),(\frac{5}{16},y_1)) \\ &= \left((\frac{1}{3}-\frac{4}{9})^2 + (y_1-y_1)^2\right)^{\frac{1}{2}} \\ &= \frac{1}{9}. \end{aligned}$$

Thus $d(y,T(y)) = \dfrac{1}{2} < \dfrac{2}{3} = 6d(x,T(x))$ *and*

$$\begin{aligned} d(x,T(y)) &\leq d(x,y) + d(y,T(y)) \\ &\leq d(x,y) + 6d(x,T(x)). \end{aligned}$$

(b) : *If* $x_1 = \frac{1}{2}$ *and* $x_2 = \frac{1}{3}$ *then* $x = (\frac{1}{2}, y_1)$ *and* $y = (\frac{1}{3}, y_2)$ *gives that*

$$\begin{aligned} d(y,T(y)) &= d((x_2,y_2),T(x_2,y_2)) \\ &= d((\frac{1}{3},y_2),(\frac{1+x_2}{3},y_2)) \\ &= d((\frac{1}{3},y_2),(\frac{4}{9},y_2)) \\ &= \left((\frac{1}{3}-\frac{4}{9})^2 + (y_2-y_2)^2\right)^{\frac{1}{2}} \\ &= \frac{1}{9}, \end{aligned}$$

and

$$\begin{aligned} d(x,T(x)) &= d(((x_1,y_1),T(x_1,y_1)) \\ &= d((\frac{1}{2},y_1),(0,y_1)) \\ &= \left((\frac{1}{2}-0)^2+(y_1-y_1)^2\right)^{\frac{1}{2}} \\ &= \frac{1}{2}. \end{aligned}$$

So $d(y,T(y)) = \frac{1}{9} < \frac{1}{2} = d(x,T(x))$ *and*

$$\begin{aligned} d(x,T(y)) &\leq d(x,y)+d(y,T(y)) \\ &\leq d(x,y)+d(x,T(x)). \end{aligned}$$

Hence the mapping T satisfies the condition (E).

Lemma 1.1. *If T is a mapping satisfying condition (E) and has a fixed point then T is a quasi-nonexpansive mapping.*

Let $\{x_n\}$ be a bounded sequence in a closed convex subset K of a Busemann space X. For $x \in X$, set;

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x_n,x).$$

The asymptotic radius of $\{x_n\}$ relative to K is given by

$$r(K,\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in K\},$$

and the asymptotic center of $\{x_n\}$ relative to K is the set

$$A(K,\{x_n\}) = \{x \in K : r(x,\{x_n\}) = r(K,\{x_n\})\}.$$

It is known that, in a Busemann space, $A(K,\{x_n\})$ consists of exactly one point.

Recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

In Busemann space it is known that every bounded sequence has a regular subsequence. Since in a Busemann space every regular sequence is a $\Delta-$convergent, we see that every bounded sequence in X has a $\Delta-$convergent subsequence.

Lemma 1.2. *Lawaong and Panyanak (2010) Let X be a Busemann space and $x \in X$, $\{t_n\}$ a sequence in $[b, c]$, for some $b, c \in (0, 1)$. If $\{x_n\}$, and $\{y_n\}$ are sequences in X satisfying*

$$\begin{aligned} &\lim_{n\to\infty} \sup d(x_n, x) \leq r, \quad \lim_{n\to\infty} \sup d(y_n, x) \leq r \\ &\text{and} \qquad \lim_{n\to\infty} \sup d(t_n x_n \oplus (1 - t_n) y_n, x) = r, \end{aligned}$$

for some $r \geq 0$, then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Lemma 1.3. *Let X be a Busemann space, $\{x_n\}$ a bounded sequence in X and K a subset of X. Then $\{x_n\}$ has a subsequence which is regular in K.*

Lemma 1.4. *(Dhompongsa et al., 2009, Proposition 2.1) If K is a closed convex subset of a uniformly convex Busemann space X and $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

Definition 1.4. *A sequence $\{x_n\}$ in Busemann space X is said to be Δ-convergent if there exists some $x \in X$ such that x is the unique asymptotic center of $\{u_x\}$ for every subsequence $\{u_x\}$ of $\{x_n\}$. In this case we write Δ-$lim_n x_n = x$ and call x the Δ-lim of $\{x_n\}$.*

Lemma 1.5. *Kirk and Panyanak (2008) Every bounded sequence in a complete Busemann space always has a Δ-convergent subsequence.*

Lemma 1.6. *(Kirk and Panyanak, 2008, Proposition 3.7) Suppose that K is closed convex subset of a Busemann space X and $T : K \to Y$ satisfies the condition (E). Then the conditions $\{x_n\}$ Δ-converges to x and $d(Tx_n, x_n) \to 0$, imply that $x \in K$ and $Tx = x$.*

It is clear that any strong convergent sequence is Δ-convergent. Also, if K is a closed convex subset of a Busemann space, then Δ-convergence of any bounded sequence to x implies that $x_n \to x$ (that is, the asymptotic center of $\{x_n\}$ with respect to K is x).

2. The K^* Iteration Process and Convergence Results

The well-known Banach contraction theorem use Picard iteration process for approximation of fixed point. By time, many iterative processes have been developed to approximate fixed point of contraction type of mapping in a Busemann type of ground spaces. Some of the other well-known iterative processes are Mann Mann (1953), Ishikawa Ishikawa (1974), S Agarwal et al. (2007), Noor Noor (2000), Abbas Abbas and Nazir (2014), SP Phuengrattana and Suantai (2011), S* Karahan and Ozdemir (2013), CR Chugh et al. (2012), Normal-S Sahu and Petrusel (2011), Picard Mann Khan (2013), Thakur-New Thakur et al. (2016), M Ullah and Arshad (2018) and so on. Recently Ullah and Arshad Ullah and Arshad (2018), introduce new three steps iteration process known as K^* iteration process, which can be written in the language of Busemann space as follows:

$$\begin{cases} x_0 \in K \\ z_n = (1-\beta_n)x_n \oplus \beta_n T x_n \\ y_n = T((1-\alpha_n)z_n \oplus \alpha_n T z_n) \\ x_{n+1} = T y_n, \end{cases} \tag{1}$$

where $n \geq 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1]$. We have the following example to show the efficiency of K^* iteration process. We compare K^* iteration process with S Agarwal et al. (2007), Thakur-New Thakur et al. (2016) and M Ullah and Arshad (2018) iteration processes.

Example 2.1. *Let $K = [0, 50]$ be endowed with absolute valued norm. Define $T : K \to K$ by $T(x) = (2x+3)^{\frac{1}{2}}$, for all $x \in K$. We see that, T is contraction with $F_T = \{3\}$. Take $\alpha_n = 0.70$ and $\beta_n = 0.65$. The iterative values for $x_1 = 3.5$ are given in Table 1. Figure 1 shows the convergence graph. The efficiency of the K^* iteration process is clear.*

Now we prove some convergence results for mappings satisfying Condition (E) using K^* iteration process in the setting of Busemann spaces.

Theorem 2.1. *Let K be a nonempty closed convex subset of a complete Busemann space X, and let $T : K \to K$ be a mapping satisfying condition (E) with $F(T) \neq \emptyset$. For arbitrary chosen $x_0 \in K$, let the sequence $\{x_n\}$ be generated by (1), then $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in F(T)$.*

Proof. Let $p \in F(T)$ and $z \in K$. Since T is a mapping satisfying condition (E), so

$$d(p, Ty) \leq \mu d(p, Tp) + d(p, y)$$

$$\begin{aligned} d(z_n, p) &= d(((1-\beta_n)x_n \oplus \beta_n T x_n), p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(Tx_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n[\mu d(p, Tp) + d(x_n, p] \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{2}$$

Table 1: Sequences generated by K^*, M, $Thakur - New$ and S iteration processes for mapping T of Example 2.1.

	K^*	M	$Thakur - New$	S
x_1	3.5	3.5	3.5	3.5
x_2	3.01633259231845	3.02873387531123	3.03753378464499	3.11313955067547
x_3	3.00054793920171	3.00169963610076	3.00289847467759	3.02611511985139
x_4	3.000018399354 99	3.00010070825475	3.00022432199869	3.00605599221187
x_5	3.00000061785423	3.00000596785823	3.00001736393623	3.00140587685381
x_6	3.0000000207477	3.00000035365072	3.00000134409580	3.00032645109197
x_7	3.00000000069671	3.00000002095708	3.00000010404296	3.00007580786309
x_8	3.0000000000234	3.0000000012419	3.00000000805370	3.00001760419844
x_9	3.00000000000079	3.00000000007359	3.00000000062342	3.00000408808220
x_{10}	3.00000000000003	3.00000000000436	3.00000000004826	3.00000094934332
x_{11}	3	3.00000000000026	3.00000000000374	3.00000022045861
x_{12}	3	3.00000000000002	3.00000000000029	3.00000005119539
x_{13}	3	3	3.00000000000002	3.00000001188871
x_{14}	3	3	3	3.00000000276082
x_{15}	3	3	3	3.00000000064112

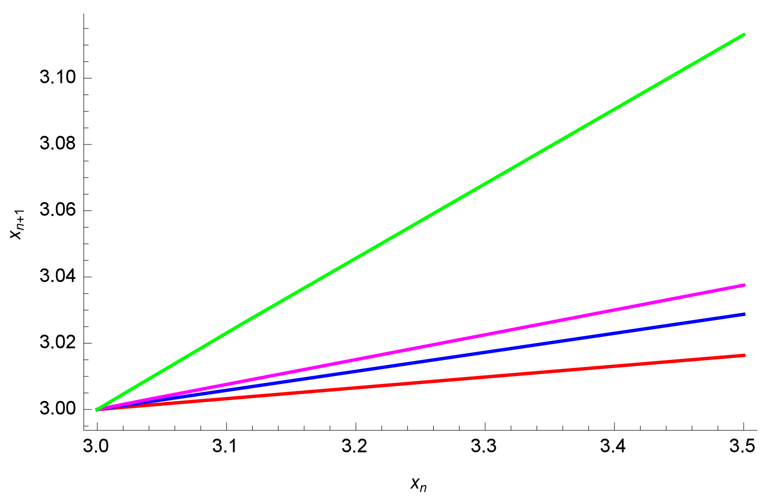

Figure 1: Convergence of K^*, M, Thakur-New and S iteration processes to the fixed point of the mapping define in Example 2.1 where $x_0 = 3.5$.

Using (2) we get

$$\begin{aligned} d(y_n,p) &= d((T(1-\alpha_n)z_n \oplus \alpha_n T z_n),p) \\ &\le d(((1-\alpha_n)z_n \oplus \alpha_n T z_n),p) \\ &\le (1-\alpha_n)d(z_n,p) + \alpha_n d(Tz_n,p) \\ &\le (1-\alpha_n)d(z_n,p) + \alpha_n[\mu d(p,Tp) + d(z_n,p)] \\ &\le (1-\alpha_n)d(x_n,p) + \alpha_n d(z_n,p) \\ &\le (1-\alpha_n)d(x_n,p) + \alpha_n d(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{3}$$

Similarly by using (3) we have

$$\begin{aligned} d(x_{n+1},p) &= d(Ty_n,p) \\ &\le \mu d(p,Tp) + d(y_n,p) \\ &\le d(y_n,p) \\ &\le d(x_n,p). \end{aligned} \tag{4}$$

This implies that $\{d(x_n,p)\}$ is bounded and non-increasing for all $p \in F(T)$. Hence $\lim_{n\to\infty} d(x_n,p)$ exists, as required. □

Theorem 2.2. *Let K be a nonempty closed convex subset of a complete Busemann space X, and let $T : K \to K$ be a mapping satisfying condition (E). For arbitrary chosen $x_0 \in K$, let the sequence $\{x_n\}$ be generated by* (1) *for all $n \ge 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a,b]$ for some a,b with $0 < a \le b < 1$. Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n,x_n) = 0$.*

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Theorem 2.1 , $\lim_{n\to\infty} d(x_n,p)$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} d(x_n,p) = r. \tag{5}$$

From (2) and (5), we have

$$\limsup_{n\to\infty} d(z_n,p) \le \limsup_{n\to\infty} d(x_n,p) = r. \tag{6}$$

By (3) we have

$$\limsup_{n\to\infty} d(y_n,p) \le \limsup_{n\to\infty} d(x_n,p) = r. \tag{7}$$

On the other hand by using (2), we have

$$\begin{aligned} d(x_{n+1},p) &= d(Ty_n,p) \\ &\leq d(y_n,p) \\ &= d((T(1-\alpha_n)z_n \oplus \alpha_n Tz_n),p) \\ &\leq d(((1-\alpha_n)z_n \oplus \alpha_n Tz_n),p) \\ &\leq (1-\alpha_n)d(z_n,p)+\alpha_n d(Tz_n,p) \\ &\leq (1-\alpha_n)d(z_n,p)+\alpha_n[\mu d(Tp,p)+d(z_n,p)] \\ &\leq (1-\alpha_n)d(x_n,p)+\alpha_n d(z_n,p) \\ &= d(x_n,p)-\alpha_n d(x_n,p)+\alpha_n d(z_n,p). \end{aligned}$$

This implies that

$$\frac{d(x_{n+1},p)-d(x_n,p)}{\alpha_n} \leq d(z_n,p)-d(x_n,p).$$

So

$$d(x_{n+1},p)-d(x_n,p) \leq \frac{d(x_{n+1},p)-d(x_n,p)}{\alpha_n} \leq d(z_n,p)-d(x_n,p),$$

implies that

$$d(x_{n+1},p) \leq d(z_n,p).$$

Therefore

$$r \leq \liminf_{n\to\infty} d(z_n,p). \tag{8}$$

By (6) and (8) we get

$$\begin{aligned} r &= \lim_{n\to\infty} d(z_n,p) \\ &= \lim_{n\to\infty} d(((1-\beta_n)x_n+\beta_n Tx_n),p) \\ &= \lim_{n\to\infty} d(\beta_n(Tx_n,p)+(1-\beta_n)(x_n,p)). \end{aligned} \tag{9}$$

From (5), (7), (9) and Lemma 1.2 we have that $\lim_{n\to\infty} d(Tx_n,x_n)=0$.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Then, by Lemma 1.3, $\{x_n\}$ has a subsequence which is regular with respect to K. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ such that $A_K(u_n) = x$. Therefore,

$$\limsup_{n\to\infty} d(u_n, Tp) \le \limsup_{n\to\infty} [\mu d(u_n, Tu_n) + d(u_n, x)] = \limsup_{n\to\infty} d(u_n, x).$$

Thus the uqiueness of asymptotic center implies that x is a fixed point of T and this completes the proof. ☐

Now we are in the position to prove Δ-convergence theorem.

Theorem 2.3. *Let K be a nonempty closed convex subset of a complete Busemann space X, and let $T : K \to K$ be a mapping satisfying condition (E) with $F(T) \neq \emptyset$. Let $\{t_n\}$ and $\{s_n\}$ be sequences in $[0,1]$ so that $\{t_n\} \in [a,b]$ and $\{s_n\} \in [0,b]$ or $\{t_n\} \in [a,1]$ and $\{s_n\} \in [a,b]$ for some a, b with $0 < a \le b < 1$. From orbitrary $x_0 \in K$, let the sequence $\{x_n\}$ generated by (1) for all $n \ge 1$. Then $\{x_n\}$ Δ-converges to a fixed point of T .*

Proof. Since $F(T) \neq \emptyset$, by Theorem 2.2 we have that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. We now let $w_w\{x_n\} := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $w_w\{x_n\} \subset F(T)$. Let $p \in w_w\{x_n\}$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{p\}$. By Lemma 1.4 and Lemma 1.5 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n \{v_n\} = p' \in K$. Since $\lim_{n\to\infty} d(v_n, Tv_n) = 0$, then $p' \in F(T)$ by Lemma 1.6. We claim that $p = p'$. Suppose not, since T is a mapping satisfying condition (E) and $p' \in F(T)$, $\lim_n d(x_n, p')$ exists by Theorem 2.1. Then by uniqueness of asymptotic centers,

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, p') &< \limsup_{n\to\infty} d(v_n, p) \\
&\le \limsup_{n\to\infty} d(u_n, p) \\
&< \limsup_{n\to\infty} d(u_n, p') \\
&= \limsup_{n\to\infty} d(x_n, p') \\
&= \limsup_{n\to\infty} d(v_n, p')
\end{aligned}$$

a contradiction, and hence $p = p' \in F(T)$. To show that $\{x_n\}$ Δ-converges to a fixed point of T, it is sufices to show that $w_w\{x_n\}$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 1.4 and Lemma 1.5 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n \{v_n\} = p' \in K$. Let $A(\{u_n\}) = \{p\}$ and $A(\{x_n\}) = \{p''\}$. We have seen that $c \in F(T)$. We can complete the proof by showing that $p'' = p'$. Suppose not, since $\{d(x_n, p')\}$ is convergent, then by the uniqueness of asymptotic centers,

$$\begin{aligned}
\limsup_{n\to\infty} d(v_n, p') &< \limsup_{n\to\infty} d(v_n, p'') \\
&\leq \limsup_{n\to\infty} d(x_n, p'') \\
&< \limsup_{n\to\infty} d(x_n, p') \\
&= \limsup_{n\to\infty} d(v_n, p')
\end{aligned}$$

a contradiction, and hence the conclusion follows. □

Next we prove the strong convergence theorem.

Theorem 2.4. *Let K be a nonempty compact convex subset of a Busemann space X, and let $T : K \to K$ be a mapping satisfying condition (E) such that $F(T) \neq \emptyset$. For arbitrary chosen $x_0 \in K$, let the sequence $\{x_n\}$ be generated by (1) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T .*

Proof. By Theorems 2.2 and 2.3, $\{x_n\}$ is bounded and Δ-converges to $x \in F(T)$. Suppose on the contrary that $\{x_n\}$ does not converge strongly to x. By the boundedly compact assumption, passing to subsequences if necessary, we may assume that there exists $x' \in K$ with $x' \neq x$ such that $\{x_n\}$ converge strongly to x'. Therefore,

$$\lim_{n\to\infty} d(x_n, x') = 0 \leq \lim_{n\to\infty} d(x_n, x)$$

Since x is the unique asymptotic center of $\{x_n\}$, it follows that $x' = x$, which is a contradiction. □

Senter and Dotson (1974) introduced the notion of a mappings satisfying condition (I) as.

Definition 2.1. *A mapping $T : K \to K$ is said to satisfy condition (I), if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in K$, where $d(x, F(T)) = \inf_{p \in F(T)} d(x, p)$*

Now we prove the strong convergence theorem using condition (I).

Theorem 2.5. *Let K be a nonempty closed convex subset of a Busemann space X, and let $T : K \to K$ be a mapping satisfying condition (E). For arbitrary chosen $x_0 \in K$, let the sequence $\{x_n\}$ be generated by* (1) *for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$ such that $F(T) \neq \emptyset$. If T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Theorem 2.1, we have $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$ and so $\lim_{n\to\infty} d(x_n, F(T))$ exists. Assume that $\lim_{n\to\infty} d(x_n, p) = r$ for some $r \geq 0$. If $r = 0$ then the result follows. Suppose $r > 0$, from the hypothesis and condition (I),

$$f(d(x_n, F(T))) \leq d(Tx_n, x_n). \tag{10}$$

Since $F(T) \neq \emptyset$, by Theorem 2.2, we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. So (10) implies that

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0. \tag{11}$$

Since f is a nondecreasing function, from (11) we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus, we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\} \subset F(T)$ such that

$$d(x_{n_k}, y_k) < \frac{1}{2^k} \text{ for all } k \in \mathbb{N}.$$

So using (5), we get

$$d(x_{n_{k+1}}, y_k) \leq d(x_{n_k}, y_k) < \frac{1}{2^k}.$$

Hence

$$\begin{aligned} d(y_{k+1}, y_k) &\leq d(y_{k+1}, x_{k+1}) + d(x_{k+1}, y_k) \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}} \to 0, \text{ as } k \to \infty. \end{aligned}$$

This shows that $\{y_k\}$ is a Cauchy sequence in $F(T)$ and so it converges to a point p. Since $F(T)$ is closed, $p \in F(T)$ and then $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} d(x_n, p)$ exists, we have that $x_n \to p \in F(T)$. $\square$

3. Conclusions

The extension of the linear version of fixed point results to nonlinear domains has its own significance. To achieve the objective of replacing a linear domain with a nonlinear one, Takahashi (1970) introduced the notion of a convex metric space and studied fixed point results of nonexpansive mappings in this framework. This initiated the study of different convexity structures on a metric space. Here we extend a linear version of convergence results to the fixed point of a mapping stisfying condition (E) for the newly intoduced K^* iteration process Ullah and Arshad (2018) to nonlinear Busemann spaces.

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