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Generalized Geodesic Convex Functions on Riemannian Manifolds

Kılıçman, A. *[1,2] and Saleh, W.[3]

[1] *Department of Mathematics, Faculty of Science, Universiti Putra Malaysia*
[2] *Institute for Mathematical Research, Universiti Putra Malaysia, Malaysia*
[3] *Department of Mathematics, Taibah University, Al- Medina, Saudi Arabia*

E-mail: akilic@upm.edu.my
* *Corresponding author*

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ABSTRACT

In the present paper, we introduce the generalized geodesic convex functions on Riemannian manifolds and present some of their properties. Based on these properties, the generalized geodesic star-shaped functions are established. Results obtained in this paper may inspire future research in convex analysis on manifolds.

Keywords: Geodesic convex sets, Geodesic convex functions, Riemannian manifolds.

1. Introduction

A function $f : U \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is called convex function on U, if and only if

$$f(\lambda a_1 + (1-\zeta)a_2) \leq \zeta f(a_1) + (1-\lambda)f(a_2),$$

for each $a_1, a_2 \in U$ and $\zeta \in [0,1]$.

The above function has much application in many fields, for example, in a biological system, economy Kılıçman and Saleh (2018a,b), Li et al. (2015), Szenthe, Wang et al. (2015).

In the literature, many authors studied generalized convex functions see Iqbal et al. (2010, 2012), Kılıçman and Saleh (2014) especially in Riemannian manifolds such as Rapcsák (2013) and Udriste (1994) who introduced geodesic convexity on a Riemannian manifold.

Furthermore, one of the important part in mathematics is fractional calculus which has many properties of fiunctions on fractal space (Babakhani and Daftardar-Gejji (2002), Erden and Sarikaya (2015), Set and Tomar (2016), Yang (2012), Yang et al. (2014, 2013), Zhao et al. (2013)).

2. Preliminaries

Mo and Xin (2014) presented the generalized convex function on fractal space as follows: let $S\colon M \subset \mathbb{R} \longrightarrow \mathbb{R}^{\alpha}$, $\forall \rho_1, \rho_2 \in M$ and $\gamma \in [0,1]$ if

$$S(\gamma\rho_1 + (1-\gamma)\rho_2) \leq \gamma^{\alpha} S(\rho_1) + (1-\gamma)^{\alpha} S(\rho_2)$$

holds, then S is a generalized convex function on M.

Now, consider that (N, g) is a complete m-dimensional Riemannian manifold with Riemannian connection $\bigtriangledown$.

Definition 2.1. *Udriste (1994). A subset $B \subseteq N$ is t-convex if and only if B contains every geodesic γ_{a_1,a_2} of N whose endpoints a_1 and a_2 are in B.*

Remark 2.1. *If B_1 and B_2 are t-convex sets, then $B_1 \cap B_2$ is t-convex set , but $B_1 \cup B_2$ is not necessarily t-convex set.*

Definition 2.2. *Udriste (1994). A function $g : S \subset N \longrightarrow \mathbb{R}$ is geodesic convex iff for all geodesic arcs γ_{a_1,a_2}, then*

$$g(\gamma_{a_1,a_2}(\lambda)) \leq \lambda g(a_1) + (1-\lambda)g(a_2)$$

for each $a_1, a_2 \in S$ and $\lambda \in [0,1]$.

3. Generalized Geodesic Convex Functions

Definition 3.1. *A function $g : S \longrightarrow \mathbb{R}^\alpha$ is generalized geodesic convex function on a t-convex set $S \subset N$ if and only if $\forall \gamma_{a_1,a_2}$, we have*

$$g(\gamma_{a_1,a_2}(\lambda)) \leq \lambda^\alpha g(a_1) + (1-\lambda)^\alpha g(a_2) \tag{1}$$

for each $a_1, a_2 \in S, \lambda \in [0,1]$ and $\alpha \in [0,1]$.

If (1) is strict $\forall a_1 \neq a_2 \in B, \lambda \in (0,1)$ and $\alpha \in (0,1)$, then S is a generalized strictly geodesic convex.

Remark 3.1. 1. *A generalized strictly geodesic convex function is also generalized geodesic convex, but the converse is not generally true.*

2. *In the event that inequality (1) is reversed, then S is a generalized geodesic concave function.*

Example 3.1. *Let $h \colon \mathbb{R} \longrightarrow \mathbb{R}^\alpha$ where $h(a) = -|a^\alpha|$ and the geodesic γ is given as*

$$\gamma_{a_1,a_2}(\lambda) = \begin{cases} -a_2 + \lambda(a_2 - a_1), & a_1 a_2 \geq 0; \\ -a_2 + \lambda(a_1 - a_2), & a_1 a_2 < 0. \end{cases}$$

Then

$$h(\gamma_{a_1,a_2}(\lambda)) \leq \lambda^\alpha h(a_1) + (1-\lambda)^\alpha h(a_2), \quad \forall \lambda \in [0,1]$$

which means that h is a generalized geodesic convex function, but not generalized strictly geodesic convex function.

While if $h(a) = a^{2\alpha}$, then h is a generalized strictly geodesic convex function.

Theorem 3.1. *A function $f : B \longrightarrow \mathbb{R}^\alpha$ is generalized geodesic convex iff every geodesic $\gamma : [n_1, n_2] \longrightarrow B$, the function $f o \gamma : [n_1, n_2] \longrightarrow \mathbb{R}^\alpha$ is a generalized convex, which means that,*

$$(fo\gamma)(\zeta a_1 + (1-\zeta)a_2) \leq \zeta^\alpha (fo\gamma)(a_1) + (1-\zeta)^\alpha (fo\gamma)(a_2)$$

for each $a_1, a_2 \in [n_1, n_2], \zeta \in [0,1]$ and $\alpha \in [0,1]$.

Proof. Assume that $g_{a_1,a_2} = fo\gamma_{a_1,a_2}$ and $g_{a_1,a_2} : [0,1] \longrightarrow \mathbb{R}^\alpha$ is a generalized convex, then

$$g_{a_1,a_2}(\zeta b_1 + (1-\zeta)b_2) \leq \zeta^\alpha g_{a_1,a_2}(b_1) + (1-\zeta)^\alpha g_{a_1,a_2}(b_2)$$

for each $b_1, b_2 \in [0,1], \zeta \in [0,1]$ and $\alpha \in [0,1]$. In particular, for $b_1 = 1$ and $b_2 = 0$, we get

$$g_{a_1,a_2}(\zeta) \leq \zeta^\alpha g_{a_1,a_2}(1) + (1-\zeta)^\alpha g_{a_1,a_2}(0), \forall \zeta \in [0,1], \alpha \in [0,1].$$

Hence

$$\begin{aligned} f(\gamma_{a_1,a_2}(\zeta)) &\leq \zeta^\alpha f(\gamma_{a_1,a_2}(1)) + (1-\zeta)^\alpha f(\gamma_{a_1,a_2}(0)) \\ &= \zeta^\alpha f(a_1) + (1-\zeta)^\alpha f(a_2), \quad a_1, a_2 \in B, \zeta \in [0,1], \quad \alpha \in [0,1]. \end{aligned}$$

Conversely, consider that f is a generalized geodesic convex function. If $\gamma_{a_1,a_2} : [0,1] \longrightarrow B$ is a geodesic joining a_1 and a_2, then the restriction of γ_{a_1,a_2} to $[b_1, b_2]$ joins $\gamma_{a_1,a_2}(b_1)$ and $\gamma_{a_1,a_2}(b_2)$. This restriction can be reparametrized as

$$\eta(\zeta) = \gamma_{a_1,a_2}(b_1 + \zeta(b_2 - b_1)), \zeta \in [0,1].$$

Since

$$f(\eta(\zeta)) \leq \zeta^\alpha f(\eta(1)) + (1-\zeta)^\alpha f(\eta(0)),$$

then

$$f(\gamma_{a_1,a_2}(\zeta b_1 + (1-\zeta)b_2)) \leq \zeta^\alpha f(\gamma_{a_1,a_2}(b_1)) + (1-\zeta)^\alpha f(\gamma_{a_1,a_2}(b_2)).$$

Hence

$$g_{a_1,a_2}(\zeta b_1 + (1-\zeta)b_2) \leq \zeta^\alpha g_{a_1,a_2}(b_1) + (1-\zeta)^\alpha g_{a_1,a_2}(b_2)$$

which implies that g_{a_1,a_2} is a generalized convex function on $[0,1]$. □

Now, let $\gamma^1_{a_1,a_2}(\lambda)$ where $\lambda \in [1, n_2]$ be a restriction of the natural extension of a geodesic $\gamma_{a_1,a_2} : [0,1] \longrightarrow B$ where $\gamma^1_{a_1,a_2}(\lambda) \in B, \forall \lambda \in [1, n_2]$. Let us give the following theorem:

Theorem 3.2. *$f : S \subset N \longrightarrow \mathbb{R}^\alpha$ is generalized geodesic convex iff for each $a_1, a_2 \in S, \lambda \geq 0$ and $\alpha \in [0,1]$ such that $\gamma^1_{a_1,a_2}(\lambda) \in S$, then*

$$f(\gamma^1_{a_1,a_2}(\lambda)) \geq \lambda^\alpha f(a_1) + (1-\lambda)^\alpha f(a_2).$$

Proof. Assume that $\gamma_{a_1,a_2} : [0,1] \longrightarrow S$ is a geodesic joining a_1 and a_2. Let $\gamma^1_{a_1,a_2}(u)$ where $u \in [0,\lambda]$ and $\lambda \geq 1$ a natural extension of γ_{a_1,a_2} beyond a_2, so that $\gamma^1_{a_1,a_2}(\lambda) \in S$. By taking $u = z\lambda$ where $z \in [0,1]$, then the reparametrization $\gamma^1_{a_1,a_2}(z\lambda)$. Due to f be a generalized geodesic convex function, then

$$f(\gamma^1_{a_1,a_2}(z\lambda)) \leq (1-z)^\alpha f(a_2) + z^\alpha f(\gamma^1_{a_1,a_2}(\lambda)), \forall z \in [0,1], \alpha \in [0,1].$$

Let $z\lambda = 1$, then

$$f(\gamma^1_{a_1,a_2}(\lambda)) \geq \lambda^\alpha f(a_1) + (1-\lambda)^\alpha f(a_2), \forall \lambda \geq 1, \alpha \in [0,1].$$

□

Assume that $\gamma^0_{a_1,a_2}(\lambda)$ where $\lambda \in [n_1, 0]$ is a restriction of the natural extension of a geodesic $\gamma_{a_1,a_2} : [0,1] \longrightarrow S$ where $\gamma^0_{a_1,a_2}(\lambda) \in S$, then the following theorem is obtained:

Theorem 3.3. *$f : B \subset N \longrightarrow \mathbb{R}^\alpha$ is generalized geodesic convex iff for each $a_1, a_2 \in B, \forall \lambda \leq 0$ and $\alpha \in [0,1]$, such that $\gamma^0_{a_1,a_2}(\lambda) \in B$, then*

$$f(\gamma^0_{a_1,a_2}(\lambda)) \geq \lambda^\alpha f(a_1) + (1-\lambda)^\alpha f(a_2).$$

Theorem 3.4. *Assume that $f_1 : S \longrightarrow \mathbb{R}^\alpha$ is a generalized geodesic convex function on S. If $f_2 : N \longrightarrow N$ is a diffeomorphism, then $f_1 o f_2^{-1}$ is a generalized geodesic convex function on the set $f_2(S)$.*

Proof. Assume that $a_1, a_2 \in S$ and $\gamma_{a_1,a_2}(\lambda)$ is a geodesic joining a_1 and a_2. The set $f_2(S)$ is a t-convex and the geodesic $f_2 o \gamma_{a_1,a_2}$ joins the point $f_2(a_1)$ and $f_2(a_2)$. Then

$$\begin{aligned}(f_1 o f_2^{-1})(f_2(\gamma_{a_1,a_2}(\lambda))) &= f_1(\gamma_{a_1,a_2}(\lambda)) \\ &\leq \lambda^\alpha f_1(a_1) + (1-\lambda)^\alpha f_1(a_2) \\ &= \lambda^\alpha (f_1 o f_2^{-1})(f_2(a_1)) + (1-\lambda)^\alpha (f_1 o f_2^{-1})(f_2(a_2)).\end{aligned}$$

Hence $f_1 o f_2^{-1}$ is a generalized geodesic convex function on the set $f_2(S)$. □

Theorem 3.5. *Assume that $f_1 : S \longrightarrow \mathbb{R}$ is a geodesic convex function. If $f_2 : M \longrightarrow \mathbb{R}^\alpha$ is a non-decreasing generalized convex function where rang of $(f_1) \subset M$, then $f_2 o f_1$ is a generalized geodesic convex function on S.*

Proof. By using the hypothesis , we get

$$\begin{aligned}f_2 o f_1(\gamma_{a_1,a_2}(\lambda)) &= f_2(f_1(\gamma_{a_1,a_2}(\lambda))) \\ &\leq f_2\left[\lambda f_1(a_1) + (1-\lambda) f_1(a_2)\right] \\ &\leq \lambda^\alpha f_2(f_1(a_1)) + (1-\lambda)^\alpha f_2(f_1(a_2)) \\ &= \lambda^\alpha (f_2 o f_1)(a_1) + (1-\lambda)^\alpha (f_2 o f_1)(a_2).\end{aligned}$$

Thus, $f_2 o f_1$ is a generalized geodesic convex on S. □

Theorem 3.6. *Assume that $f_i : B \subset N \longrightarrow \mathbb{R}^\alpha, i = 1, 2, ..., n$ are functions on a t-convex set B and*

$$f = \sum_{i=1}^{n} b_i^\alpha f_i, \quad b_i^\alpha \in \mathbb{R}^\alpha, b_i^\alpha \geq o^\alpha, i = 1, 2, ..., n.$$

Then

(i) If $f_i, \forall i$ are generalized geodesic convex functions on B, then f is a generalized geodesic convex function on B.

(ii) If $f_i, \forall i$ are generalized strictly geodesic convex functions on B, then f is a generalized strictly geodesic convex function on B.

Proof. (i) Since $f_i, \forall i$ is generalized geodesic convex functions, then

$$f_i(\gamma_{a_1,a_2}(\lambda)) \leq \lambda^\alpha f_i(a_1) + (1-\lambda)^\alpha f_i(a_2).$$

Thus,

$$b_i^\alpha f_i(\gamma_{a_1,a_2}(\lambda)) \leq \lambda^\alpha b_i^\alpha f_i(a_1) + (1-\lambda)^\alpha b_i^\alpha f_i(a_2),$$

$$\sum_{i=1}^{n} b_i^\alpha f_i(\gamma_{a_1,a_2}(\lambda)) \leq \lambda^\alpha \sum_{i=1}^{n} b_i^\alpha f_i(a_1) + (1-\lambda)^\alpha \sum_{i=1}^{n} b_i^\alpha f_i(a_2).$$

(ii) Similar to proof the above result.

□

Theorem 3.7. *Let $\{f_i\}_{i\in U}$ be a family of real-valued functions defined on B such that $\sup_{i\in U} f_i(x)$ exists in $\mathbb{R}^\alpha$. If $f_i \colon B \longrightarrow \mathbb{R}^\alpha, i \in U$ are generalized geodesic convex functions on B, then the function which is defined by $f(a) = \sup_{i\in U} f_i(a), \ \forall x \in B$ is a generalized geodesic convex function on B.*

Proof. Since $f_i, \ \forall i \in U$ are generalized geodesic convex function on B, then

$$f_i(\gamma_{a_1,a_2}(\lambda)) \leq \lambda^\alpha f_i(a_1) + (1-\lambda)^\alpha f_i(a_2),$$

$\forall a_1, a_2 \in B, \lambda \in [0,1]$ and $\alpha \in [0,1]$. Then

$$\begin{aligned} f(\gamma_{a_1,a_2}(\lambda)) &= \sup_{i\in U} f_i(\gamma_{a_1,a_2}(\lambda)) \\ &\leq \lambda^\alpha \sup_{i\in U} f_i(a_1) + (1-\lambda)^\alpha \sup_{i\in U} f_i(a_2) \\ &= \lambda^\alpha f(a_1) + (1-\lambda)^\alpha f(a_2). \end{aligned}$$

□

Definition 3.2. *Assume that $S \subset N$ is a t-convex set. A function $f : S \longrightarrow \mathbb{R}^\alpha$ is called*

(i) a generalized geodesic quasiconvex if and only if

$$f(\gamma_{a_1,a_2}(\lambda)) \leq \max\{f(a_1), f(a_2)\},$$

for each $a_1, a_2 \in S$, $\lambda \in [0,1]$ and $\alpha \in [0,1]$.

(ii) a generalized strictly geodesic quasiconvex if and only if

$$f(\gamma_{a_1,a_2}(\lambda)) < \max\{f(a_1), f(a_2)\},$$

for each $a_1, a_2 \in S$, with $\mathrm{a}_1 \neq \mathrm{a}_2$, $\lambda \in [0,1]$ and $\alpha \in [0,1]$.

In Example 3.1, if

$$h(a) = \begin{cases} \frac{|a^\alpha|}{a^\alpha}, & a \neq 0; \\ 0^\alpha, & a = 0 \end{cases}$$

then h is a generalized geodesic quasiconvex, but not generalized strictly geodesic quasiconvex.

Theorem 3.8. *Suppose the function $f_2 : B \longrightarrow \mathbb{R}$ is a geodesic quasiconvex function on a t-convex set B and $f_1 : \mathbb{R} \longrightarrow \mathbb{R}^\alpha$ is a non-decreasing function. Then $f_1 o f_2$ is a generalized geodesic quasiconvex function on B.*

Proof. Since f_2 is a geodesic quasiconvex on B and f_1 is a non-decreasing function, then

$$\begin{aligned} (f_1 o f_2)(\gamma_{a_1,a_2}(\lambda)) &= f_1(f_2(\gamma_{a_1,a_2}(\lambda))) \\ &\leq f_1[\max\{f_2(a_1), f_2(a_2)\}] \\ &= \max\{f_1(f_2(a_1)), f_1(f_2(a_2))\} \\ &= \max\{f_1 o f_2(a_1), f_1 o f_2(a_2)\}. \end{aligned}$$

Hence $f_1 o f_2$ is a generalized geodesic quasiconvex on B. □

Theorem 3.9. *Suppose that $g : S \subset N \longrightarrow \mathbb{R}^\alpha$ is a generalized geodesic quasiconvex on a t-convex S, then $b^\alpha g, b^\alpha > 0^\alpha$ is a generalized geodesic quasiconvex on S.*

Proof. For $b^\alpha > 0^\alpha$ we have

$$(b^\alpha g)(\gamma_{x,y}) \leq b^\alpha \max\{g(a_1), g(a_2)\} = \max\{b^\alpha g(a_1), b^\alpha g(a_2)\}.$$

□

Theorem 3.10. *Let $\kappa : S \subset N \longrightarrow \mathbb{R}^{\alpha}$ be a function on a t-convex set S.*

(i) If κ is a generalized geodesic convex function on S, then κ is a generalized geodesic quasiconvex function on S.

(ii) If κ is a generalized strictly geodesic convex function on S, then κ is a generalized strictly geodesic quasiconvex function on S.

(iii) If κ is a generalized strictly geodesic quasiconvex function on S, then κ is a generalized geodesic quasiconvex function on S.

Proof. (i) Since κ is a generalized geodesic convex on S. Then

$$\begin{aligned} \kappa(\gamma_{a_1,a_2}(\lambda)) &\leq \lambda^{\alpha}\kappa(a_1) + (1-\lambda)^{\alpha}\kappa(a_2) \\ &\leq \lambda^{\alpha}\max\{\kappa(a_1),\kappa(a_2)\} + (1-\lambda)^{\alpha}\max\{\kappa(a_1),\kappa(a_2)\} \\ &= \max\{\kappa(a_1),\kappa(a_2)\}. \end{aligned}$$

(ii) The second and third parts can be proven as the first part, while the last part followed directly by definition.

□

Remark 3.2. *From Definitions 3.1 and 3.2, we get*

$$\begin{array}{ccc} \text{Generalized Strictly Geodesic Convex} & \Longrightarrow & \text{Generalized Strictly Geodesic Quasiconvex} \\ \Downarrow & & \Downarrow \\ \text{Generalized Geodesic Convex} & \Longrightarrow & \text{Generalized Geodesic Quasiconvex} \end{array}$$

Theorem 3.11. *A function $g : S \subset N \longrightarrow \mathbb{R}^{\alpha}$ is a generalized geodesic quasiconvex on a t-convex S iff $L = \{a \in S : g(a) \leq n^{\alpha}, n^{\alpha} \in \mathbb{R}^{\alpha}\}$ is a t-convex.*

Proof. Let g be a generalized geodesic quasiconvex and assume that $a_1, a_2 \in L$, then $g(a_1) \leq n^{\alpha}$ and $g(a_2) \leq n^{\alpha}$. $g(\gamma_{a_1,a_2}(\lambda)) \leq \max\{g(a_1), g(a_2)\} \leq n^{\alpha}$, hence $\gamma_{a_1,a_2(\lambda)} \in L$.

Now, assume that L is a t-convex and $n^{\alpha} = \max\{f(a_1), f(a_2)\}$ for all $a_1, a_2 \in S$. Then $a_1, a_2 \in L$ implies that $\gamma_{a_1,a_2}(\lambda) \in L$. Hence

$$g(\gamma_{a_1,a_2}(\lambda)) \leq n^{\alpha} = \max\{g(a_1), g(a_2)\}$$

which means that g is a generalized geodesic quasiconvex. □

We give the following result as application of Theorem 3.11:

Theorem 3.12. *Assume that $g : S \subset N \longrightarrow \mathbb{R}^\alpha$ is a generalized geodesic quasiconvex on a t-convex S and let S^* be the set of all global minimum points of g. Then S^* is a t-convex.*

Proof. Assume that

$$S^* = \{a \in S : g(a) = n^\alpha\} = \{a \in S : g(a) \leq n^\alpha\}$$

where n^α is the minimum value of g on S, by Theorem 3.11, then S^* is a t-convex. □

Now, let us taking $S \subset N$ is a star-shaped set at a_0, then a function $f : S \longrightarrow \mathbb{R}^\alpha$ is called a generalized geodesic convex at a_0 if

$$g(\gamma_{a_0,a}(\lambda)) \leq \lambda^\alpha g(a_0) + (1-\lambda)^\alpha g(a),$$

for each $a \in S, \lambda \in (0,1)$ and $\alpha \in [0,1]$.

Remark 3.3. *Let $-g$ be a generalized geodesic convex at a_0, then g is called a generalized geodesic concave at a_0.*

Definition 3.3. *A program of type $\min_{a\in B} g(a)$, where $g : B \subset N \longrightarrow \mathbb{R}^\alpha$, is called a generalized geodesic convex if there is a Riemannian metric p on N such that the Riemannian manifold (N,p) is a complete, the set S is a t-convex in (N,p) and g is a generalized geodesic function.*

Theorem 3.13. *Let $a \in S \subset N$ be a local minimum point of $g : S \longrightarrow \mathbb{R}^\alpha$. If S is a star-shaped at a and g is a generalized geodesic convex function at a, then a is a global minimum point.*

Proof. Since $a \in S$ is a local minimum point of g , then there is neighborhood μ of a such that $g(a) \leq g(c), \ \forall c \in \mu \cap S$. Let $b \in S$ and $g(b) < g(a)$ and consider that $c = \gamma_{a,b}(\lambda), \ \lambda \in (0,1)$. Since S is a star-shaped at a and g is a generalized geodesic convex function at a, then

$$g(\gamma_{a,b}(\lambda)) \leq \lambda^\alpha g(a) + (1-\lambda)^\alpha g(b) < g(a).$$

On the other hand, $c = \gamma_{a,b}(\gamma) \in \mu \cap S$ for some $\lambda \in (0,1)$, which implies that $g(c) \geq g(a)$. This is a contradiction and hence $g(b) \geq g(a), \ \forall b \in S$. □

Corollary 3.1. *(i) For a generalized geodesic convex function the set of local minimum points and the set of global minimum points coincides.*

(ii) If $g : S \longrightarrow \mathbb{R}^\alpha$ is a generalized geodesic convex function, then g has the same minimum value on S.

Theorem 3.14. *The set of minimum points for a generalized convex program $\min_{a\in S} f(a)$ is a t-convex.*

Proof. Assume that $\min_{a\in S} f(a) = \phi$, then the result is obvious. Since the $\cap_i S_i$ is a t-convex set where S_i are t-convex, then $\min_{a\in S} f(a) = S \cap \{a \in S, f(a) \leq n^\alpha\}$ is a t-convex. □

Corollary 3.2. *Assume that $\min_{a\in S} f(a)$ has at least two distinct points, then it has an infinity and the function that we minimize is not generalized strictly geodesic convex.*

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