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Interval Shrinkage Estimators of Scale Parameter of Exponential Distribution in the Presence of Outliers

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ABSTRACT

The purpose of this paper is to find an estimation for scale parameter of exponential distribution with some extreme observations which are named outliers. We assume a guess value to find the shrinkage estimators and an interval for unknown parameter to obtain a new estimator that is called interval shrinkage estimator. It is shown that the mean square error of the interval shrinkage estimator is less than the shrinkage estimator base on the performing simulation study. Finally, we come to this conclusion that for some samples, interval shrinkage estimator gives better results in estimation of an unknown parameter.

Keywords: Interval information, mean square error, shrinkage estimator, two parameter exponentioal distribution, outliers.

1. Introduction

The exponential distribution is known for its essential role in life testing and reliability theory. The reciprocal of the scale parameter is the failure rate. Epstein and Sobel (1954) obtained the minimum variance unbiased estimator (MVUE) for its scale parameter and location parameter, respectively. The shrinkage procedure has been applied to a number of different problems. Bhattacharia and Srivastava (1974) proposed the shrinkage estimator for scale parameter. A prominent approach which allows utilizing additional (non-sample) information in the shrinkage estimators introduced by Stein (1955). The idea of shrinkage provides a balanced trade-off between a conventional estimator and a shrinkage target. Hawkins (1980) believes that an outlier as an observation that deviates so much from other observations as to arouse suspicions that it was generated by a different mechanism. Pandey (1983) uses the shrinkage estimation for location parameter of exponential distribution. Dixit and Nasiri (2001) considered estimation of parameters of exponential distribution in the presence of outliers generated from uniform distribution. Nasiri and Jabbari (2010) estimate the parameters of the generalized exponential distribution in the presence of outliers. Golosony (2011) introduces an interval for shrinkage estimator and Nasiri (2016) estimates the interval shrinkage for location parameter of exponential distribution.

The most widely life distribution use is the exponential distribution with probability density function as:

$$f(x,\theta) = \frac{1}{\theta}e^{\frac{-x}{\theta}}, \qquad x > 0, \theta > 0 \tag{1}$$

Let $(X_1, X_2, ..., X_n)$ be the random sample of size n taken form exponential distribution. The parameter θ is called the scale parameter, better known as average life. Proposed shrinkage estimator and its properties following Thompson (1968), a shrinkage estimator for the parameter and , a guess values of it is available, is defined as

$$\hat{\theta}_{Sh} = \theta_g + \omega(\hat{\theta} - \theta_g) \tag{2}$$

A shrinkage factor is defined based on guessed value. Let $\hat{\theta}_{Sh}$ be the shrinkage estimation of scale parameter. Then

$$\hat{\theta}_{Sh} = \theta_g + \omega(\hat{\theta} - \theta_g) \tag{3}$$

where $\hat{\theta} = \bar{X}$ such that $E(\hat{\theta}) = E(\bar{X}) = \theta$ and $V(\hat{\theta}) = V(\bar{X}) = \frac{\theta^2}{n}$

To find ω we have to consider MSE of estimator as:

$$MSE(\hat{\theta}_{Sh}) = [\hat{\theta}_{Sh} - \theta]^2 \tag{4}$$
$$= E[\theta_g + \omega(\hat{\theta} - \theta_g) - \theta]^2 \tag{5}$$
$$= E[\omega(\hat{\theta} - \theta) + (\omega - 1)(\theta - \theta_{Sh})]^2 \tag{6}$$
$$= \omega^2 MSE(\hat{\theta}, \theta) + (\omega - 1)^2 * (\theta - \theta_g)^2 + 2\omega(\omega - 1)(\theta - \theta_g)E(\hat{\theta} - \theta) \tag{7}$$
$$= \frac{\omega^2\theta^2}{n^2} + (\omega - 1)^2(\theta - \theta_g)^2 \tag{8}$$

Now, we have to minimize the MSE,

$$\frac{dMSE(\hat{\theta}_{Sh})}{d\omega} = \frac{2\omega\theta^2}{n} + 2(\theta_g - \theta)^2(\omega - 1) = 0 \tag{9}$$
$$\omega^* = \frac{(\theta_g - \theta)^2}{\frac{\theta^2}{n} + (\theta_g - \theta)^2} \tag{10}$$

So the shrinkage estimator is given by

$$\hat{\theta}_{Sh} = \theta_g + [\frac{(\theta_g - \theta)^2}{\frac{\theta^2}{n} + (\theta_g - \theta)^2}](\hat{\theta} - \theta_g) \tag{11}$$

2. Joint distribution of $(X_1, X_2, ..., X_n)$ in the presence of outliers

Dixit and Nasiri(2001) consider estimation of parameters of exponential distribution in the presence of outliers generated from uniform distribution. So, if we have random variables $(X_1, X_2, ..., X_n)$ such that k of them are distribution with pdf $g(x, \theta, \beta)$

$$g(x, \theta, \beta) = \frac{\beta}{\theta} e^{-\frac{\beta x}{\theta}}, x > 0, \beta > 0, \theta > 0 \tag{12}$$

and the remaining $(n - k)$ random variables are distributed with pdf $f(x, \theta)$ function

$$f(x, \theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, x > 0, \theta > 0 \tag{13}$$

Then the joint distribution of is $(X_1, X_2, ..., X_n)$ is

$$f(x_1, x_2, ..., x_n) = \frac{k!(n-k)!}{n!} \prod_{i=1}^{n} f(x_i, \alpha). \sum^{*} \prod_{j=1}^{k} \frac{g(x_{A_i})}{f(x_{A_i})} \tag{14}$$

where $\sum^{*} = \sum_{A_i=1}^{n} \sum_{A_2=A_1+1} \cdots = \sum_{A_k=A_{k-1}}^{n-k}$ For the $g(x, \theta, \beta)$ and $f(x, \theta)$, $f(x_1, x_2, ..., x_n)$ is

$$f(x_1, x_2, ..., x_n) = \frac{k!(n-k)!}{n!} . \frac{e^{\frac{-\sum x_i}{\theta}}}{\theta^n} \sum^{*} \prod_{i=1}^{k} \frac{\beta e^{-\beta x_{A_j}}}{e^{\frac{-x_{A_j}}{\theta}}} \tag{15}$$

$$= \frac{k!(n-k)!\beta^k}{n!\theta^n} . e^{\frac{-\sum x_i}{\theta}} \sum^{*} \prod_{i=1}^{k} \frac{e^{-\beta x_{A_j}}}{e^{\frac{-x_{A_j}}{\theta}}} \tag{16}$$

$$= \frac{k!(n-k)!\beta^k}{n!\theta^n} . e^{\frac{-\sum x_i}{\theta}} \sum^{*} \prod_{i=1}^{k} e^{-(\beta-1)\frac{x_{A_j}}{\theta}} \tag{17}$$

From (17) the marginal distribution of X in obtained as

$$f(x, \theta, \beta) = \frac{k}{n} g(x, \theta, \beta) + \frac{n-k}{n} f(x, \theta) \tag{18}$$

$$= \frac{k}{n} e^{\frac{-\beta x}{\theta}} + \frac{n-k}{n} e^{\frac{-x}{\theta}}, x > 0 \tag{19}$$

So we have

$$E(\bar{X}) = \Big[\frac{k}{n\beta} + \frac{(n-k)}{n}\Big]\theta \tag{20}$$

$$V(\bar{X}) = \frac{\theta^2}{n^2}\Big[\frac{k^2}{\beta^2} + (n-k)^2\Big] \tag{21}$$

It is easy to show that

$$\hat{\theta} = \frac{n\beta\bar{X}}{k + \beta(n-k)} \tag{22}$$

$$\hat{\beta} = \frac{-k(1 \pm \sqrt{1-M^2})}{M(n-k)^2} \tag{23}$$

where

$$M = \frac{n\bar{X}^2}{n\Big(\overline{X^2} - \bar{X}^2\Big)} - 1$$

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3. Shrinkage estimator of θ with present of outlier

From (2) and (22) we have

$$\hat{\theta}_{Sh-outlier} = \theta_g + \omega\big(A\bar{X} - \theta_g\big) \tag{24}$$

Such that $E\big(A\bar{X}\big) = \theta$ and $V\big(A\bar{X}\big) = A^2C\big(\frac{\theta}{n}\big)^2$, where $A = \frac{n\beta}{k+\beta(n-k)}$ and $C = \frac{k^2+\beta^2(n-k)^2}{n\beta^2}$.

The ω which minimize the MSE is obtained by

$$\begin{aligned}
MSE(\hat{\theta}_{Sh-outlier}) &= E(\hat{\theta}_{Sh-outlier} - \theta)^2 \\
&= E\Big(\theta_g + \omega(A\bar{X} - \theta_g) - \theta\Big)^2 \\
&= E\Big(\omega(A\bar{X} - \theta) + (\omega - 1)(\theta - \theta_g)\Big)^2 \\
&= E\Big(\omega^2(A\bar{X} - \theta)^2 + (\omega - 1)^2(\theta - \theta_g)^2 + 2\omega(\omega - 1)(A\bar{X} - \theta)(\theta - \theta_g)\Big) \\
&= \omega^2 A^2.\frac{C\theta^2}{n^2} + (\omega - 1)^2(\theta - \theta_g)^2
\end{aligned}$$

Thus

$$\frac{dMSE(\bar{\theta}_{Sh-outlier})}{d\omega} = 2\omega A^2\frac{C\theta^2}{n^2} + 2(\omega - 1)(\theta - \theta_g)^2 = 0 \tag{25}$$

and

$$\omega^* = \frac{(\theta - \theta_g)^2}{\frac{A^2C\theta^2}{n^2} + (\theta - \theta_g)^2} \tag{26}$$

Therefore,

$$\hat{\theta}_{Sh-outlier} = \theta_g + \frac{(\theta - \theta_g)^2}{\frac{A^2C\theta^2}{n^2} + (\theta - \theta_g)^2}(A\bar{X} - \theta_g) \tag{27}$$

4. Feasible interval shrinkage estimator

In 2011, Golosony and Liesenfeld show the shrinkage estimator towards the interval $\theta \in [\theta_0, \theta_1] \subset \Theta$ for unbiased conventional sample estimator of $\hat{\theta}$ with $E(\hat{\theta}) = \theta$ is given by

$$\widetilde{\theta}_{Sh} = \hat{\theta} + \sqrt{V(\hat{\theta})}.\frac{\theta - \hat{\theta}}{\theta_1 - \theta_0}\Big[arctan\Big(\frac{\theta_1 - \theta}{\sqrt{V(\hat{\theta}}}\Big) - arctan\Big(\frac{\theta_0 - \theta}{\sqrt{V(\hat{\theta}}}\Big) + \frac{V(\hat{\theta})}{2(\theta_1 - \theta_0)} ln\Big[\frac{V(\hat{\theta}) + (\theta_1 - \theta)^2}{V(\hat{\theta}) + (\theta_0 - \theta)^2}\Big]\Big] \tag{28}$$

and

$$E(\widetilde{\theta}_{Sh}) = \hat{\theta} + \frac{V(\hat{\theta})}{2(\theta_1 - \theta_0)} ln\Big[\frac{V(\hat{\theta}) + (\theta_1 - \hat{\theta})^2}{V(\hat{\theta}) + (\theta_0 - \hat{\theta})^2}\Big] \tag{29}$$

For $E(\hat{\theta}) = \theta$, we have

$$\widetilde{\theta}_{Sh} = \hat{\theta} + \frac{V(\hat{\theta})}{2(\theta_1 - \theta_0)} ln\Big[\frac{V(\hat{\theta}) + (\theta_1 - \hat{\theta})^2}{V(\hat{\theta}) + (\theta_0 - \hat{\theta})^2}\Big] \tag{30}$$

For different values of lower and upper bound of the interval in (30), we can see when θ_1 is far from θ_0 or $V(\hat{\theta})$ approaches zero, the $MSE(\hat{\theta})$ decreases. Furthermore, if $\hat{\theta}$ is considered as the median of the interval, $\theta_m = (\theta_0 + \theta_1)/2$, then $\widetilde{\theta}_{Sh}$ approaches $\hat{\theta}$.

Note that expectation and variance of $\widetilde{\theta}_{Sh}$ is not easy since $\widetilde{\theta}_{Sh}$ is not linear $\hat{\theta}$. Golosony and Liesenfeld(2011) suggest to find $\widetilde{\theta}_{Sh}$ by using first order Taylor expansion around the median point θ_m. We also define $\theta_d = (\theta_1 - \theta_0)/2$ and (30) would be as follows:

$$\widetilde{\widetilde{\theta}}_{Sh} = \hat{\theta}\Big[1 - \frac{V(\hat{\theta})}{V(\hat{\theta}) + \theta_d^2}\Big] + \theta_m \frac{V(\hat{\theta})}{V(\hat{\theta}) + \theta_d^2} \tag{31}$$

Since $\frac{V(\hat{\theta})}{V(\hat{\theta})+\theta_d^2}$ is constant, its variance is equal zero. So we can easily show that

$$E(\widetilde{\widetilde{\theta}}_{Sh}) = \theta - (\theta - \theta_m)\frac{V(\hat{\theta})}{V(\hat{\theta}) + \theta_d^2}$$

and

$$V(\tilde{\tilde{\theta}}_{Sh}) = V(\hat{\theta})\Big(1 - \frac{V(\hat{\theta})}{V(\hat{\theta}) + \theta_d^2}\Big)^2$$

and it can be shown that $MSE(\tilde{\tilde{\theta}}_{Sh}) \leq V(\hat{\theta})$.

Substitute in equation (31) we have

$$\tilde{\tilde{\theta}}_{Sh} = A\bar{X} - (A\bar{X} - \theta_m)\frac{A^2C\theta^2}{A^2C\theta^2 + n^2\theta_d^2} \tag{32}$$

where $A = \frac{n\beta}{k+\beta(n-k)}$ and $C = \frac{k^2+\beta^2(n-k)^2}{n\beta^2}$.

The estimator in (32) is linear point shrinkage in which $\hat{\theta}$ represents the shrinkage target. However the shrinkage weights are different from those of the point shrinkage estimator in (13).

5. Numerical Study

In this section, we generate samples with different sizes to compare the estimators that we introduced in pervious sections. Since each sample has outliers, we should consider k out n as outliers where n is the sample size. Then we generate different percentage of k outliers corresponding to different sample size n. We choose k as %5, %10, and %15 of n and generate k and $n-k$ data from density functions (12) and (13) respectively. Simulation study are carried out by using R software in order to compare the performance of bias and mean square error (MSE) of shrinkage $\hat{\theta}_{Sh}$ and shrinkage interval $\tilde{\tilde{\theta}}_{Sh}$ estimations. We generate data for $\theta = 4$ and $\beta = 1.5$, guessed value is 3.2 and interval shrinkage is [3.5 , 5]. The results are shown in Tables 1 to Table 3. Also, we change the parameters to $\theta = 2$, $\beta = 1.25$, $\theta_g = 1.6$, $\theta_0 = 1.75$, and $\theta_1 = 2.25$ and the results are arranged in Table 2 to 6. The MSE of both estimators decrease when sample size increase. Comparing the MSE of two estimators, states that the interval shrinkage estimator generally works better than shrinkage estimator;however, they are very close in most parts

Table 1: $k = \%5n,\quad \theta = 4,\quad \beta = 1.5,\quad \theta_g = 3.2,\quad \theta_0 = 3.75,\quad \theta_1 = 4.25$

n	$bias(\hat{\theta}_{Sh_outlier})$	$mse(\hat{\theta}_{Sh_outlier})$	$bias(\widetilde{\widetilde{\theta}}_{Sh})$	$mse(\widetilde{\widetilde{\theta}}_{Sh})$
30	2.337470	0.0007650744	0.3938728	0.0000300553
60	2.833982	0.0006394076	0.7270532	0.0000496252
100	3.103195	0.0004962448	1.0733880	0.0000656247
150	3.276657	0.0003129219	1.4113920	0.0000620766
200	3.366822	0.0003023896	1.6695760	0.0000781985
300	3.481230	0.0001949713	2.0529360	0.0000701129
500	3.577005	0.0001234335	2.5071690	0.0000618694

Table 2: $k = \%10n,\quad \theta = 4,\quad \beta = 1.5,\quad \theta_g = 3.2,\quad \theta_0 = 3.75,\quad \theta_1 = 4.25$

n	$bias(\hat{\theta}_{Sh_outlier})$	$mse(\hat{\theta}_{Sh_outlier})$	$bias(\widetilde{\widetilde{\theta}}_{Sh})$	$mse(\widetilde{\widetilde{\theta}}_{Sh})$
30	2.399378	0.0008176889	0.4220489	0.00003422966
60	2.817419	0.0007115986	0.7523316	0.0000593847
100	3.098809	0.0005173746	1.11403	0.00007349949
150	3.284544	0.0003678885	1.465511	0.00007798323
200	3.390227	0.0002929805	1.73586	0.00008049775
300	3.475403	0.0002143150	2.104501	0.00008109343
500	3.556843	0.0001519202	2.542427	0.00007910057

Table 3: $k = \%15n,\quad \theta = 4,\quad \beta = 1.5,\quad \theta_g = 3.2,\quad \theta_0 = 3.75,\quad \theta_1 = 4.25$

n	$bias(\hat{\theta}_{Sh_outlier})$	$mse(\hat{\theta}_{Sh_outlier})$	$bias(\widetilde{\widetilde{\theta}}_{Sh})$	$mse(\widetilde{\widetilde{\theta}}_{Sh})$
30	2.412799	0.0010953410	0.4420462	0.00004899848
60	2.860933	0.0007575479	0.7972043	0.00006810692
100	3.139813	0.0005546759	1.1744100	0.00008472827
150	3.298356	0.0004301953	1.5245390	0.00009745860
200	3.380164	0.0003361580	1.7862310	0.00009810429
300	3.478568	0.0002463646	2.1622050	0.00009802098
500	3.562721	0.0001533496	2.5960360	0.00008286679

Table 4: $k = \%5n,\ \ \theta = 2,\ \ \beta = 1.25,\ \ \theta_g = 1.6,\ \ \theta_0 = 1.75,\ \ \theta_1 = 2.25$

n	$bias(\hat{\theta}_{Sh_outlier})$	$mse(\hat{\theta}_{Sh_outlier})$	$bias(\widetilde{\widetilde{\theta}}_{Sh})$	$mse(\widetilde{\widetilde{\theta}}_{Sh})$
30	2.387207	0.0009436327	1.825147	0.0008430596
60	2.847388	0.0006760202	2.467896	0.0006289976
100	3.131686	0.0005215278	2.866995	0.0004967368
150	3.286639	0.0003944997	3.095003	0.0003810647
200	3.383795	0.0002895099	3.233487	0.0002818287
300	3.476014	0.0002161778	3.371132	0.0002121999
500	3.565923	0.0001471763	3.500497	0.0001454973

Table 5: $k = \%10n,\ \ \theta = 2,\ \ \beta = 1.25,\ \ \theta_g = 1.6,\ \ \theta_0 = 1.75,\ \ \theta_1 = 2.25$

n	$bias(\hat{\theta}_{Sh_outlier})$	$mse(\hat{\theta}_{Sh_outlier})$	$bias(\widetilde{\widetilde{\theta}}_{Sh})$	$mse(\widetilde{\widetilde{\theta}}_{Sh})$
30	0.767028	0.003355659	0.3650393	0.001237274
60	1.012017	0.002371847	0.6444473	0.001196376
100	1.132958	0.001743157	0.8346432	0.001074964
150	1.236198	0.001294352	0.9941109	0.000908537
200	1.271316	0.001076939	1.0722310	0.000814546
300	1.308489	0.000793063	1.1621020	0.000651630
500	1.370634	0.000481055	1.2733400	0.000425242

Table 6: $k = \%15n,\ \ \theta = 2,\ \ \beta = 1.25,\ \ \theta_g = 1.6,\ \ \theta_0 = 1.75,\ \ \theta_1 = 2.25$

n	$bias(\hat{\theta}_{Sh_outlier})$	$mse(\hat{\theta}_{Sh_outlier})$	$bias(\widetilde{\widetilde{\theta}}_{Sh})$	$mse(\widetilde{\widetilde{\theta}}_{Sh})$
30	0.8418695	0.003581749	0.4231912	0.001371443
60	0.9601013	0.002549009	0.6226864	0.001331104
100	1.1404420	0.001922833	0.8562380	0.001219990
150	1.2040800	0.001494765	0.9811700	0.001073303
200	1.2530300	0.001161111	1.0687020	0.000894844
300	1.2967730	0.000829634	1.1612760	0.000691196
500	1.3411430	0.000547258	1.2524430	0.000488175

6. Conclusion

The exponential distribution is absolutely important for its variety demands is Statistics. Estimation of its scale parameter in the presence of outliers can be closer to real value by using an interval shrinkage. Simulation Study accomplished to confirm it.

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