Analytical and Numerical Studies of Resonant Wave Run-up on a Plane Beach

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ABSTRACT

Wave run up is the vertical extent of wave up rushed on a beach. In the case of monochromatic wave run-up, it was a common belief that the leading wave will reach the maximum run-up. However, this is not always the case, when the incident wave is of normal mode frequency, resonance phenomena might appear. In this article, the normal mode frequency of a semi-enclosed basin on a plane beach is derived using the variable separation technique. Simulation results demonstrate that when the incoming wave frequency is close to the normal mode frequency, the largest run-up height is not the leading wave, but the second, third or fourth waves, which indicates the occurrence of resonance phenomena. Sensitivity analysis was applied to show the dependence of the maximum run-up height to the beach slope, as well as the incident wave frequency. Further, a simulation using a real bathymetry was conducted to examine whether the resonance phenomenon appears in the actual tsunami events.

Keywords: The Staggered Conservative Scheme, Wave Run-up, Resonance
1. Introduction

Wave run-up on a sloping beach has been studied in the past fifty years. One of the challenges in the run-up studies is on coping with moving shoreline. In 1958, an important contribution came from Carrier and Greenspan (1958) which used the hodograph transformation to find analytical solutions of the shallow water equations (SWE). Since then, many researches have used this Carrier and Greenspan transformation to study run-up characteristics of various waves, such as soliton in Kâno§lu and Synolakis (2006), Synolakis (1987), as well as N-wave in Tadepalli and Synolakis (1994). Later, this transformation is also used to study run-up characteristics of monochromatic waves as in Pelinovsky and Mazova (1992) and its relation with surf-similarity parameter as in Madsen and Fuhrman (2008). In 2007, Antuono in Antuono and Broccini (2007) and Matteo and Broccini (2008) used this transformation to solve the boundary value problem of SWE in order to find the run-up height formula.

However, most studies on wave run-up do not discuss resonance phenomena, whilst this phenomena may arise in certain situations, namely when the incoming wave frequency is close to beach natural frequency. The occurrence of this phenomena may worsen tsunami impact on beaches. Stefanakis et al. (2011) found that resonance may enhance the run-up of nonleading wave. This causes the run-up height of the following wave is higher than the leading wave. Resonance of long waves on the composite beach was also encountered by Stefanakis et al. (2015), as well as by Ezersky et al. (2013) which includes a discussion of tsunami application.

The staggered conservative scheme as proposed by Stelling and Duinmeijer (2003) is known to be a robust method for simulating various shallow water flows. This scheme can handle simulations involving wet-dry areas. Implementation of this scheme to several test cases, such as transcritical flow and dam break problem can be found in Magdalena et al. (2013), Pudjprasetya and Magdalena (2014). The main result of this paper is on the direct implementation of the staggered conservative scheme in simulating resonance phenomena of wave run up on a plane beach. The non-dissipative property of the scheme enable us to test the analytically derived natural frequency of the sloping beach system.

Our discussion in this paper is organized as follows. In Section 2, natural frequencies for the semi-enclosed beach is derived analytically. In Section 3, the staggered conservative scheme is reviewed. Run-up height of monochromatic waves and the occurrence of resonance phenomena are investigated in Section 4. Conclusions are given in the last section.
2. Natural Frequency of Long Waves Over a Sloping Beach

In this section we will derive the natural frequency for waves on a plane beach using the variable separation technique. Our discussion start with considering the following linear shallow water equations

\begin{align}
\eta_t + (d(x)u_x) &= 0 \\
u_t + g\eta_x &= 0
\end{align}

where \(\eta\) is surface elevation, \(u\) horizontal velocity, \(d(x)\) undisturbed water depth, and \(g\) gravitational acceleration. Eliminating \(u\) from (1) and (2) will give us

\[\eta_{tt} - (gd(x)\eta_x)_x = 0.\] (3)

Equation (3) is the homogeneous linear wave equation. If we restrict to a beach with a constant slope \(\alpha = \tan \theta\), to be explicit \(d(x) = -\alpha x\), the equation (3) becomes

\[\eta_{tt} + \alpha g\eta_x + \alpha gx\eta_{xx} = 0.\] (4)

The natural frequency for waves on a plane beach can be obtained via variable separation technique. Next we look for solutions in the form of

\[\eta(x,t) = X(x)e^{-i\omega t}\] (5)

where \(\omega\) is the wave frequency.

Consider a sloping beach with semi-enclosed boundary condition on a domain \(-L \leq x \leq 0\). Using (5), equation (4) reduces to an equation of \(X(x)\) reads

\[g\alpha(xX_{xx} + X_x) - \omega^2 X = 0, \quad -L < x < 0.\] (6)

Solutions of (6) are

\[X(x) = C_1J_0\left(\sqrt{-\frac{4\omega^2x}{g\alpha}}\right) + C_2Y_0\left(\sqrt{-\frac{4\omega^2x}{g\alpha}}\right),\]

where \(J_0\) and \(Y_0\) are the zero-th order of the first and second kind of Bessel functions, respectively. Moreover, under the assumption that wave amplitude is bounded, we should have \(\lim_{x \to 0} \eta(x,t)\) is finite. That means the constant \(C_2\) should be zero. For semi enclosed seas connected to the ambient ocean, as recorded by Kämpf (2009), the natural frequency is determined by the normal modes that has a node at the vicinity of the entrance. So, here we adopt
the boundary condition \( \eta(-L, t) = 0 \), from which we get a relation \( X(-L) = C_1 J_0 \left( 2 \sqrt{\frac{\omega^2 L}{gL}} \right) = 0 \). Let \( 2z_k \) be the \( k \)-root of \( J_0(x) \) on the interval \( -L \leq x \leq 0 \), to be explicit they read

\[
 z_1 = 1.2025, \quad z_2 = 2.760, \quad z_3 = 4.3270, \quad \cdots.
\]  

(7)

Hence, the natural frequency for waves on a plane beach are

\[
 \omega_k = z_k \sqrt{\frac{g\alpha}{L}}, \quad k = 1, 2, 3, \cdots
\]  

(8)

In Section 4 this natural frequency formulas (8) with \( z_k \) given in (7) is tested with numerical simulations of wave run-up on a sloping beach. Indeed, for a beach with slope \( \alpha \) and length \( L \), incident wave with frequency \( \omega_k \) will experience oscillation with increasing amplitude.

3. The Staggered Conservative Scheme of SWE

In this section we review the staggered conservative scheme of Stelling and Duimmeijer (2003) that will be used in all simulations conducted in this paper. Consider the following shallow water model, which holds for relatively long waves in a shallow region:

\[
 h_t + (hu)_x = 0, \quad \text{(9)}
\]

\[
 u_t + uu_x + g\eta_x = 0. \quad \text{(10)}
\]

In the above equations, \( u(x, t) \) denotes the depth-averaged horizontal velocity, \( h(x, t) \) denotes the total water layer thickness such that \( h(x, t) = \eta(x, t) + d(x) \), where \( \eta(x, t) \) denotes the free surface elevation and \( d(x) \) describes the depth of the bottom topography relative to the undisturbed free surface height, see Figure 1.

On the computational domain \([-L, L]\) which is divided into \( N_x \) cells of homogeneous length \( \Delta x \), a staggered partition points are

\[
x_{1/2} = -L, \quad x_{3/2} = \Delta x, \quad \cdots, \quad x_{j+1/2} = j\Delta x, \quad \cdots, \quad x_{N_x+1/2} = N_x\Delta x = L
\]  

(11)

On this staggered partitions, \( u \) are computed on staggered grid, whereas \( h \) (which also means \( \eta \)) are computed on full grids, as illustrated in Figure 2.
Here, we consider a staggered conservative scheme for solving the above shallow water equations, for which a discrete formulation of (9) (10) is given by

\[
\frac{h_i^{n+1} - h_i^n}{\Delta t} = -\left( \frac{\ast h_{i+\frac{1}{2}}^n u_{i+\frac{1}{2}}^n - \ast h_{i-\frac{1}{2}}^n u_{i-\frac{1}{2}}}{\Delta x} \right)
\]  

(12)

\[
\frac{u_{i+\frac{1}{2}}^{n+1} - u_{i+\frac{1}{2}}^n}{\Delta t} = -\frac{1}{h_{i+\frac{1}{2}}} \left( \frac{\bar{q}_{i+1} \ast u_{i+1} - \bar{q}_i \ast u_i}{\Delta x} - u_{i+\frac{1}{2}} \frac{\bar{q}_{i+1} - \bar{q}_i}{\Delta x} - g \left( \frac{\eta_{i+1}^{n+1} - \eta_i^{n+1}}{\Delta x} \right) \right),
\]

respectively. In the above equations

\[
\bar{h}_{i+\frac{1}{2}} = \frac{h_i + h_{i+1}}{2}, \quad \bar{q}_i = \frac{q_{i+\frac{1}{2}} + q_{i-\frac{1}{2}}}{2}, \quad q_{i+\frac{1}{2}} = \ast h_{i+\frac{1}{2}} u_{i+\frac{1}{2}}.
\]

Upwind approximations for \( \ast h \) and \( \ast u \) are given by

\[
\ast h_{i+\frac{1}{2}} = \begin{cases} 
  h_i, & \text{if } u_{i+\frac{1}{2}} \geq 0, \\
  h_{i+1}, & \text{if } u_{i+\frac{1}{2}} < 0
\end{cases}, \quad \ast u_i = \begin{cases} 
  u_{i-\frac{1}{2}}, & \text{if } \bar{q}_i \geq 0, \\
  u_{i+\frac{1}{2}}, & \text{if } \bar{q}_i < 0
\end{cases}
\]

(14)

For simulation of wave over a sloping beach, the numerical scheme should be able to accommodate wet-dry region. Here the wet-dry procedure is simply, computing the discrete formula (13) only if the water depth at the momentum cell \([x_{j-1/2}, x_{j+1/2}] \) is greater than a minimum threshold depth \( h_{\text{min}} \). Ideally,
this threshold depth $h_{\text{min}}$ is zero, but to avoid computation difficulties of division using small number, we often need to adopt $h_{\text{min}} \approx 10^{-2}$, or other small number depending on the particular problem we are dealing with. By adopting this simple requirement, the staggered scheme (12, 13) can handle simulations that involve dry areas. A more detailed discussion of this staggered scheme can be seen in Pudjaprasetya and Magdalena (2014), Stelling and Zijlema (2003), Stelling and Duinmeijer (2003).

3.1 Simulation of wave oscillation over a sloping beach

In this section, the staggered scheme (12, 13) is implemented on a computational domain $[-L, L]$, to simulate waves run-up over a sloping topography $d(x) = -\alpha x$. Sketch of notations and variables are depicted in Figure 1.

Our simulation use the initial still water level and the following parameters $L = 12.5$ m, $\alpha = 0.13$, and gravitational acceleration $g = 9.81$ m/s$^2$. Further, we employ a monochromatic wave influx that enters from the left by taking

$$\eta(-L, t) = \eta_0 \sin \omega t, \quad (15)$$

where $\eta_0$ and $\omega$ denote wave amplitude and frequency, respectively. Computation were conducted using $\Delta x = 0.1$ m, and $\Delta t = \frac{\Delta x}{2\sqrt{g\eta_0}} = 0.0124$ s, to maintain stability. This first simulation uses $\eta_0 = 0.025$ m, and $\omega = 0.6$ s$^{-1}$, snapshots of surface wave at subsequent times are given in Figure 3.

As depicted in Figure 1, the vertical extend of shoreline during run up process is denoted as $R$, whereas $R_{\text{max}}$ denotes its maximum run up. When shoreline position of the previous simulation, is recorded, the result is plotted on Figure 4 (top). From the inset figure we can observe that the leading wave hit the shoreline and reach the vertical extent of $R/\eta_0 = 4.68$. Wave up rush to the sloping beach is followed with wave run-down. As time progresses, this run-up and run-down process is continued. We observed that the following run up height is less than the first run up by the leading wave. The second run up height is $R/\eta_0 = 4.16$. In this numerical experiment the maximum run up is the fifth run up with $R/\eta_0 = 5.2$. In most simulations, the maximum run up height is achieved by the first wave. However, when the resonance takes place, run up of the second wave is larger, and run up of the third or later waves may even larger.
4. Resonance on a Plane Beach

Under the same set up like the previous, \( L = 12.5 \, m \), \( \alpha = 0.13 \), and \( \eta_0 = 0.025 \, m \), but now we conduct a simulation using monochromatic wave influx (15) with frequency \( \omega = 0.4 \, s^{-1} \). This time the frequency of the wave influx matches with the natural frequency \( \omega_1 \) as in (8). Shoreline position as a function of time was recorded, and the result is given in Figure 4 (bottom). This shoreline motion clearly show that this time the wave influx gives rise to a resonance phenomenon. As shown in Figure 4 that while run up height of wave \( \omega = 0.6 \, s^{-1} \) remains within the normal range of \( \frac{R}{\eta_0} \in [-4.16, 5.20] \), run up height of wave \( \omega = 0.4 \, s^{-1} \) exhibits resonance behavior: its value is increases from the first wave, the second and so on. This value continues to increase until the seventh wave that reaches the maximum of \( \frac{R_{\text{max}}}{\eta_0} = 27.04 \), which then slightly decreases afterwards. Thus, resonance phenomenon is clearly observable in the case of wave oscillation over a sloping beach with \( \omega = 0.4 \, s^{-1} \).
Figure 4: Plot of wave run-up $R/\eta_0$ as a function of time for beach with slope $\alpha = 0.13$ with wave frequency (top) $\omega = 0.6 \, s^{-1}$, (bottom) $\omega = 0.4 \, s^{-1}$.

Further, for both cases $\omega = 0.4 \, s^{-1}$ and $\omega = 0.6 \, s^{-1}$, we compute the total fluid volume $V = \int h \, dx$ as time progresses. It is shown in Figure 5 that indeed in the case of resonance, the total volume increases as time progresses, which means more and more fluid involve in the oscillation. Whereas in the case of non-resonance, the total fluid volume remains about the same as the initial volume $V_0$ up to 5% accuracy.
Moreover, total energy of the system is also computed. Here total energy is defined as the sum of potential energy $E_p$ and kinetic energy $E_k$ as follows

\begin{align}
E_p &= \frac{1}{2} \rho g \int_x \eta^2 \, dx, \\ E_k &= \frac{1}{2} \rho \int_x hu^2 \, dx. \tag{16, 17}
\end{align}

In the non-resonance case, the computed total energy remains constant, whereas in the case of resonance, the kinetic energy $E_k$ is oscillating with increasing amplitude. This increase in amplitude occurs as time progresses, but after sometime it slightly decreases. Something similar applies to the potential energy $E_p$, which means it applies to the total energy $E_k + E_p$ as well, see Figure 6.
4.1 Maximum run up

In this section, we study the dependence of maximum run-up with respect to wave frequency, and the beach slope.

For this study, three beach slopes were used, i.e. \( \alpha = 0.13, 0.26, 0.3 \), with the same beach length \( L = 12.5 \text{ m} \). For each slope, we conduct simulations using various wave frequency. For every beach slope, run up height \( R \) of each wave frequency \( \omega \) is measured, and the result is plotted in Figure 7. In Figure 7 the maximum run up is plotted with respect to the normalized frequency \( z = \omega / (g \alpha / L)^{1/2} \). It is shown that for all three slopes \( \alpha_1 = 0.13 \), \( \alpha_2 = 0.26 \), and \( \alpha_3 = 0.3 \), the first amplification of maximum run up is achieved at the first normalized frequency 1.2, followed by the second and third amplification that achieved at 2.8 and 4.4, respectively. Our numerical results clearly confirm the analytical formula of the normalized frequency as derived in Section 2 which is half of the zeroes of the Bessel function \( J_0 \) as provided in (7).
Further, the dependence of run-up height \( R_{\text{max}}/\eta_0 \) to \( \lambda_0 \) is studied by the following sensitivity analysis. In Figure 8 the maximum run up is plotted against the normalized wave length \( \lambda_0/L \), where \( \lambda_0 \) is associated to \( \omega \) through the shallow water relation

\[
\frac{2\pi}{\lambda_0} \sqrt{g\alpha} = \omega \quad \text{or} \quad \frac{\lambda_0}{L} = \frac{2\pi}{\omega} \sqrt{g\alpha}.
\]

Several simulations with various wave length \( \lambda_0 \) of monochromatic wave were conducted for beaches with three different slopes \( \alpha = 0.13, 0.26 \) dan 0.3. The highest value for the maximum wave run-up occur when \( \lambda_0/L = 5 \). Another high run up value is achieved when \( \lambda_0/L = 2 \).
4.2 Wave run-up on Mentawai beach

In this section we use real data to study whether similar resonant occur during real tsunami. We consider the Mentawai Island tsunami on October 25th 2010. Following Stefanakis et al. (2011) approach, here we use a tsunami influx from a virtual wave-gage located at Lon = 100.24°E, Lat = −3.4°N, on the water depth 120 m of a plane beach with slope $\alpha = 0.03$. Note that this slope is a good approximation of the actual topography near the location of the wave-gage. Simulation is conducted using our staggered numerical scheme, and run up height is measured and the result is plotted in Figure 9. It is shown that our result is nicely comparable with Stefanakis et al. (2011). The simulated shoreline position as plotted in Figure 9 show that the highest run up is achieved not by the leading wave, but by the third wave. This simple observation suggests that similar resonant phenomena may occur during real tsunami.

![Figure 9: Run up height during the first 2000 s on Mentawai beach, computed using our staggered scheme (solid) and the finite volume characteristic flux scheme by Stefanakis (dashed).](image)

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References

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