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Formulate a Relationship Between Saddle Points on Surfaces and Inflection Points on Curves

Mohd, I. *[1] and Dasril, Y. [2]

[1,2] *Institute for Mathematical Research, Universiti Putra Malaysia, Malaysia*

[2] *Center for Telecommunication Research and Innovation, Universiti Teknikal Malaysia Melaka, Malaysia*

E-mail: ismail_ayah_irma@yahoo.com

* *Corresponding author*

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ABSTRACT

It is known that there is a much closed relationship between saddle and inflection points. It was shown in one of the research papers that a connection between the saddle points of functions of two variables with the inflection points of functions of one variable and the researcher claimed that he has not found any references to this result in the literature. However, the author himself worried by asking whether there always exists such a one variable function that is differentiable at the saddle point or not. In this paper, it will be proposed two results for relationship between the saddle and inflection points through the quadratic functions of two variables and two linear and non-linear functions of one variable. These results will be supported with several numerical examples.

Keywords: Saddle, inflection, differentiation, quadratic, linear.

1. Introduction

The results of the research of surfaces of $f : \mathbb{R}^2 \to \mathbb{R}$, show that there exist some similarities between inflection points on curves and saddle points on surfaces.

In this paper, a direct connection between the two concepts are explained deeply. For this purpose, it is assumed that $f(x, y)$ has continuous second partial derivatives in an open set of the plane and that (a, b) is a critical point in that set. This means that the first derivative of $f(x, y)$ with respect to x and y are zeros. Therefore

$$\frac{\partial}{\partial x} f(x,y)\Big|_{(a,b)} = f_x(a,b) = 0, \quad \text{and} \quad \frac{\partial}{\partial y} f(x,y)\Big|_{(a,b)} = f_y(a,b) = 0$$

For convenience, suppose that

$$A = f_{xx}(a,b), \quad B = f_{xy}(a,b), \quad \text{and} \quad C = f_{yy}(a,b).$$

Then (Hass and Weir (2018)), the standard second derivative test for extreme points of functions of two variables, uses the determinant

$$D = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2$$

are (i) f has a local maximum point at (a, b) if $D > 0$ and $A < 0$; (ii) f has a local minimum point at (a, b) if $D > 0$ and $A > 0$; (iii) f has a saddle point at (a, b) if $D < 0$, and (iv) the second derivative test is inconclusive if $D = 0$.

A sufficient existence condition for a point of inflection is "if $f(x)$ is k times continuously differentiable in a certain neighborhood of a point x_0 with k odd and $k \geq 3$, while $f^{(n)}(x_0) = 0$ for $n = 2, ..., k-1$, and $f^{(k)}(x_0) \neq 0$, then $f(x)$ has a point of inflection at x_0" ((Dixon and Szego (1975)), (Wilde (1964)), (Horst and Tuy (1990))).

In our discussion, it is easier if we use the quadratic form

$$F(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= Ax^2 + 2Bxy + Cy^2.$$

where if (a,b) is a saddle point of f, then $F(x,y)$ is indefinite in the sense that there exist two points (x_1,y_1) and (x_2,y_2) with $x_1, x_2 \neq 0$ for which $F(x_1,y_1) > 0$ and $F(x_2,y_2) < 0$.

2. Result

In (de la Rosa (2007)), we are given the following theorem.

Theorem 2.1. *If (1) $f(x,y)$ is a function with continuous second partial derivatives in an open set U in the plane, (2) (a,b) is a saddle point in U, then there exists a continuous function $y = g(x)$ with $g(a) = b$ for which the projection on the xz-plane of the intersection of the surface $z = f(x,y)$ and the cylindrical surface $y = g(x)$ has an inflection point at $x = a$.*

In his proof, for satisfying the hypothesis of the theorem, the author has used a continuous linear function defined by

$$g(x) = \begin{cases} g_1(x) & (x < a) \\ g_2(x) & (a \leq x) \end{cases}$$

where

$$g_j(x) = \frac{y_j}{x_j}(x-a) + b, (j = 1,2).$$

By taking $u_j(x) = f(x, g_j(x)), (j = 1,2)$, the author has shown that (i) $u_1^{'}(a) = 0$ and $u_1^{''}(a) > 0$ for (x_1,y_1) which satisfies $F(x_1,y_1) > 0$ and (ii) $u_2^{'}(a) = 0$ and $u_2^{''}(a) < 0$ for (x_2,y_2) which satisfies $F(x_2,y_2) < 0$.

However, according to the formula of $g(x)$, we have

$$g_1(a) = g_2(a) = b, \quad g_1'(a) = \frac{y_1}{x_1} \text{ and } g_2'(a) = \frac{y_2}{x_2}.$$

Therefore, if $(y_1 \div x_1) \neq (y_2 \div x_2)$, then $g(x)$ satisfies the Theorem 2.1 but is not differentiable at $x = a$ since (x_1, y_1) and (x_2, y_2) are different points such that $F(x_1, y_1) > 0$ and $F(x_2, y_2) < 0$. If $(y_1 \div x_1) = (y_2 \div x_2)$, then

$$F(x_1, y_1) = \frac{x_1^2}{x_2^2} F(x_2, y_2).$$

Therefore both $F(x_1, y_1)$ and $F(x_2, y_2)$ have the same sign. Therefore, $F(x, y)$ is not indefinite. Whence, (a, b) is not a saddle point.

The above discussion proves the following theorem.

Theorem 2.2. *If (1) $f(x, y)$ is a function with continuous second partial derivatives in an open set U in the plane, (2) (a, b) is a saddle point in U, and (3) a piecewise linear continuous function $y = g(x)$ with $g(a) = b$ and not differentiable at $x = a$, then the projection on the xz-plane of the intersection of the surface $z = f(x, y)$ and the cylindrical surface $y = g(x)$ has an inflection point at $x = a$.*

Furthermore, in this paper, we proposed one more theorem to answer the question in de la Rosa (2007) which asking whether there always exists such a function $g(x)$ that is differentiable at $x = a$ for Theorem 2.1.

Theorem 2.3. *If (1) f is a function with continuous second partial derivatives in an open set U in the plane, (2) (a, b) is a saddle point in U, then there exists a continuous function $y = g(x) = x^2 + qx + r$ with $g(a) = b$ for which the projection on the xz-plane of the intersection of the surface $z = f(x, y)$ and the cylindrical surface $y = g(x)$ has an inflection point at $x = a$.*

Proof. By hypothesis, there exist points (x_1, y_1) and (x_2, y_2) with $x_1, x_2 \neq 0$ for which $F(x_1, y_1) > 0$ and $F(x_2, y_2) < 0$.

Suppose that $u_1(x) = f(x, g(x))$. The first derivative of $u_1(x)$ is given by

$$u_1^{'}(x) = f_x(x, g(x)) + f_g(x, g(x))g^{'}(x)$$

and the second derivative of $u_1(x)$ is given by

$$\begin{aligned} u_1^{''}(x) &= f_{xx}(x, g(x)) + f_{xg}(x, g(x))g^{'}(x) \\ &\quad + g^{'}(x)(f_{xg}(x, g(x)) + f_{gg}(x, g(x))g^{'}(x)) \\ &\quad + f_g(x, g(x))g^{''}(x) \end{aligned}$$

Since $f_g(a, g(a)) = 0$, we obtain

$$
\begin{aligned}
u_1^{''}(a) &= f_{xx}(a, g(a)) + 2f_{xg}(a, g(a))g^{'}(a) + f_{gg}(a, g(a))(g^{'}(a))^2 \\
&= f_{xx}(a, b) + 2f_{xg}(a, b)g^{'}(a) + f_{gg}(a, b)(g^{'}(a))^2
\end{aligned}
$$

By

$$g(a) = a^2 + qa + r = b,$$

we obtain $r = b - a^2 - qa$. Thus

$$g(x) = x^2 + qx + b - (a^2 + qa)$$

and

$$g^{'}(a) = 2a + q.$$

But, by the previous discussion, we can put that $g'(x) = (y_1 \div x_1)$. Therefore, we obtain $g'(a) = 2a + q = (y_1 \div x_1)$ and

$$
\begin{aligned}
u_1^{''}(a) &= f_{xx}(a, b) + 2f_{xg}(a, b)\frac{y_1}{x_1} + f_{gg}(a, b)(\frac{y_1}{x_1})^2 \\
&= \frac{1}{x_1^2}(f_{xx}(a, b)x_1^2 + 2f_{xg}(a, b)x_1y_1 + f_{gg}(a, b)y_1^2) \\
&= \frac{1}{x_1^2}F(x_1, y_1) > 0.
\end{aligned}
$$

Furthermore, by taking $u_2(x) = f(x, g(x))$ and $y_2 = (2a + q)x_2$, we will obtain

$$
\begin{aligned}
u_2^{''}(a) &= f_{xx}(a, b) + 2f_{xg}(a, b)\frac{y_2}{x_2} + f_{gg}(a, b)(\frac{y_2}{x_2})^2 \\
&= \frac{1}{x_2^2}(f_{xx}(a, b)x_2^2 + 2f_{xg}(a, b)x_2y_2 + f_{gg}(a, b)y_2^2) \\
&= \frac{1}{x_2^2}F(x_2, y_2) < 0.
\end{aligned}
$$

Then, $g(x)$ satisfies the theorem and differentiable at $x = a$. Therefore g is continuous at $x = a$ and $g(a) = b$. Thus the theorem valid. □

3. Numerical Results

Example 3.1. *In (de la Rosa, 2007), according to Theorem 2.1, the example to be considered is given by*

$$z = f(x, y) = -x^2 + y^2 \text{ and } y = x - x^2.$$

By substituting y *into* z*, the projection on the xz-plane, is given by* $z = x^3(x-2)$ *which has a point of inflection at* $x = 0$*.*

Example 3.2. *According to Theorem 2.3, the example to be considered is given by*

$$z = f(x, y) = -x^2 + y^2$$

which has a saddle point at $(0, 0)$*. For* $(x_1, y_1) = (1, 2)$ *and* $(x_2, y_2) = (2, 1)$*,* $F(x_1, y_1) > 0$ *and* $F(x_2, y_2) < 0$*. From*

$$g(x) = x^2 + qx + r,$$

and substitute x *with* 0 *to give*

$$g(0) = 0 + q(0) + r$$

which gives $r = 0$*. Thus we have* $g(x) = x^2 + qx$*. By intersection between the surfaces* $z = f(x, y)$ *and* $g(x) = x^2 + qx$*, to give*

$$z(x) = -x^2 + (x^2 + qx)^2$$

from which we obtain

$$z^{''}(x) = -2 + 2((2x + q)^2 + 2(x^2 + qx)).$$

At $x = 0$*,* $z^{''}(0) = 0$ *will yields* $q = 1$ *and* $q = -1$*. Therefore, by substituting* $q = 1$ *and* $q = -1$ *into* $z(x)$ *separately, we can observe that the projections on the xz-plane have a point of inflection at* $x = 0$ *for both* $q = 1$ *and* $q = -1$ *with* $z = x^3(x + 2)$ *and* $z = x^3(x - 2)$ *(see Example 3.1) respectively.*

4. Discussion

As mentioned in Theorem 2.2, we would like to make some note on a piecewise linear function $y = g(x)$ as follows. For Example 3.1, if we let $(x_1, y_1) = (1, m)$ and $(x_2, y_2) = (m, 1)$ where $m \neq -1, 0, 1$, then

$$F(1,m) = \begin{cases} > 0 & ((m > 1) \vee (m < -1)) \\ < 0 & ((0 < m < 1) \vee (-1 < m < 0)) \end{cases}$$

and

$$F(m,1) = \begin{cases} < 0 & ((m > 1) \vee (m < -1)) \\ > 0 & ((0 < m < 1) \vee (-1 < m < 0)) \end{cases}$$

Therefore, if we let

$$g(x) = \begin{cases} mx & (((x < 0) \wedge (m > 1)) \vee ((x < 0) \wedge (m < -1))) \\ \frac{1}{m}x & (((0 \leq x) \wedge (m > 1)) \vee ((0 \leq x) \wedge (m < -1))) \end{cases}$$

then we obtain

$$z_1(x) = \begin{cases} (m^2 - 1)x^2 & (((x < 0) \wedge (m > 1)) \vee ((x < 0) \wedge (m < -1))) \\ (\frac{1}{m^2} - 1)x^2 & (((0 \leq x) \wedge (m > 1)) \vee ((0 \leq x) \wedge (m < -1))) \end{cases}$$

Furthermore, if we let

$$g(x) = \begin{cases} mx & (((x < 0) \wedge (0 < m < 1)) \vee ((x < 0) \wedge (-1 < m < 0))) \\ \frac{1}{m}x & (((0 \leq x) \wedge (0 < m < 1)) \vee ((0 \leq x) \wedge (-1 < m < 0))) \end{cases}$$

then we obtain

$$z_2(x) = \begin{cases} (m^2 - 1)x^2 & ;\ (((x < 0) \wedge (0 < m < 1)) \vee ((x < 0) \wedge (-1 < m < 0))) \\ (\frac{1}{m^2} - 1)x^2 & ;\ (((0 \leq x) \wedge (0 < m < 1)) \vee ((0 \leq x) \wedge (-1 < m < 0))) \end{cases}$$

By observing $z_1(x)$ and $z_2(x)$, it is clear that the projection on the xz-plane has a point of inflection at $x = 0$ since $z_1(x)$ is convex upward for $(x < 0) \wedge (m > 1)$ and is convex downward for $(0 \leq x) \wedge (m > 1)$. In contrast, $z_2(x)$ is convex downward for $(x < 0) \wedge (0 < m < 1)$ and is convex upward for $(0 \leq x) \wedge (0 < m < 1)$.

5. Conclusion

In this paper, it has been shown that there is a strong relationship between saddle point of quadratic function of two variables with inflection point of linear piecewise function of one variable. The relationship is also can be extended to a quadratic function of one variable. By these remarks, it can be concluded that there exists such a function $g(x)$ that is differentiable at $x = a$.

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References

de la Rosa, F. M. (2007). Saddle points and inflection points. *The College Mathematics Journal.*, 38(5):380–383.

Dixon, L. C. W. and Szego, G. P. (1975). *Towards Global Optimization.* North-Holland Publishing Company, Amsterdam.

Hass, J. R. and Weir, C. H. M. D. (2018). *Thomas' Calculus: Early Transcendental.* Pearson, 14th edition.

Horst, R. and Tuy, H. (1990). *Global Optimization : Deterministic Approaches.* Springer-Verlag, Berlin Heidelberg New York.

Wilde, D. J. (1964). *Optimum Seeking Methods.* Prentice-Hall, New Jersey.