Malaysian Journal of Mathematical Sciences 14(2): 295–304 (2020)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

A Description of Derivations of a Class of Nilpotent Evolution Algebras

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Received: 23 March 2019
Accepted: 10 March 2020

ABSTRACT

As a system of abstract algebra,evolution algebras are non associative algebras. There is no deep structure theorem for general non associative algebra. However, there are deep structure theorem and classification theorem for evolution algebras because it has been introduced concepts of dynamical systems to evolution algebras.In the Mukhamedov et al. (2019) ,they have been studied some properties of nilpotent evolution algebra $\mathbf{E}$ with $\dim \mathbf{E^2} = \dim \mathbf{E} - 1$(maximal nilindex). In this paper, we describe derivations of nilpotent finite-dimensional evolution algebras $\mathbf{E}$ with $\dim \mathbf{E^2} = \dim \mathbf{E} - 2$.

Keywords: Nilpotent , evolution algebra and derivation.

1. Introduction

Recently in Tian and Vojtechovsky (2006) a new type of evolution algebra is introduced. This algebra also describes some evolution laws of the genetics. The study of evolution algebras constitutes a new subject both in algebra and the theory of dynamical systems. There are many related open problems to promote further research in this subject (for more details we refer to Tian (2008)).

We notice that evolution algebras are not defined by identities, and therefore they do not form a variety of non-associative algebras, like Lie, Jordan or alternative algebras. Hence, the investigation of such kind of algebras needs a different approach (see Camacho et al. (2013, 2010), Casas et al. (2014)).

In Casas et al. (2014) the equivalence between nil, right nilpotent evolution algebras and evolution algebras, which are defined by an upper triangular matrix of structural constants, have been established. A classification of low dimensional evolution algebras have been carried out in Casado et al. (2017), Elduque and Labra (2016), Hegazi and Abdelwahab (2015a,b). However, a full classification of nilpotent evolution algebras is far from its solution. Therefore, in the present paper we are going to investigate certain properties of nilpotent evolution algebras with maximal nilindex.

It is known that in the theory of non-associative algebras, particularly, in genetic algebras, the Lie algebra of derivations of a given algebra is one of the important tools for studying its structure. There has been much work on the subject of derivations of genetic algebras (Costa (1982), Costa (1983), Gonshor (1988), Holgate (1987), Mukhamedov and Qaralleh (2014)).

In fact, in Camacho et al. (2013) the authors investigate several properties of derivations of n-dimensional complex evolution algebras, depending on the rank of the appropriate matrices. In the present paper we explicitly describe the space of derivations of evolution algebras with nilindex $2^{n-2}+1$ which allows us to study further properties of the evolution algebras.

2. Evolution algebras

Recall the definition of evolution algebras. Let $\mathbf{E}$ be a vector space over a field $\mathbb{K}$. In what follows, we always assume that $\mathbb{K}$ has characteristic zero. The vector space $\mathbf{E}$ is called *evolution algebra* w.r.t. *natural basis* $\{\mathbf{e}_1, \mathbf{e}_2, ...\}$ if a multiplication rule ·Â· on **E** satisfies

$$\mathbf{e}_i \cdot \mathbf{e}_j = \mathbf{0}, \ i \neq j,$$
$$\mathbf{e}_i \cdot \mathbf{e}_i = \sum_k a_{ik}\mathbf{e}_k, \ i \geq 1.$$

From the above definition it follows that evolution algebras are commutative (therefore, flexible).

We denote by $A = (a_{ij})_{i,j=1}^n$ the matrix of the structural constants of the finite-dimensional evolution algebra **E**. Obviously, $rankA = \dim(\mathbf{E}\cdot\mathbf{E})$. Hence, for finite-dimensional evolution algebra the rank of the matrix does not depend on choice of natural basis.

In what follows for convenience, we write **uv** instead $\mathbf{u}\cdot\mathbf{v}$ for any $\mathbf{u}, \mathbf{v} \in \mathbf{E}$ and we shall write $\mathbf{E^2}$ instead $\mathbf{E}\cdot\mathbf{E}$.

A linear map $\psi : \mathbf{E}_1 \to \mathbf{E}_2$ is called an *homomorphism* of evolution algebras if $\psi(\mathbf{uv}) = \psi(\mathbf{u})\psi(\mathbf{v})$ for any $\mathbf{u}, \mathbf{v} \in \mathbf{E}_1$. Moreover, if ψ is bijective, then it is called an *isomorphism*. In this case, the last relation is denoted by $\mathbf{E}_1 \cong \mathbf{E}_2$.

For an evolution algebra **E** we introduce the following sequence, $k \geq 1$

$$\mathbf{E}^k = \sum_{i=1}^{k-1} \mathbf{E}^i\mathbf{E}^{k-i}. \tag{1}$$

Since **E** is commutative algebra we obtain

$$\mathbf{E}^k = \sum_{i=1}^{\lfloor k/2 \rfloor} \mathbf{E}^i\mathbf{E}^{k-i},$$

where $\lfloor x \rfloor$ denotes the integer part of x.

Definition 2.1. *An evolution algebra* **E** *is called nilpotent if there exists some* $n \in \mathbb{N}$ *such that* $\mathbf{E}^m = \mathbf{0}$. *The smallest* m *such that* $\mathbf{E}^m = \mathbf{0}$ *is called the index of nilpotency.*

Due to Theorem (3.1) Casas et al. (2014) any nilpotent evolution algebra **E** with $\dim(\mathbf{E}^2) = n - 2$ has the following form:

$$\mathbf{e}_i^2 = \begin{cases} \sum\limits_{j=i+1}^{n} a_{ij}\mathbf{e}_j, & i \leq n-2; \\ \mathbf{0}, & i \in \{n-1, n\}, \end{cases} \tag{2}$$

where $a_{ij} \in \mathbb{K}$ and $a_{i,i+1} \neq 0$ for any $i < n-1$.

In this paper, we consider only nilpotent evolution algebras with $2^{n-2}+1$ index of nilpotency. Therefore, we only consider evolution algebras with multiplication table given by (2).

Lemma 2.1. *Let* $\mathbf{E}_1$, $\mathbf{E}_2$ *be two isomorphic evolution algebras. Then* $Der(\mathbf{E}_1) \cong Der(\mathbf{E}_2)$.

Lemma 2.2. *Let* $\mathbf{E}$ *and* $\mathbf{E}'$ *be evolution algebras with basis* $\{\mathbf{e}_i\}_{i=1}^n$ *and* $\{\mathbf{f}_i\}_{i=1}^n$ *respectively, defined by*

$$\boldsymbol{e}_i^{\,2} = \begin{cases} a_{i,i+1}\boldsymbol{e}_{i+1} + a_{in-1}\boldsymbol{e}_{n-1} + a_{in}\boldsymbol{e}_n, & i < n-1; \\ 0, & i \in \{n-1, n\}. \end{cases}$$

$$\boldsymbol{f}_i^{\,2} = \begin{cases} \boldsymbol{f}_{i+1}, & i < n-; \\ \boldsymbol{0}, & i \in \{n-1, n\}. \end{cases}$$

If $a_{i,i+1} \neq 0$ *for every* $i < n-1$*, then* $\mathbf{E} \cong \mathbf{E}'$.

Proof. Let $a_{i,i+1} \neq 0$ for every $i < n-1$. If $n = 3$ after changing the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{f}_1 = \mathbf{e}_1$, $\mathbf{f}_2 = \mathbf{e}_1^2$ and $\mathbf{f}_3 = \mathbf{e}_3$ we immediately get $\mathbf{E}'$.

So, let us suppose $n \geq 4$. Then the linear mapping $\varphi : \mathbf{E} \to \mathbf{E}'$ defined by

$$\varphi : \begin{cases} \mathbf{f}_1 = \mathbf{e}_1 \\ \mathbf{f}_2 = \mathbf{e}_1^2 \\ \mathbf{f}_{i+1} = \prod\limits_{k=1}^{i-1} a_{k,k+1}^{2^{i-k}} \mathbf{e}_i^2, \quad 2 \leq i < n-1 \\ \mathbf{f}_n = \mathbf{e}_n \end{cases} \tag{3}$$

is an isomorphism from $\mathbf{E}$ to $\mathbf{E}'$. □

3. Derivations

In this section, we consider derivations of nilpotent evolution algebras with $2^{n-2}+1$ index of nilpotency.

Recall that derivation of an evolution algebra $\mathbf{E}$ is a linear mapping $d : \mathbf{E} \to \mathbf{E}$ such that $d(\mathbf{uv}) = d(\mathbf{u})\mathbf{v} + \mathbf{u}d(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbf{E}$.

We note that for any algebra, the space $Der(\mathbf{E})$ of all derivations is a Lie algebra w.r.t. the commutator multiplication,

$$[d_1, d_2] = d_1 d_2 - d_2 d_1, \quad \forall d_1, d_2 \in Der(\mathbf{E}),$$

for a given structural matrix $A = (a_{ij})_{i,j\geq 1}^n$ of nilpotent evolution algebra $\mathbf{E}$ with $dim(\mathbf{E}^2) = n-2$ we denote

$$I_A = \{(i,j) : i+1 < j < n-1,\ a_{ij} \neq 0\}. \tag{4}$$

Theorem 3.1. *Let* $\mathbf{E}$ *be an evolution algebra with structural matrix* $A = (a_{ij})_{i,j\geq 1}^n$ *in a natural basis* $\{\mathbf{e}_i\}_{i=1}^n$*. If* $\mathbf{E}$ *is a nilpotent with* $rankA = n-2$*, then the following statements hold*

(i) *if* $I_A \neq \emptyset$ *then*

$$Der(\mathbf{E}) = \left\{ \begin{pmatrix} 0 & 0 & \dots & d_{1n-1} & d_{1n} \\ 0 & 0 & \dots & d_{2n-1} & d_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1n-1} & d_{n-1n} \\ 0 & 0 & \dots & d_{nn-1} & d_{nn} \end{pmatrix} : \ d_{i,n-1}, d_{i,n} \in \mathbb{K}, \right\}$$

where

$$\begin{aligned} d_{n-1,n-1} &= -a_{n-2,n}d_{n,n-1}; \\ d_{n-1,n} &= -a_{n-2,n}d_{nn}; \\ d_{im} &= -\sum_{k=1}^{n-i} a_{i-1,k+i}d_{k+i,m},\ m \in \{n-1,n\} \end{aligned}$$

(ii) *if* $I_A = \emptyset$ *then*

$$Der(\mathbf{E}) = \left\{ \begin{pmatrix} \alpha & 0 & \dots & \beta & \gamma \\ 0 & 2\alpha & \dots & d_{2,n-1} & d_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-2,n-1} & d_{n-2,n} \\ 0 & 0 & \dots & d_{n-1,n-1} & d_{n-1,n} \\ 0 & 0 & \dots & s & t \end{pmatrix} : \ \alpha, \beta, \gamma, s, t \in \mathbb{K} \right\}$$

where

$$\begin{aligned} d_{n-1,n-1} &= 2^{n-2}\alpha - a_{n-2,n}s \\ d_{n-1,n} &= (2^{n-2}-t)a_{n-2,n} \\ d_{i,n-1} &= (2^{i-1}-2^{n-2})\alpha a_{i-1,n-1} + (a_{i-1,n-1}a_{n-2,n} - a_{i-1,n})s \\ d_{i,n} &= a_{i-1,n-1}d_{n-1,n} + a_{n-2,n}(2^{i-1}\alpha - t), \quad 2 \leq i < n-1. \end{aligned}$$

Proof. The (i) and (ii) are easy to check for $n = 3, 4$. So, we consider only the case $n > 4$. Let d be a derivation. We represent d in a matrix form in the basis $\{\mathbf{e}_i\}_{i=1}^n$ as follows $d(\mathbf{e}_i) = \sum_{j=1}^n d_{ij}\mathbf{e}_j$. Then, we have $d_{ji}\mathbf{e}_i^2 + d_{ij}\mathbf{e}_j^2 = \mathbf{0}$ for all $1 \le i < j \le n$. Since $\mathbf{e}_i^2$ and $\mathbf{e}_j^2$ are linearly independent, then $d_{ij} = d_{ji} = 0$ for any $1 \le i < j < n-1$. If we take $m \in \{n-1, n\}$, then taking into account that $\mathbf{e}_{n-1}^2 = \mathbf{e}_n^2 = \mathbf{0}$ from $d_{mi}\mathbf{e}_i^2 + d_{im}\mathbf{e}_m^2 = \mathbf{0}$ one has $d_{mi} = 0$ for any $i < m$.

Hence, we have shown the following:

$$d_{ij} = 0, \quad \text{if } i \neq j,\ i \le n,\ j < n-1. \tag{5}$$

On the other hand, we have $d(\mathbf{e}_i^2) = 2d_{ii}\mathbf{e}_i^2$ for any $i \le n$. Then, for $i = n-2$ using (2) we obtain $d(\mathbf{e}_{n-1} + a_{n-2,n}\mathbf{e}_n) = 2d_{n-2,n-2}\mathbf{e}_{n-2}^2$, then we have the following system,

$$\begin{aligned} d_{n-1,n-1} + a_{n-2,n}d_{n,n-1} &= 2d_{n-2,n-2} \\ d_{n-1,n} + a_{n-2,n}d_{n,n} &= 2a_{n-2,n}d_{n-2,n-2}. \end{aligned} \tag{6}$$

Furthermore, assume that $i < n-2$. Then, one finds

$$\begin{aligned} d(\mathbf{e}_i^2) &= d\left(\sum_{j=i+1}^{n} a_{ij}\mathbf{e}_j\right) = \sum_{j=i+1}^{n} a_{ij}d(\mathbf{e}_j) \\ &= \sum_{j=i+1}^{n-2} a_{ij}d_{jj}\mathbf{e}_j + \sum_{j=i+1}^{n} a_{ij}d_{j,n-1}\mathbf{e}_{n-1} + \sum_{j=i+1}^{n} a_{ij}d_{jn}\mathbf{e}_n. \end{aligned} \tag{7}$$

On the other hand, from

$$d(\mathbf{e}_i^2) = 2d_{ii}\mathbf{e}_i^2 = 2d_{ii}\sum_{j=i+1}^{n} a_{ij}\mathbf{e}_j,$$

with (7) one finds

$$2d_{ii} = d_{i+1,i+1}, \quad 1 \le i < n-2 \tag{8}$$

$$a_{ij}d_{jj} = 2a_{ij}d_{ii}, \quad i+2 \le j \le n-2 \tag{9}$$

$$\sum_{j=i+1}^{n} a_{ij}d_{j,n-1} = 2d_{ii}a_{i,n-1}, \quad 1 \le i < n-2 \tag{10}$$

$$\sum_{j=i+1}^{n} a_{ij}d_{jn} = 2d_{ii}a_{in}, \quad 1 \le i < n-2. \tag{11}$$

From (8),(9) we can easily derive

$$d_{jj} = 2^{j-1} d_{11}, \qquad 2 \le j \le n-2 \tag{12}$$
$$a_{ij} d_{11} = 0, \quad i+2 \le j \le n-1. \tag{13}$$

Now we consider (10), (11).

$$\begin{aligned} d_{i+1,n-1} &= 2a_{i,n-1} d_{ii} - \sum_{k=1}^{n-i} a_{i-1,k+i} d_{k+i,n-1} \\ d_{i+1,n} &= 2a_{i,n} d_{ii} - \sum_{k=1}^{n-i} a_{i-1,k+i} d_{k+i,n}. \end{aligned} \tag{14}$$

So, from (5),(6),(12),(13) and (14) we conclude that d is a derivation of evolution algebra given by (2) if and only if

$$d_{ij} = d_{n-1,i} = d_{ni} = 0, \quad 1 \le i \ne j \le n-2 \tag{15}$$

$$d_{jj} = 2^{j-1} d_{11}, \quad 2 \le j \le n-2 \tag{16}$$

$$a_{ij} d_{11} = 0, \quad i+2 \le j \le n-2 \tag{17}$$

$$d_{i+1,n-1} = 2a_{i,n-1} d_{ii} - \sum_{k=1}^{n-i} a_{i-1,k+i} d_{k+i,n-1} \tag{18}$$

$$d_{i+1,n} = 2a_{i,n} d_{ii} - \sum_{k=1}^{n-i} a_{i-1,k+i} d_{k+i,n}. \tag{19}$$

Case $I_A \neq \emptyset$. In this case, we have $a_{i_0 j_0} \neq 0$ for some pair (i_0, j_0) satisfying $i_0 + 2 \le j_0 < n-1$. Then, from (17) one finds $d_{11} = 0$. Plugging this fact into (16) ,(18) and (19) we obtain

$$Der(\mathbf{E}) = \left\{ \begin{pmatrix} 0 & 0 & \dots & d_{1n-1} & d_{1n} \\ 0 & 0 & \dots & d_{2n-1} & d_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1n-1} & d_{n-1n} \\ 0 & 0 & \dots & d_{nn-1} & d_{nn} \end{pmatrix} : \; d_{i,n-1}, d_{in} \in \mathbb{K} \right\},$$

where

$$\begin{aligned} d_{n-1,n-1} &= -a_{n-2,n}d_{n,n-1}; \\ d_{n-1,n} &= -a_{n-2,n}d_{nn}; \\ d_{im} &= -\sum_{k=1}^{n-i} a_{i-1,k+i}d_{k+i,m}, \;\; m \in \{n-1, n\} \end{aligned}$$

Case $I_A = \emptyset$. In this case (17) is true for any $d_{11} \in \mathbb{K}$. So, from (15),(16), (18) and (19) we conclude that

$$Der(\mathbf{E}) = \left\{ \begin{pmatrix} \alpha & 0 & \dots & \beta & \gamma \\ 0 & 2\alpha & \dots & d_{2,n-1} & d_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-2,n-1} & d_{n-2,n} \\ 0 & 0 & \dots & d_{n-1,n-1} & d_{n-1,n} \\ 0 & 0 & \dots & s & t \end{pmatrix} : \alpha, \beta, \gamma, s, t \in \mathbb{K} \right\}$$

where

$$\begin{aligned} d_{n-1,n-1} &= 2^{n-2}\alpha - a_{n-2,n}s \\ d_{n-1,n} &= (2^{n-2} - t)a_{n-2,n} \\ d_{i,n-1} &= (2^{i-1} - 2^{n-2})\alpha a_{i-1,n-1} + (a_{i-1,n-1}a_{n-2,n} - a_{i-1,n})s \\ d_{i,n} &= a_{i-1,n-1}d_{n-1,n} + a_{n-2,n}(2^{i-1}\alpha - t), \quad 2 \le i < n-1. \end{aligned}$$

This completes the proof. □

Remark 3.1. *From the proved theorem we infer that* $1 \le \dim Der(\mathbf{E}) \le 5$. *This kind of result could be proved using Jacobson (1989). But the advantage of Theorem 3.1 is that it fully describes structure of the derivations in the natural basis.*

4. Conclusion

The description of space of derivation of evolution algebra is a crucial task. In this paper the space of derivation of finite dimensional nilpotent evolution algebras with index of nilpoteincy $2^{n-2} + 1$ is presented.

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