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Dynamic Behaviors of p-adic Ising-Vannimenus Model on the Cayley Tree of Order Three

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ABSTRACT

Recently, Ising-Vannimenus model on the Cayley tree of order $k = 3$ has been studied by Akin (2017) in real case. In this study we continue to investigate Ising-Vannimenus model on the Cayley tree of order $k = 3$ in p-adic sense. We investigate the dynamic aspects of p-adic Ising-Vannimenus model on the Cayley tree of order $k = 3$. We show that the recurrent equation 26, is associated to the model, has four non-trivial fixed points. And one of the fixed points lies in $\mathcal{E}_p$ and the rest of fixed points lie in $\mathbb{Z}_p^\star$. As a main result of the paper, we show that the fixed point u_0 is attractive and the other fixed points u_1, u_2, u_3 are repellent when $u_i = p - 1$, and neutral when $u_i \neq p - 1$.

Keywords: p-adic Gibbs measures, p-adic dynamical systems, Ising-Vannimenus model and Cayley tree.

1. Introduction

It is clear that Ising Vannimenus (see Vannimenus (1981)) model is one of the most crucial models in statistical mechanics. In this work we continue the investigation of Ising-Vannimenus model on the Cayley tree of order $k = 3$. In recent studies, existence of p-adic quasi Gibbs measures and phase transition were studied in (Mukhamedov et al. (2016), Mukhamedov et al. (2014)) for the p-adic Ising-Vannimenus model with competing interactions of nearest, next-nearest and prolonged next-nearest neighbors on the Cayley tree of order $k = 2$. In this work, we study the dynamic behaviors of fixed points of p-adic Ising-Vannimenus model with competing interactions of nearest and prolonged next-nearest neighbors on the Cayley tree of order $k = 3$.

We consider Hamiltonian ($H_n : \Omega_{V_n} \to \mathbb{Q}_p$) of the p-adic Ising-Vannimenus model as follows;

$$H(\sigma) = -J \sum_{<x,y>} \sigma(x)\sigma(y) - J_p \sum_{>x,y<} \sigma(x)\sigma(y), \tag{1}$$

and $J, J_p \in \mathbb{Q}_p$ are coupling constants of nearest-neighbor, and prolonged next-nearest-neighbors potentials, respectively. In this paper we cosider $J, J_p \neq 0$, i.e. $J \cdot J_p \neq 0$. The uniqueness of p-adic Gibbs measures in real case was studied in Akin (2017) for the model. In this work, we prove the existence of translation invariant p-adic Gibbs measures and dynamic behaviors of fixed points in p-adic setting by analyzing the fixed points of dynamical system;

$$g(u) = \left(\frac{1 + cdu}{d + cu}\right)^3. \tag{2}$$

Note that the results of this paper fails in real setting.

2. Preliminaries

2.1 p-Adic Numbers

In what follows p is a fixed prime number, and $\mathbb{Q}_p$ denotes the field of p-adic numbers, established by completion of $\mathbb{Q}$ with respect to p-adic absolute value $|\cdot|_p$. This norm is called non-Archimedean, and satisfies the ultrametric triangle inequality;

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}. \tag{3}$$

Any nonzero p-adic number $x \in \mathbb{Q}_p$ can be uniquely shown as

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...), \tag{4}$$

where $\gamma = \gamma(x) \in \mathbb{Z}$, and x_j are integers such that $0 \leq x_j \leq p-1$, $x_0 > 0$, $j = 0, 1, 2, \ldots$. Here the norm of x is defined by $|x|_p = p^{-\gamma(x)}$.

The p-adic logarithm is defined by

$$\log_p(x) = \log_p(1 + (x-1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}, \tag{5}$$

and converges when $x \in B(1, 1)$,

and p-adic exponentials are defined by

$$\exp_p(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}, \tag{6}$$

and converges when $x \in B(0, p^{-1/(p-1)})$.

Lemma 2.1. *Koblitz (1977), Vladimirov et al. (1994) Let $x \in B(0, p^{-1/(p-1)})$ then we have*

$$|\exp_p(x)|_p = 1, \quad |\exp_p(x) - 1|_p = |x|_p < 1, \quad |\log_p(1+x)|_p = |x|_p < p^{-1/(p-1)}$$

and

$$\log_p(\exp_p(x)) = x, \quad \exp_p(\log_p(1+x)) = 1 + x.$$

Lemma 2.2. *Khrennikov et al. (2007) If $|a_i|_p \leq 1$, $|b_i|_p \leq 1$, $i = 1, \ldots, n$, then*

$$\left| \prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i \right|_p \leq \max_{i \leq n} \{|a_i - b_i|_p\}. \tag{7}$$

We denote the following set,

$$\mathcal{E}_p = \{x \in \mathbb{Q}_p : \ |x|_p = 1, \ \ |x-1|_p < p^{-1/(p-1)}\}. \tag{8}$$

So, from Lemma-2.1 one concludes that if $x \in \mathcal{E}_p$, then there is an element $h \in B(0, p^{-1/(p-1)})$ such that $x = \exp_p(h)$. Note that the fundamentals of p-adic analysis, p-adic mathematical physics were explained in Koblitz (1977), Mahler (1981), Rozikov (2013), Schikhof (1984), Vladimirov et al. (1994).

The p-adic integers are defined by

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : | \, x \, |_p \leq 1\}, \tag{9}$$

and the set $\mathbb{Z}_p^{\star} = \mathbb{Z}_p - p\mathbb{Z}_p$ is called p-adic units.

2.2 Dynamical Systems in $\mathbb{Q}_p$

In this subsection we briefly recall some standard terminology of theory of dynamical systems (see Khrennikov and Nilsson (2004)). We define following sets,

$$B_r(a) = \{x \in \mathbb{Q}_p : \ |x-a|_p < r\}, \quad \bar{B}_r(a) = \{x \in \mathbb{Q}_p : |x-a|_p \leq r\}, \tag{10}$$
$$B_r(a) = \{x \in \mathbb{Q}_p : \rho < |x-a|_p < s\}. S_r(a) = \{x \in \mathbb{Q}_p : |x-a|_p = r\}, \tag{11}$$

for $r, s > 0$ $(r < s)$ and $a \in \mathbb{Q}_p$. It is clear that $\bar{B}_r(a) = B_r(a) \cup S_r(a)$.

A function $f : B_r(a) \to \mathbb{Q}_p$ is said to be *analytic* if it can be represented by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad f \in \mathbb{Q}_p \tag{12}$$

and converges uniformly on the ball $B_r(a)$.

Consider a dynamical system (f, B) in $\mathbb{Q}_p$, where $f : x \in B \to f(x) \in B$ is an analytic function and $B = B_r(a)$ or $\mathbb{Q}_p$ (as given above). We denote $x^{(n)} = f^n(x^{(0)})$, where $x^0 \in B$ and $f^n(x) = \underbrace{f \circ \cdots \circ f(x)}_{n}$.

If $f(x^{(0)}) = x^{(0)}$ then $x^{(0)}$ is called a *fixed point*.

Let $x^{(0)}$ be a fixed point of an analytic function $f(x)$ then,

$$\lambda = \frac{d}{dx}(f(x^{(0)})),$$

is a formal derivative of f. Then the fixed point $x^{(0)}$ is called *attractive* when $0 \leq |\lambda|_p < 1$, *indifferent or neutral* when $|\lambda|_p = 1$, and *repellent* when $|\lambda|_p > 1$.

2.3 p-Adic Measure

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a *p-adic measure* if the following equality holds;

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j), \tag{13}$$

for any $A_1, \dots, A_n \subset \mathcal{B}$, and $A_i \cap A_j = \emptyset$ $(i \neq j)$.

A p-adic measure is called a *probability measure* if $\mu(X) = 1$.

2.4 Cayley Tree

Let $\Gamma^k_+ = (V, L)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root $x^{(0)}$ (Each vertex has exactly $k+1$ edges, except $x^{(0)}$ which has k edges). Here V, L are the set of vertices and the set of edges respectively. The vertices x and y are called *nearest neighbors* if there exists an edge connecting them i.e. $d(x, y) = 1$, and they are denoted by $l =< x, y >$. Two vertices $x, y \in V$ are called *next-nearest neighbors*, if $d(x, y) = 2$. And the next-nearest-neighbors x and y are called *prolonged next-nearest neighbors* whenever $x \in W_{n-2}$ and $y \in W_n$, and denoted by $> x, y <$, or *one-level next-nearest-neighbors*, if $x, y \in W_n$ for some n and denoted by $> \overline{x, y} <$. Here W_n is the *nth* level of Cayley tree, and we determine following sets;

$$W_n = \left\{x \in V \mid d(x, x^{(0)}) = n\right\}, \bigcup_{m=1}^{n} W_m, L_n = \{l =< x, y >\in L \mid x, y \in V_n\}.$$

The set of direct successors of x is defined by

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1, x \in W_n\}.$$

Recall that any vertex $x \neq x^{(0)}$ has $k+1$ direct successors except $x^{(0)}$ which has k.

3. p-adic Ising-Vannimenus (IV) model and p-adic Gibbs measures

In this section, we consider the p-adic Ising-Vannimenus model such that spin values $\sigma(x)$ are from the set $\Phi = \{-1, +1\}$, (Φ is called a *state space*), and these values are assigned to the vertices of Cayley tree $\Gamma^k = (V, \Lambda)$. A configuration σ on V is defined as a function such that $f : x \in V \to \sigma(x) \in \Phi$. Using a similar manner one can be defined configurations σ_n and ω on V_n and W_n, respectively. The set of all configurations on V (resp. V_n, W_n) coincides with $\Omega = \Phi^V$ (resp. $\Omega_{V_n} = \Phi^{V_n}, \Omega_{W_n} = \Phi^{W_n}$). One can see that $\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_n}$, and we define their concatenations as follows;

$$(\sigma_{n-1} \vee \omega)(x) = \begin{cases} \sigma_{n-1}(x), & \text{if } x \in V_{n-1}, \\ \omega(x), & \text{if } x \in W_n, \end{cases}$$

for configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\omega \in \Omega_{W_n}$. It is clear that $\sigma_{n-1} \vee \omega \in \Omega_{V_n}$.

The Hamiltonian ($H_n : \Omega_{V_n} \to \mathbb{Q}_p$) of the p-adic Ising-Vannimenus model is defined as follows;

$$H(\sigma) = -J \sum_{<x,y>} \sigma(x)\sigma(y) - J_p \sum_{>x,y<} \sigma(x)\sigma(y), \tag{14}$$

with interactions of the nearest-neighbors, and next-nearest-neighbors respectively. Here $J, J_p \in \mathbb{Q}_p$ are coupling constants.

Let $\mathbf{h} :< x, y >\to \mathbf{h}_{xy} = (h_{xy,++}, h_{xy,+-}, h_{xy,-+}, h_{xy,--}) \in \mathbb{Q}_p^4$ be a vector valued function on each edge, L. We consider a p-adic probability measure $\mu_{\mathbf{h}}^{(n)}(\sigma)$ on Ω_{V_n} is defined by

$$\mu_{\mathbf{h}}^{(n)}(\sigma) = \frac{1}{Z_n} \exp_p[-\beta H_n(\sigma) + \sum_{x\in W_{n-1}} \sum_{y\in S(x)} \sigma(x)\sigma(y)h_{xy,\sigma(x)\sigma(y)}], \tag{15}$$

for an $n \in \mathbb{N}$, and $\beta = \frac{1}{kT}$ (Boltzmann constant, k, and temperature, T). Here, $\sigma_n : x \in V_n \to \sigma_n(x)$ is a function, and Z_n is a partition function as follows;

$$Z_n = \sum_{\sigma_n\in\Omega_{V_n}} \exp_p[-\beta H(\sigma_n) + \sum_{x\in W_{n-1}} \sum_{y\in S(x)} \sigma(x)\sigma(y)h_{xy,\sigma(x)\sigma(y)}]. \tag{16}$$

We consider increasing subsets of the set of states for one dimensional lattices Fannes and Verbeure (1984) as $\mathfrak{G}_1 \subset \mathfrak{G}_2 \subset ... \subset \mathfrak{G}_n \subset ...$, where $\mathfrak{G}_n$ is the set of states corresponding to non-trivial correlations between n-successive lattice points. $\mathfrak{G}_1$ is the set of main field states; and $\mathfrak{G}_2$ is the set of Bethe-Peierls states, the latter extending to the so-called Bethe lattices. In the probability theory, all these states correspond to so-called Markov chains with memory of length n. In this paper, we will discuss a new method of defining Markov chains with memory length of two on a Cayley tree of order three in the p-adic sense using the methods described by Akin (2017) (see for details Fannes and Verbeure (1984)).

Let $x \in W_{n-1}$ for some $n \in \mathbb{N}$ and $S(x) = \{y, z, w\}$, where $y, z, w \in W_n$ are the direct successors of x. Note that $B_1(x) = \{x, y, z, w\}$ is a unit semi-ball with a center x, where $S(x) = \{y, z, w\}$. We denote the set of all spin configurations on V_n by Φ^{V_n} and the set of all configurations on unit semi-ball $B_1(x)$ by $\Phi^{B_1(x)}$. One can get that the set $\Phi^{B_1(x)}$ consists of sixteen configurations;

$$\Phi^{B_1(x)} = \left\{ \begin{pmatrix} l & k & j \\ & i & \end{pmatrix} : i, j, k, l \in \{-1, +1\} \right\}. \tag{17}$$

Briefly, we do an appropriate definition for the quantities $h \begin{pmatrix} z, y, w \\ x \end{pmatrix}$ as $h_{B_1(x)}$.

In this work we consider $\mathbf{h} : V \setminus \{x^{(0)}\} \times V \setminus \{x^{(0)}\} \times V \setminus \{x^{(0)}\} \to \mathbb{Q}_p^{\Phi}$ is a mapping such that

$$\mathbf{h} :< x, y, z, w > \to \mathbf{h}_{B_1(x)} = (h_{B_1(x),\sigma(x)\sigma(y)\sigma(z)\sigma(w)} : \sigma(i) \in \{\pm 1\}), \tag{18}$$

where $h_{B_1(x),\sigma(x)\sigma(y)\sigma(z)\sigma(w)} \in \mathbb{Q}_p$, $x \in W_{n-1}$ and $y, z, w \in S(x)$. As a result, we use the function $h_{xyzw,\sigma(x)\sigma(y)\sigma(z)\sigma(w)}$ to define the Gibbs measure of any configuration $\begin{pmatrix} \sigma(z) & \sigma(y) & \sigma(w) \\ & \sigma(x) & \end{pmatrix}$ that belongs to $\Phi^{B_1(x)}$.

In this section, we present the general structure of Gibbs measures with memory length of two on the Cayley tree of order $k = 3$. An arbitrary edge $< x^{(0)}, x^1 >= \ell \in L$ deleted from a Cayley tree Γ_1^k and Γ_0^k splits into two components: semi-infinite Cayley tree Γ_1^k and semi-infinite Cayley tree Γ_0^k.

We define the finite-dimensional Gibbs probability distributions on the configuration space $\Omega^{V_n} = \{\sigma_n = \{\sigma(x) = \pm 1, x \in V_n\}\}$ as follows;

$$\mu_{\mathbf{h}}^{(n)}(\sigma) = \frac{1}{Z_n} \exp_p[-\beta H_n(\sigma) + \sum_{x \in W_{n-1}} \sum_{y,z,w \in S(x)} \sigma(x)\sigma(y)\sigma(z)\sigma(w) h_{B_1(x),\sigma(x)\sigma(y)\sigma(z)\sigma(w)}]. \tag{19}$$

with corresponding partition function which is defined by

$$Z_n = \sum_{\sigma_n \in \Omega_{V_n}} \exp_p[-\beta H(\sigma_n) + \sum_{x \in W_{n-1}} \sum_{y,z,w \in S(x)} \sigma(x)\sigma(y)\sigma(z)\sigma(w) h_{B_1(x),\sigma(x)\sigma(y)\sigma(z)\sigma(w)}],$$

where $\beta = \frac{1}{kT}$. We obtain a new set of p-adic Gibbs measures which is different from previous studies Akin (2017), Ganikhodjaev et al. (2011). We consider a construction of an infinite volume distribution with given finite-dimensional distributions. More exactly, we will attempt to find a probability measure μ on Ω which is compatible with given measures $\mu_{\mathbf{h}}^{(n)}$, *i.e.*

$$\mu(\sigma \in \Omega : \sigma|_{V_n} = \sigma_n) = \mu_{\mathbf{h}}^{(n)}(\sigma_n), \quad \text{for all} \;\; \sigma_n \in \Omega_{V_n}, \; n \in \mathbb{N}. \tag{20}$$

Kolmogorov consistency condition for $\mu_{\mathbf{h}}^n(\sigma_n)$, $n \geq 1$ is defined as follows

$$\sum_{\omega \in \Omega_{W_n}} \mu_{\mathbf{h}}^{(n)}(\sigma_{n-1} \vee \omega) = \mu_{\mathbf{h}}^{(n-1)}(\sigma_{n-1}), \tag{21}$$

for any $\sigma_{n-1} \in \Omega_{V_{n-1}}$.

This condition implies the existence of a unique measure $\mu_{\mathbf{h}}$ defined on Ω with a required condition (20). Such a measure $\mu_{\mathbf{h}}$ is called a Gibbs measure with memory length of two for the considered model. We define interaction energy on V with inner configuration $\sigma_{n-1} \in V_{n-1}$ and boundary condition $\eta \in W_n$ as follows;

$$\begin{aligned} H_n(\sigma_{n-1} \vee \eta) &= -J \sum_{<x,y>\in V_{n-1}} \sigma(x)\sigma(y) - J \sum_{x\in W_{n-1}} \sum_{y\in S(x)} \sigma(x)\eta(y) \\ &\quad -J_p \sum_{>x,y<\in V_{n-1}} \sigma(x)\sigma(y) - J_p \sum_{x\in W_{n-2}} \sum_{z\in S^2(x)} \sigma(x)\eta(z) \\ &= H_n(\sigma_{n-1}) - J \sum_{x\in W_{n-1}} \sum_{y\in S(x)} \sigma(x)\eta(y) \\ &- J_p \sum_{x\in W_{n-2}} \sum_{z\in S^2(x)} \sigma(x)\eta(z). \end{aligned} \tag{22}$$

If we follow the same processes as in Akin (2017), then easily we can get the following basic equations. And we express the vector-valued function given in (18) as follows:

$$\mathbf{h}(x) = (h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8). \tag{23}$$

Assume that $a = p^{\beta J}$ and $b = p^{\beta J_p}$, $u_i' = p^{h_{B_1(x)}}$ for $x \in W_{n-1}$ and $u_i = p^{h_{B_1(x)}}$ for $x \in W_n$.

For simplicity, when we apply the same technique like in Akin (2017), then we get the following nonlinear dynamical function;

$$f(x) = \left(\frac{1 + (ab)^2 v_4^4}{b^2 + a^2 v_5^4} \right)^3. \tag{24}$$

To reduce (24), we consider that $p^{\frac{2J}{T}} = a^2 = c$ and $p^{\frac{2J_p}{T}} = b^2 = d$, where T is an absolute temperature. Therefore, (24) is conjugate to the following function;

$$g(x) = \left(\frac{1 + cdx}{d + cx} \right)^3. \tag{25}$$

Hereafter we analyze the equation (25) for the existence of transition invariant p-adic Gibbs measures for the considered model.

4. Translation-invariant p-adic Gibbs measures

In this section, we investigate the existence of translation-invariant p-adic Gibbs measures (TIpGM) through analyzing the equation (25). Note that a function $\mathbf{h} = \{h_{B_1(x),\sigma_i^{(1)}} : i \in \{1,2,\ldots,16\}\}$ is considered as translation-invariant, if $h_{B_1(x),\sigma_i^{(1)}} = h_{B_1(y),\sigma_i^{(1)}}$ for all $y \in S(x)$ and $i \in \{1,2,\ldots,16\}$. A translation-invariant Gibbs measure is defined as a measure $\mu_{\mathbf{h}}$ corresponding to a translation-invariant function $\mathbf{h}$ (see for details Ganikhodjaev et al. (2011), Rozikov (2013)).

After obtaining the equation (25), the existence of p-adic Gibbs measures is reduced to the existence of the fixed points of (25). Therefore, to show the existence of p-adic Gibbs measures for all $p > 3$, we analyze the following equation (26) which is obtained from (25) with assuming $x = u$,

$$g(u) = \left(\frac{1+cdu}{d+cu}\right)^3 \tag{26}$$

where $c, d \in \mathcal{E}_p$.

We state the following Proposotion 4.1 to show the existence of p-adic Gibbs measures.

Proposition 4.1. *Let $p > 3$, if $u, v \in \mathcal{E}_p$ then following statements hold;*

i. $g(\mathcal{E}_p) \subset \mathcal{E}_p$, *and*

ii. $\mid g(u) - g(v) \mid_p \leq \frac{1}{p} \mid u - v \mid_p$.

Proof. Let $c, d, u \in \mathcal{E}_p$, and using strong triangle inequality, Lemma 2.1 and Lemma 2.2, then easily we get the following items.

i. Let

$$\begin{aligned} \mid g(u) \mid_p = \left| \left(\frac{1+cdu}{d+cu}\right)^3 \right|_p &= \frac{|1+cdu|_p^3}{|d+cu|_p^3} \\ &= \frac{|(cdu-1)+2|_p^3}{|(d-1)+(cu-1)+2|_p^3} = 1. \end{aligned}$$

Then $\mid g(u) \mid_p = 1, u \in \mathcal{E}_p$.

We show the inequality below,

$$\begin{aligned} |g(u)-1|_p &= |(\frac{1+cdu}{d+cu})^3 - 1|_p \\ &= |\frac{|1+3cdu+3c^2d^2u^2+c^3d^3u^3-d^3-3cd^2u-3c^2du^2-c^3u^3|_p}{|(d-1)+(cu-1)+2|_p^3} \leq \frac{1}{p} < 1. \end{aligned}$$

Hence $g(u) \in \mathcal{E}_p$ since (8).

ii. Let $u, v \in \mathcal{E}_p$ then the following inequality holds;

$$\begin{aligned} |g(u) - g(v)|_p &= \left| \left(\frac{1+cdu}{d+cu} \right)^3 - \left(\frac{1+cdv}{d+cv} \right)^3 \right|_p \\ &= \left| (1+cdu)^3 (d+cv)^3 - (1+cdv)^3 (d+cu)^3 \right|_p \\ &= \left| (cd^2 - c)(u-v) \right|_p \\ \leq \tfrac{1}{p} & \ | \ u - v \ |_p \ . \end{aligned}$$

Hence the proof is completed. □

From the Propostion4.1, the function g satisfies the Banach contraction principle. This means that (26) has a fixed point $u_0 \in \mathcal{E}_p$. To find out the other fixed points of (26), let $u = g(u)$ then one gets;

$$c^3u^4 + (3c^3d - c^3d^3)u^3 + (3cd^2 - 3c^2d^2)u^2 + (d^3 - 3cd)u - 1 = 0 \tag{27}$$

where $c, d \in \mathcal{E}_p$. It is clear that (27) has a solution $u_0 \in \mathcal{E}_p$ since Proposition 4.1 then we rewrite (27) as follows;

$$(u - u_0)[c^3u^3 + (3c^2d - c^3d^3 - c^3u_0)u^2 + Au + B] = 0 \tag{28}$$

where

$$A = 3d^2c + 3c^2du_0 - 3c^2d^2 - c^3d^3u_0 - c^3u_0^2$$

and

$$B = d^3 + 3d^2cu_0 + 3c^2du_0^2 - 3cd - 3c^2d^2u_0 - c^3d^3u_0^2 - c^3u_0^3.$$

One of the fixed points of (27) is $u_0 \in \mathcal{E}_p$ and to find the other fixed points of (27) we solve the equation below;

$$c^3u^3 + (3c^2d - c^3d^3 - c^3u_0)u^2 + Au + B = 0, \tag{29}$$

where $c, d, u_0 \in \mathcal{E}_p$. When we divide both sides of (29) by c^3 then one gets;

$$u^3 + (\frac{3c^2d - c^3d^3}{c^3})u^2 + \frac{A}{c^3}u + \frac{B}{c^3} = 0. \tag{30}$$

In (30) let $a = 3c^{-1}d - d^3 - u_0$, $b = 3c^{-2}d^2 + 3c^{-1}du_0 - 3c^{-1}d^2 - d^3u_0 - u_0^2$ and

$$e = \frac{d^3 + 3cd^2u_0 + 3c^2du_0^2 - 3cd - 3c^2d^2u_0 - c^3d^3u_0^2 - u_0^3}{c^3},$$

then we obtain the following equation;

$$u^3 + au^2 + bu + e = 0. \tag{31}$$

When we apply the transformation of $u = u - \frac{a}{3}$ then we get the depressed cubic equation;

$$u^3 + (b - \frac{1}{3}a^2)u - (\frac{2}{27}a^3 - \frac{1}{3}ab + e) = 0. \tag{32}$$

Let $\tilde{a} = b - \frac{1}{3}a^2$, $\tilde{b} = \frac{2}{27}a^3 - \frac{1}{3}ab + e$ then one gets

$$u^3 + \tilde{a}u - \tilde{b} = 0. \tag{33}$$

Hereafter we investigate the existence and number of solutions of depressed equation (33) in $\mathbb{Z}_p^\star$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ with $p > 3$ as in Mukhamedov and Omirov (2014b) and Saburov and Khameini (2015).

Any non-zero p-adic number $x \in \mathbb{Q}_p$ can be uniquely represented by $x = \frac{x^\star}{|x|_p}$. Hence, we can represent $\tilde{a} = \frac{\tilde{a}^\star}{|\tilde{a}|_p}$, and $\tilde{b} = \frac{\tilde{b}^\star}{|\tilde{b}|_p}$ where $\tilde{a}^\star = a_0 + a_1p + a_2p^2 + ..., \tilde{b}^\star = b_0 + b_1p + b_2p^2 + ...$, for any non-zero $\tilde{a}$, $\tilde{b}$.

Let $D_0 = -4a_0^3 - 27b_0^2$ and $u_{n+3} = b_0u_n - a_0u_{n+1}$ with $u_1 = 0, u_2 = -a_0$, and $u_3 = b_0$ for $n = \overline{1, p-3}$. Accordingly we find the fixed points of (33) through following proposition.

Proposition 4.2. *Mukhamedov and Omirov (2014a) Let $p > 3$ be a prime. If $|\tilde{a}|_p = |\tilde{b}|_p = 1$ and $|D|_p = 1$ then (33) has three solutions in $\mathbf{Z}_\mathbf{p}^\star$, where $\tilde{a}, \tilde{b} \in \mathbf{Q}_\mathbf{p}$ with $\tilde{a}\tilde{b} \neq 0$ and $D = -4\tilde{a}^3 - 27\tilde{b}^2$.*

Proof. Using strong triangle inequality and Lemma 2.1

i. Let

$$\begin{aligned} |\tilde{a}|_p &= |b - \frac{1}{3}a^2|_p \\ &= |\frac{9c^4d^2 + 9c^5du_0 - 9c^5d^2 - 3c^6d^3u_0 - 3c^6u_0^2 - 9c^2d^2 - c^6d^6 + 6c^4d^4}{3c^6}|_p \\ &= \frac{|9c^4d^2 + 9c^5du_0 - 9c^5d^2 - 3c^6d^3u_0 - 3c^6u_0^2 - 9c^2d^2 - c^6d^6 + 6c^4d^4|_p}{|3c^6|_p} \\ &= 1. \end{aligned}$$

ii. Let

$$
\begin{aligned}
|\tilde{b}|_p =& |\frac{2}{27}a^3 - \frac{1}{3}ab + e|_p \\
=& |\frac{2}{27}(\frac{3cd - c^3d^3}{c^3}^3) \\
&+ \frac{d^3 + 3cd^2u_0 + 3c^2du_0^2 - 3cd - 3c^2d^2u_0 - c^3d^3u_0 - u_0^3}{c^3} \\
&- \frac{1}{3}\frac{9c^2d^3 + 9c^3d^2u_0 - 9c^3d^3 - 3c^4d^4u_0 - 3c^4du_0^2 - 3c^4d^5}{c^6}|_p \\
&+ \frac{1}{3}\frac{-3c^5d^4u_0 + 3c^5d^5 + c^6d^6u_0 + c^6d^3u_0^2}{c^6}|_p \\
=& \max\{|\frac{2}{27}(\frac{3cd - c^3d^3}{c^3}^3)|_p, | \\
& \frac{d^3 + 3cd^2u_0 + 3c^2du_0^2 - 3cd - 3c^2d^2u_0 - c^3d^3u_0 - u_0^3}{c^3}|_p, | \\
& \frac{1}{3}\frac{9c^2d^3 + 9c^3d^2u_0 - 9c^3d^3 - 3c^4d^4u_0 - 3c^4du_0^2}{c^6} \\
&+ \frac{1}{3}\frac{-3c^4d^5 - 3c^5d^4u_0 + 3c^5d^5 + c^6d^6u_0 + c^6d^3u_0^2}{c^6}|_p \\
=& 1.
\end{aligned}
$$

iii. Let

$$
\begin{aligned}
|D|_p &= |-4\tilde{a}^3 - 27\tilde{b}^2|_p \\
&= \max\{|-4\tilde{a}^3|_p, |27\tilde{b}^2|_p\} \\
&= 1
\end{aligned}
$$

From [i], [ii] and [iii] we obtain the required one. □

Consequently, the depressed equation (33) has three non-trivial fixed points in $\mathbb{Z}_p^\star$. So (26) has one fixed point in $\mathcal{E}_p$ and three non-trivial fixed points in $\mathbb{Z}_p^\star$. This result yields the existence of translation invariant p-adic quasi Gibbs measures.

5. Dynamic Behavior of Fixed Points of Dynamical System

In this section, we investigate the dynamic behavior of the fixed points of (26). In the previous section we proved that the equation (26) has one fixed point in $\mathcal{E}_p$ and three fixed points in $\mathbb{Z}_p^\star$. Here we are going to determine that the fixed points of (26) are attractive, repellent or neutral. To achieve this we state following theorem.

Theorem 5.1. *Let $c, d \in \mathcal{E}_p$ and $p > 3$. The function f (26) has four fixed points u_0, u_1, u_2, u_3 such that u_0 is attractive, the fixed points u_1, u_2, u_3 are repellent, if $u_i = p-1$, and neutral, if $u_i \neq p-1$, for $i = 1, 2, 3$, where $u_0 \in \mathcal{E}_p$ and $u_1, u_2, u_3 \in \mathbb{Z}_p^\star$.*

Before starting the proof of theorem, we need to state the lemma below.

Lemma 5.2. *If $u_1, u_2, u_3 \in \mathbb{Z}_p^\star$ then following statements hold;*

i. $|1+u_i|_p \leq \frac{1}{p}$ *if* $u_i = p-1$,

ii. $|1+u_i|_p = 1$ *if* $u_i \neq p-1$,

where $|u_1|_p = |u_2|_p = |u_3|_p = 1$ *and* $i = 1, 2, 3$.

Proof. i. Let $u_i = p-1$ then it is clear that

$$|1+u_i|_p = |p|_p = \frac{1}{p}.$$

ii. Let $u_i \neq p-1$ then it is clear that

$$|1+u_i|_p = 1.$$

since $p \nmid (1+u_i)$ □

Now we are ready to prove Theorem 5.1.

Proof. Since (26) let

$$g(u) = \left(\frac{1+cdu}{d+cu}\right)^3$$

then we get derivative of g as follows;

$$g'(u) \quad = \quad 3\frac{cd^2 + 2c^2d^3u + c^3d^4u^2 - c - 2c^2du - c^3d^2u^2}{(d+cu)^4}.$$

where $c, d \in \mathcal{E}_p$. Thereafter one gets;

$$\begin{aligned}
|g'(u_0)|_p &= \frac{|3cd^2 + 6c^2d^3u_0 + 3c^3d^4u_0^2 - 3c - 6c^2du_0 - 3c^3d^2u_0^2|_p}{|d+cu_0|_p^4} \\
&= |3(cd^2-1) + 3 + 6(c^2d^3u_0 - 1) + 6 + 3(c^3d^4u_0^2 - 1) + 3 \\
&- 3(c-1) - 3 - 6(c^2du_0 - 1) - 6 - 3(c^3d^2u_0^2 - 1) - 3|_p \\
&= |3 + 6 + 3 - 3 - 6 - 3|_p = 0.
\end{aligned}$$

Therefore, u_0 is an attractive.

Now let us look for the dynamic behaviors of other fixed points $u_1, u_2, u_3 \in \mathbb{Z}_p^{\star}$. From Lemma 5.2 and ultrametric triangle inequality we get the items below.

i. Let us take $u_i \in \mathbb{Z}_p^{\star}, i = 1, 2, 3$ and $u_i = p - 1$ then we get

$$\begin{aligned}
|g'(u_i)|_p &= \frac{|3cd^2 + 6c^2d^3u_i + 3c^3d^4u_i^2 - 3c - 6c^2du_i - 3c^3d^2u_i^2|_p}{|d+cu_i|_p^4} \\
&= \frac{|(1+u_i)^2 - (1+u_i)|_p}{|1+u_i|_p^4} \\
&= \frac{|(1+u_i)|_p}{|1+u_i|_p^4} \\
&= \frac{1}{|1+u_i|_p^3} > 1.
\end{aligned}$$

Therefore u_i are repellent in the considered case $u_i = p - 1$.

ii. Let us take $u_i \in \mathbb{Z}_p^{\star}, i = 1, 2, 3$ and $u_i \neq p - 1$ then we easily get

$$\begin{aligned}
|g'(u_i)|_p &= \frac{|3cd^2 + 6c^2d^3u_i + 3c^3d^4u_i^2 - 3c - 6c^2du_i - 3c^3d^2u_i^2|_p}{|d+cu_i|_p^4} \\
&= \frac{|(1+u_i)^2 - (1+u_i)|_p}{|1+u_i|_p^4} \\
&= \frac{|(1+u_i)|_p}{|1+u_i|_p^4} = 1.
\end{aligned}$$

Hence u_i are neutral in the case of $u_i \neq p-1$.

□

Consequently, we conclude that u_0 is an attractive, u_i are repellent when $u_i = p-1$ and u_i are neutral when $u_i \neq p-1$ for the function (26).

6. Conclusion

In the present paper, we obtained the dynamic function (26) as in Akin (2017) and then we proved the existence of the translation invariant p-adic Gibbs measures for p-adic Ising-Vannimenus model on the Cayley tree of order $k=3$. We found out that one of fixed points of (26) lies in $\mathcal{E}_p$ and the other three fixed points lie in $\mathbb{Z}_p^{\star}$. As the dynamic behaviours of the model, we proved that the fixed point $u_0 \in \mathcal{E}_p$ is attractive and the other fixed points $u_i \in \mathbb{Z}_p^{\star}, i = 1,2,3$ are repellent when $u_i = p-1$, and are neutral when $u_i \neq p-1$.

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