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Introducing Fuzzy Almost m-Ideals, Fuzzy Generalized Almost m-Ideals and Related Objects in Semigroups

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ABSTRACT

In this article, we present the concepts of almost m-ideals, generalized almost m-ideals, fuzzy almost m-ideals and fuzzy generalized almost m-ideals. We present the results illustrating their mutual differencs, and similarities. Along these lines, we bring about their examples, and then characterize semigroups through their properties. One of the distinguished properties of the almost m-ideals respectively the related objects is that their intersection is not an almost m-ideal respectively related onjects, while their union is. A subsemigroup of a semigroup is the almost m-ideals if and only if its charateristic function is the fuzzy almost m-ideals. Through these four closely related concepts of m-ideals, we show that this field will open new avenues of research and applications of semigroups.

Keywords: m-ideals, maximal almost m-ideal, fuzzy almost m-ideals, irreducible almost m-ideals.

1. Introduction

A semigroup being an object consisting of a non-empty set and an associative binary operation always caught the attention of the researchers to study it due to its simplicity. Being relatively young among all major algebraic structures, it finds its applications in the most advanced and applied fields. A few to name are the automation theory, formal languages, biological sciences and sociology. The ideals being the supreme sub-objects of all algebraic structures have been the focal point of all studies. The generalizations of ideals played important role to characterize semigroups. This generalization has variant natures. One of the generalizations of ideals in semigroups is made through the given non-negative integers. This generalization through the non-negative integers is of two types which are described in the following two paragraphs.

One type consists of generalizing ideals through two non-negative integers m and n; usually called (m, n)-ideals. This generalization was started initially by Lajos (1960). His studied the essential properties in Lajos (1963, 1964, 1965, 1969, 1970). In Ansari et al. (2009), characterized the quasi ideals through two positive integers in semirings (see Chinram (2008)). The characterized the quasi ideals in semigroups through the two positive integers and characterized the non-associated structures of Kausar (2019), Shah and Kausar (2014) in his article referred as Ansari (2019). In Alam et al. (2013), gave the concept of the (m, n) semirings. Mahboob et al. (2020) defined some types of ideals like (m, n)-hyperideals in ordered semihypergroups using two positive integers m and n. The idea of the (m, n)-bi-quasi hyperideals in semihyperrings was introduced by Pibaljommee and Nakkhasen (2020). The author presented their basic and important properties. Bussaban and Changphas (2016) characterized the regular ordered semigroups through (m, n)-ideals. In Akram et al. (2013) interpreted the (m, n)-ideals in LA-semigroups (see Kausar et al. (2020a), Shah and Kausar (2014)). Khan and Mahboob (2019) presented the relevant idea of

(m, n)-filters in ordered semigroups (see Khan and Mahboob (2019)). Recently, Suebsung et al. (2019) presented the concept of almost (m, n)-ideals in the semigroups in their article.

The other class is the generalization of ideals though one positive integer m, called m-ideals. As far as this generalization of ideals through one positive integer is concerned, Munir and Shafiq (2018) generalized the bi ideals in the semiring through a positive integer m, and called them m-bi ideals. The author again presented the concept of m-bi ideals in the semigroups (see Munir (2018)). Nakkhasen and Pibaljommee (2019) gave the concept of m-bi-hyperideals in semihyperrings through the non-negative integer m. Munir and Ali (2020) again presented the idea of m-quasi ideals in semirings, and other related concepts (see Munir and Habib (2016)) like m-regular and m-intraregular semirings. In the field of fuzzification, Munir et al. (2020) characterized semigroups through introducing the concept of prime fuzzy m-bi ideals.

The above two methods of generalizing ideals in semigroups are exclusively different and are different approaches towards studying the different properties of semigroups. Taking motivation from the theory of generalization of ideals through positive integers in algebraic structures on one hand (see Munir (2018), Munir and Ali (2020), Munir et al. (2020), Munir and Shafiq (2018)), and the already available concepts of almost ideals and fuzzy almost ideals in these structures on the other hand, we were led to combine both the ideas and present the new ideas of four closely related types of m-ideals in semigroups through a positive integer m, namely, (I). Almost m-ideals, (II). Generalized almost m-ideals, (III). Fuzzy almost m-ideals, and (IV). Fuzzy generalized m-ideals.

Since these four classes of m-ideals are closely related, we study them simultaneously taking their properties of primeness, semiprimeness, irreducibility and maximality into consideration. For this purpose, we have divided our work of this paper into eight sections. In Section 1, we have given the introduction of the history of m-ideals, and the motivation to define them and the related objects in semigroups. In Section 2, we present the preliminary results necessary to understand the theory of the almost m-ideals. Section 3 presents the ideas of the almost m-ideals and generalized almost m-ideals. In Section 4, we develop the theory of the fuzzy almost m-ideals and the fuzzy generalized m-ideals. Section 5 discusses their primeness, and Section 6 presents their semiprimess properties. Section 7 studies their maximality, and minimality. Section 8 presents the conclusion of this research work.

2. Preliminaries

Let S be a semigroup, and m be a positive integer, we have, $S^m = SSS\cdots S$ (m times). In lieu of multiplications of sets, $S^m \subseteq S^l$, for $l \leq m$ (see Munir (2018)). Any function $\mu : S \longrightarrow [0,1]$ is said to be a fuzzy set on S (Munir et al. (2020)).

Let there be two fuzzy subsets μ and ν of a semigroup S, then the statement $\mu \leq \nu$ implies that for all $x \in S$, $\mu(x) \leq \nu(x)$.

Consider the fuzzy subsets λ and μ of a semigroup S, the proposition $\lambda \leq \mu$ implies that $\lambda(x) \leq \mu(x)\ \forall\ x \in S$ (Munir et al. (2020)). The conjunction (disjunction) of μ and ν is defined by $\mu \vee \nu(x) = \max[\mu(x), \nu(x)](\mu \wedge \nu(x) = \min[\mu(x), \nu(x)])$. Their composition, denoted by $\mu \circ \nu$, is defined by

$$\mu \circ \nu(z) =$$

$$\begin{cases} \sup\limits_{z=xy,\ x,y\in S} \{\min[\mu(x), \nu(y)]\} \text{ if z is represented by } z = xy, \\ 0 \quad \text{otherwise,} \end{cases} \tag{1}$$

for all $z \in S$. $\mu \vee \nu$, $\mu \wedge \nu$ and $\mu \circ \nu$ are fuzzy sets of S (Shabir and Kanwal (2007)).

The characteristic function of $A \subseteq S$ defined by

$$\chi_A(t) = \begin{cases} 1 & \text{if} \quad t \in A, \\ 0 & \text{if} \quad t \notin A. \end{cases}$$

is a fuzzy subset of S.

The fuzzy subset μ is called a fuzzy subsemigroup of S if and only if $\mu \circ \mu \leq \mu$. That $\mu \circ \mu(x) \leq \mu(x)$, for all $x \in S$.

The fuzzy subsemigroup μ is called a fuzzy left respectively fuzzy right ideal of S if $\chi_S \circ \mu \leq \mu$ respectively $\mu \circ \chi_S \leq \mu$.

μ is a fuzzy ideal of S if $\chi_S \circ \mu \circ \chi_S \leq \mu$, where χ_S is the characteristic function on S. (See Equation 1).

Every subset of S^m is called the m-subset of S. Every subsemigroup of S^m is called m-subsemigroup of S.

Definition 2.1. *Suebsung et al. (2019) For a fuzzy subset μ of S, the support of μ is defined by*

$$\text{supp}(\mu) = \{x \in S \mid \mu(x) \neq 0\}$$

.

3. Almost m-Ideals and Generalized Almost m-Ideals

This section presents the definitions of almost m-ideals and generalized almost m-ideals in semigroups which are the central blocks of this research paper.

Definition 3.1. *A subsemigroup L respectively R of a semigroup S is called an almost m-left ideal respectively an almost m-right ideal of S if $S^m L \cap L \neq \emptyset$ respectively $RS^m \cap R \neq \emptyset$, m is a positive integer. A subsemigroup I of S is called almost m-ideal of S if it is both an almost m-left and an almost m-right ideal. That is, $S^m IS^m \cap I \neq \emptyset$, or $s^m Is^m \cap I \neq \emptyset$, for all $s \in S$.*

Now, we come to definition of generalized almost m-ideals:

Definition 3.2. *A generalized almost m-left (generalized almost m-right ideal) of a semigroup S is simply a non-empty subset $A(B)$ of S which satisfies the condition $S^m A \cap A \neq \emptyset\,(BS^m \cap B \neq \emptyset)$, where m is a positive integer. A non-empty subset C of S is called a generalized almost m-ideal of S if it both a generalized almost m-left and m-right ideal of S or the condition $S^m CS^m \cap C \neq \emptyset$ is satisfied by C.*

Remarks 3.1. *The remarks given below are worhty to be stated.*

1. *The set $S^m CS^m$ is of the form $S^m CS^m = \{s^m ct^m \ \forall s, t \in S, c \in C\}$.*
2. *An almost m-ideal needs to be a subsemigroup of semigroup S, whereas a generalized almost m-ideal is a subset of the semigroup. An almost m-ideal is a generalized almost m-ideal of S, but the converse is not true. This is described in Example 3.1.*

   If $m = 1$, then the condition $S^m IS^m \cap I \neq \emptyset$ respectively ($S^m CS^m \cap C \neq \emptyset$) gives $SIS \cap I \neq \emptyset$ ($SIS \cap I \neq \emptyset$), which means that the subsemigroup I (subset C) is an almost ideal (generalized almost ideal) of S.

3. *Every almost (generalized almost) ideal of S is an almost m-ideal (generalized almost m-ideal) of S(for $m = 1$), but the converse does not hold.*

4. *The almost ideals have $m = 1$, whereas all the other almost m-ideals are to be identified by the value of m.*

Example 3.1. *Let $S = \{1, 2, 3, 4, 5, 6\}$ be the semigroup with the binary operation $\circ$ defined on its elements as given in the following Table 1. If we take $m = 2$, then $S^2 = \{1, 2, 4, 5\}$ (Grimble (1950)).*

Table 1: Almost m-Ideals *versus generalized almost m-ideals*

$\circ$	1	2	3	4	5	6
1	1	1	1	4	4	4
2	1	1	1	4	4	4
3	1	1	2	4	4	5
4	4	4	4	1	1	1
5	4	4	4	1	1	1
6	4	4	5	1	1	2

We find the almost m-ideals and the generalized almost m-ideals of S separately as follow:

1. *Almost m-ideals of S: The subsemigroup, $I = \{1,2,3,4\}$, is an almost m-ideal of S for $m = 2$, because*

$$S^2IS^2 \cap I = \{1, 2, 4, 5\}\{1, 2, 3, 4\}\{1, 2, 4, 5\} \cap \{1, 2, 3, 4\} = \{1\} \neq \emptyset.$$

2. *Generalized almost m-ideals of S: Take a subset $C = \{2, 3, 4\}$, then*

$$S^2CS^2 \cap C = \{1, 2, 4, 5\}\{2, 3, 4\}\{1, 2, 4, 5\} \cap \{2, 3, 4\} = \{4\} \neq \emptyset.$$

This gives that A is a generalized almost 2-ideal of S. This is to be noted that A is not a subsemigroup of S.

The following theorem elaborates a distinguished property of almost m-ideals or generalized almost m-ideals.

Theorem 3.1. *Let I be an almost m-ideal (generalized almost m-ideal) of S, then any superset B of I ($B \supseteq I$) is also an almost m-ideal (generalized almost m-ideal) of S.*

Proof. Since $\emptyset \neq S^m I S^m \cap I \subseteq S^m B S^m \cap B$, so $S^m B S^m \cap B \neq \emptyset$. Thus B is an almost m-ideal (generalized almost m-ideal) of S. □

Theorem 3.1 leads us towards the following important result about these ideals.

Theorem 3.2. *Let $\{A_i : i \in \Lambda\}$ be an arbitrary family of almost m-ideals (generalized almost m-ideals) of a semigroup S, Λ is an indexing set. Then their union, $U = \bigcup_{i\in\Lambda} A_i$, is an almost m-ideal (generalized almost m-ideal) of S.*

Proof. Since $U = \bigcup_{i\in\Lambda} A_i \supseteq A_i$, for some, $i \in \Lambda$, and A_i is almost m-ideal (generalized almost m-ideal) of S, so $\bigcup_{i\in\Lambda} A_i$being superset of A_i , $i \in \Lambda$, is again almost m-ideal (generalized almost m-ideal) of S. □

Remarks 3.2. *We have the following remarks.*

1. *Theorem 3.2 can also be treated as a corollary of Theorem 3.1.*
2. *This is to be noted that a subset of an almost m-ideal (generalized almost m-ideal) of S is not an almost m-ideal (generalized almost m-ideal) of S.*

In consequence of Remarks 3.2, we conclude that the intersection of any number of almost m-ideals respectively generalized almost m-ideals of S is not an almost m-ideal respectively generalized almost m-ideal. In Example 3.1, it is straightforward to check that

1. **Almost m-ideals:** $L = \{1, 2, 3, 4\}$, and $R = \{1\}$ are two almost m-ideals of $S = \{1, 2, 3, 4, 5, 6\}$, but their intersection, $L \cap R = \{2, 3, 4\}$, is not an almost m-ideal of S.

2. **Generalized almost m-ideals:** $A = \{2, 3, 4\}$, and $B = \{1, 2, 3\}$ are two generalized almost m-ideals of S, but their intersection $A \cap B = \{2, 3\}$ is not generalized almost m-ideal of S.

Remark 3.1. *If the intersection of two or more subsemigroups (subsets) of a semigroup S is an almost m-ideals (generalized almost m-ideals), then the subsemigroups (subsets) are also almost m-ideals (generalized almost m-ideals).*

The following proposition answers the question whether the product of two or more almost m-ideals is almost m-ideal or not.

Proposition 3.1. *The product of two or more almost m-ideals (generalized almost m-ideals) of a semigroup S is generally not an almost m-ideal (generalized almost m-ideals).*

Proof. We prove this theorem for the case of two objects, the extension to more than two follows by induction. Let M and N be two almost m-ideals (generalized almost m-ideals) of S, then $MN \subseteq M \cap N$. But $M \cap N$ is not an almost m-ideals (generalized almost m-ideal) of S, so MN being subset of $M \cap N$ is not an almost m-ideal (generalized almost m-ideal) of S. □

Theorem 3.3. *A semigroup S has no proper almost m-ideals (generalized almost m-ideals) if and only if for every $a \in S^m$, $S^m (S - \{a\}) S^m = \{a\}$.*

Proof. Suppose that S has no proper almost m-ideals (generalized almost m-ideals). Then $S-\{a\}$ is not an almost m-ideal (generalized almost m-ideal) of S. So, $S^m (S - \{a\}) S^m \cap (S-\{a\}) = \emptyset$. This implies that $S^m (S - \{a\}) S^m \subseteq \{a\}$. But$\{a\} \subseteq S^m (S - \{a\}) S^m$, so $S^m (S - \{a\}) S^m = \{a\}$.

Conversely suppose that, $S^m (S - \{a\}) S^m = \{a\}$, for every $a \in S^m$. This implies

$$S^m (S - \{a\}) S^m \cap (S - \{a\}) = \{a\} \cap (S - \{a\}),$$

This gives

$$S^m (S - \{a\}) S^m \cap (S - \{a\}) = \emptyset.$$

This means that $S - \{a\}$ is not an almost m-ideal of S, for every $a \in S^m$. We need to show that S has no proper almost m-ideals(generalized almost m-ideals). Let us suppose on contradiction that S has a proper almost m-ideal (generalized almost m-ideal) B. Then we have $B \subseteq S - \{s\}$, for some $s \in S^m$. By Theorem 3.1, $S - \{s\}$ is an almost m-ideal (generalized almost m-ideal) of S, which is a contradiction. Thus S has no proper almost m-ideals (generalized almost m-ideals). □

4. Fuzzy Almost m-Ideals and Fuzzy Generalized Almost m-Ideals

Definition 4.1. *A fuzzy subsemigroup μ of a semigroup S is called a fuzzy almost m-ideal of S if $\chi_{S^m} \circ \mu \circ \chi_{S^m} \wedge \mu \neq 0$, where*

$$\chi_{S^m}(x) = \begin{cases} 1 & if \quad x \in S^m, \\ 0 & if \quad x \notin S^m, \end{cases}$$

m is a positive integer. Equivalently, we say that μ is a fuzzy almost m-ideal of S if for all $x \in S^m$, $(\chi_{S^m} \circ \mu \circ \chi_{S^m} \wedge \mu)(x) \neq 0$.

Definition 4.2. *We define the fuzzy generalized almost m-ideal of S as simply a fuzzy subset δ of the semigroup S satisfying the proposition $\chi_{S^m} \circ \delta \circ \chi_{S^m} \wedge \delta \neq 0$, where*

$$\chi_{S^m}(s) = \begin{cases} 1 & if \quad s \in S^m, \\ 0 & if \quad s \notin S^m, \end{cases}$$

m is a positive integer. In other words, we say that μ is a fuzzy generalized almost m-ideal of S if for all $s \in S^m$, $(\chi_{S^m} \circ \mu \circ \chi_{S^m} \wedge \mu)(s) \neq 0$.

This is to be noted that $\circ$ is given precedence over the operations $\wedge$ or $\vee$.

Proposition 4.1. *A fuzzy subsemigroup (non-empty subset) ν of S which proceeds a fuzzy almost m-ideal (fuzzy generalized almost m-ideal) μ of S i.e., $\mu \leq \nu$, is also fuzzy almost m-ideal(fuzzy generalized almost m-ideal) of* S.

Proof. Since $\mu \leq \nu$, and μ is a fuzzy almost m-ideal (fuzzy generalized almost m-ideal), so for all $x \in S^m$, we have,

$$0 \neq (\chi_{S^m} \circ \mu \circ \chi_{S^m} \wedge \mu)(x) \leq (\chi_{S^m} \circ \nu \circ \chi_{S^m} \wedge \nu)(x).$$

So, $(\chi_{S^m} \circ \nu \circ \chi_{S^m} \wedge \nu)(x) \neq 0$, making ν a fuzzy almost m-ideal (*fuzzy generalized almost m-ideal*) of S. □

Corollary 4.1. *The union of arbitrary number of fuzzy almost m-ideals (fuzzy generalized almost m-ideals) of a semigroup is again a fuzzy almost m-ideal (fuzzy generalized almost m-ideal) of S.*

Proof. Let $\mathrm{V} = \bigvee_{\mathrm{i} \in \Lambda} \mu_{\mathrm{i}}$ be the arbitrary union of the fuzzy almost m-ideals (fuzzy generalized almost m-ideals) μ_{i}; $\mathrm{i} \in \Lambda$, of S. Then $\mu_{\mathrm{i}} \leq \bigvee_{\mathrm{i} \in \Lambda} \mu_{\mathrm{i}}$ for some $\mathrm{i} \in \Lambda$. That is $\mu_{\mathrm{i}} \leq \mathrm{V}$ implying that V is an almost m-ideal (fuzzy generalized almost m-ideal) of S. □

The disjunction (intersection) of two fuzzy almost m-ideals (generalized fuzzy almost m-ideals) μ and ν i.e., $\mu \bigwedge \nu$ is not fuzzy almost m-ideals (generalized fuzzy almost m-ideals). This is displayed in the following example.

Example 4.1. *For the semigroup given in Example 3.1, take two fuzzy almost m-ideals μ and ν defined on S as given below in Table 2.*

Table 2: Intersection of Fuzzy Almost m-Ideals

μ	ν
$\mu(1) = 0$	$\nu(1) = 0$
$\mu(2) = 0.3$	$\nu(2) = 0.7$
$\mu(3) = 0$	$\nu(3) = 0.3$
$\mu(4) = 0$	$\nu(4) = 0$
$\mu(5) = 0.6$	$\nu(5) = 0$
$\mu(6) = 0.4$	$\nu(6) = 0.2$

We check the condition for the elements of $S^2 = \{1, 2, 4, 5\}$ one by one. Consider 1$\in S^2$, then

$$\begin{aligned}
&((\chi_{S^m} \circ (\mu \wedge \nu) \circ \chi_{S^m}) \wedge (\mu \wedge \nu))(1) \\
&= \min[\chi_{S^m} \circ (\mu \wedge \nu) \circ \chi_{S^m}(4), \mu \wedge \nu(1)], \\
&= 0,
\end{aligned}$$

as

$$\mu \wedge \nu(1) = \min[\mu(1), \nu(1)] = \min[0, 0] = 0.$$

Let $2 \in S^2$, we have two representations for 2 as $2 = 3.3$, and $2 = 6.6$ as depicted in the Table 1. In both the cases, 3 and 6 do not belong to S^2, so using Equation 1, we get $(\mu \wedge \nu) \circ \chi_{S^m}(2) = 0$.

Consequently, $((\chi_{S^m} \circ (\mu \wedge \nu) \circ \chi_{S^m}) \wedge (\mu \wedge \nu))(2) = 0$.

Similarly, $((\chi_{S^m} \circ (\mu \wedge \nu) \circ \chi_{S^m}) \wedge (\mu \wedge \nu))(4) = 0$, and

$((\chi_{S^m} \circ (\mu \wedge \nu) \circ \chi_{S^m}) \wedge (\mu \wedge \nu))(5) = 0$. *We concluded that*

$(\chi_{S^m} \circ \mu \circ \chi_{S^m} \wedge \mu)(x) \leq (\chi_{S^m} \circ \nu \circ \chi_{S^m} \wedge \nu)(x) = 0, \forall x \in S^2.$

That the intersection of two almost fuzzy m-ideals (*fuzzy generalized almost m-ideals)*, $\mu \wedge \nu$, is not an almost m-ideals (*fuzzy generalized almost m-ideals).*

Proposition 4.2. *Since $\mu \circ \nu \leq \mu \wedge \nu$ (see Shabir and Kanwal (2007)), so the composition of two fuzzy almost m-ideals (fuzzy generalized almost m-ideals) is not a fuzzy almost m-ideal fuzzy generalized almost m -ideal.*

Lemma 4.1. *If P and Q are subsets of a semigroup S, then $\chi_A \circ \chi_B = \chi_{AB}$ (Shabir and Kanwal (2007)).*

Theorem 4.1. *Let P be a subsemigroup of a semigroup S, such that $P \subseteq S^m$, then the characteristics function, χ_P, of P is a fuzzy almost m-ideal of S if and only if P is almost m-ideal of S.*

Proof. Suppose that χ_A is a fuzzy almost m-ideal of S. Then

$$\Leftrightarrow (\chi_{S^m} \circ \chi_P \circ \chi_{S^m}) \wedge \chi_P \neq 0,$$
$$\Leftrightarrow (\chi_{S^m} \circ \chi_P \circ \chi_{S^m}) \wedge \chi_P (x) \neq 0, for\ x \in S^m,$$
$$\Leftrightarrow (\chi_{S^m P S^m} \wedge \chi_P)(x) \neq 0,\ for\ x \in S^m,$$
$$\Leftrightarrow S^m P S^m \cap P \neq 0,\ for\ x \in S^m.$$
$$\Leftrightarrow P \text{ is an almost } m\text{-ideal of } S.$$

□

Corollary 4.2. *If A is a subset of S, such that $A \subseteq S^m$, then the characteristics function, χ_A, of A is a fuzzy generalized almost m-ideal of S if and only if A is generalized almost m-ideal of S.*

Proof. Similar. □

Next theorem characterizes the fuzzy almost m-ideal of a semigroup in terms of its support.

Theorem 4.2. *A fuzzy subsemigroup (fuzzy subset) μ of a semigroup S is fuzzy almost m-ideal (fuzzy generalized almost m-ideal) of S if and only if the support of μ (*supp (μ)*) is an almost m-ideal (generalized almost m-ideal) of S.*

Proof. If μ is a subsemigroup, so supp(μ) is also a subsemigroup of S. Suppose directly that μ is a fuzzy almost m-ideal *(fuzzy generalized almost m-ideal)* of

S, then

$$\Leftrightarrow (\chi_{S^m} \circ \mu \circ \chi_{S^m}) \wedge \mu \neq 0,$$
$$\Leftrightarrow ((\chi_{S^m} \circ \mu \circ \chi_{S^m}) \wedge \mu)(x) \neq 0, \quad for \;\; x \in S^m.$$

So there exist $s_1, \cdots, s_m$, $t_1, \cdots, t_m \in S$, such that $x = s_1 \cdots s_m t_1 \cdots t_m$, and $\mu(s_1) \neq 0, \cdots, \mu(s_m) \neq 0, \mu(t_1) \neq 0, \cdots, \mu(t_m) \neq 0$.

Therefore, $x, \;\; s_1, \; \cdots, s_m, t_1, \;\; \cdots, \;\; t_m \in \text{supp}(\mu)$.

Thus,

$\Leftrightarrow \left(\chi_{M^m} \circ \chi_{\text{supp}(\mu)} \circ \chi_{M^m}\right)(x) \neq 0$, and$\chi_{\text{supp}(\mu)}(x) \neq 0$.
$\Leftrightarrow (\chi_{M^m} \circ \chi_{\text{supp}(\mu)} \circ \chi_{M^m}) \cap \chi_{\text{supp}(\mu)}(x) \neq 0$.
$\Leftrightarrow \chi_{\text{supp}(\mu)}$ is a fuzzy almost m-ideal of S.
$\Leftrightarrow$ *supp*(μ) is an almost m-ideal (generalized almost m-ideal) of S.

□

5. Primeness of Almost m-Ideals, Generalized Almost m-Ideals, Fuzzy Almost m-Ideals and Fuzzy Generalized Almost m-Ideals

In this section of the article, we describe the primeness, semiprimeness and completely primeness properties of the relevant types of the almost m-ideals of the semigroup.

Definition 5.1. *For a semigroup S.*

1. *An almost m-ideal (generalized almost m-ideal) A of S is said to be prime if we take any two almost m-ideals(generalized almost m-ideals) B, C of S, and $BC \subseteq A$ implies $B \subseteq A$ or $C \subseteq A$.*
2. *An almost m-ideal (generalized almost m-ideal) A of S is said to be completely prime if $\forall x, \; y \in S$ such that $xy \in A$. Then either $x \in A$ or $y \in A$.*
3. *A fuzzy almost m-ideal (fuzzy generalized almost m-ideal) μ of S is called prime if for any two fuzzy almost m-ideals(fuzzy generalized almost m-ideals) ν, λ of S, $\nu \circ \lambda \leq \mu$ implies $\nu \leq \mu$ or $\lambda \leq \mu$.*

4. *A fuzzy almost m-ideal (fuzzy generalized almost m-ideal) μ of S is called completely prime if*

$$\forall\ x, y\ \in S,\ \mu(xy) \leq \max[\mu\,(x)\,,\quad \mu(y)].$$

5. *A fuzzy almost m-ideal (fuzzy generalized almost m-ideal) μ of S is called weakly completely prime if* $\forall\ x, y\ \in S,\ \ \mu\,(xy) = \max[\mu\,(x)\,,\quad \mu(y)]$.

Remarks 5.1. *The following remarks follow immediately.*

1. *A completely prime almost m-ideal (resp. generalized almost m-ideal) is a prime almost m-ideal, but the converse holds if the semigroup is commutative Park and Kim (1992). Similar statements hold for the other classes of almost m-ideals.*

2. *A completely prime almost m-ideal (resp. generalized almost m-ideal) is weakly completely prime almost m-ideal (resp. generalized almost m-ideal), but the converse holds if the semigroup is commutative Park and Kim (1992).*

Example 5.1. *Let* $S = \ \{0,\ a,\ b,\ c, d\}$ *with the binary operation* $*$ *defined in Table 3.*

Table 3: Completely Prime Almost m-Ideal

$*$	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	b	b
d	0	a	b	b	c

Taking $m = 3$*, we get* $S^3 = \{0, a, b\}$*. Subsemigroup* $\{0, a\}$ *is the only completely prime almost* 3*-ideal of* S*. It is also prime(semiprime) almost* 3*-ideal of* S*. It is also weakly completely prime almost* 3*-ideal of* S*.*

Theorem 5.1. *A subsemigroup (non-empty) A of a semigroup S is a completely prime almost m-ideal (a completely prime generalized almost m-ideal) of S $\Leftrightarrow$ the characteristics function defined on A, i.e., χ_A is a completely prime fuzzy almost m-ideal (a completely prime fuzzy generalized almost m-ideal) of S.*

Proof. If A is a completely prime almost m-ideal (a completely prime generalized almost m-ideal) of S, then by Theorem 4.1, the characteristic function, χ_A is a fuzzy almost m-ideal (generalized almost m-ideal) of S. In order to show that χ_A is completely prime fuzzy almost m-ideal (completely prime fuzzy generalized almost m-ideal), we take $x,\ y\ \in S$.

Now, if $xy\ \in\ A$, then by our hypothesis either $x\ \in\ A$ or $y\ \in\ A$. Thus $\chi_A\ (x) = 1$, and $\chi_A\ (y) = 1$. Consequently, $max\{\chi_A\ (x),\ \chi_A\ (y)\}\ =\ 1\ \geq\ \chi_A\ (xy)$. This makes χ_A *a completely prime fuzzy almost m-ideal (completely prime fuzzy generalized almost m-ideal) of S.* if $xy\ \notin\ A$, then, $\chi_A\ (xy) = 0 \leq\ max\{\chi_A\ (x),\ \chi_A\ (y)\}$, making again χ_A a completely prime fuzzy almost m-ideal(a completely prime generalized almost m-ideal). So, in both cases, χ_A is completely prime.

For the converse, if χ_A is assumed a completely prime fuzzy almost m-ideal (completely prime fuzzy generalized almost m-ideal) of S, then A becomes an almost m-ideal (generalized almost m-ideal) of S by Theorem 4.1. Suppose that for $x,\ y\ \in\ S$, $xy\ \in\ A$. Then $\chi_A\ (xy)= 1$. Using our hypothesis,

$$\chi_A\ (xy)\ \leq \max[\chi_A\ (x)\,,\ \chi_A\ (y)].$$

Therefore, $max\{\chi_A\ (x),\ \chi_A\ (y)\}=1$. Hence, $x\ \in\ A$ or $y\ \in\ A$. Thus, A is completely prime almost m-ideal(completely prime *generalized almost m-ideal*) of S. □

Corollary 5.2: *If B is a subsemigroup (non-empty subset) of S, then B is a weakly completely prime almost m-ideal(weakly completely prime generalized almost m-ideal) of S if and only if χ_A.is a weakly completely prime fuzzy almost m-ideal(weakly completely prime fuzzy generalized almost m-ideal) of S.*

Proof. Similar. □

6. Semi-Primeness of Almost m-Ideals, Generalized Almost m-Ideals, Fuzzy Almost m-Ideals and Fuzzy Generalized Almost m-Ideals

Definition 6.1. *Below are the following definitions.*

1. *An almost m-ideal (generalized almost m-ideal) A of S is called semiprime if for any almost m-ideals (generalized almost m-ideal) B of S, $B^2 \subseteq A$ implies $B \subseteq A$ (Park and Kim (1992)).*

2. *An almost m-ideal (generalized almost m-ideal) A of S is termed to be a completely semiprime if for all $x \in S$, $x^2 \in A \; \forall \; x \in A$.*

3. *A fuzzy almost m-ideal (fuzzy generalized almost m-ideal) μ of S is said to semiprime in case for a fuzzy almost m-ideals(fuzzy generalized almost m-ideals) λ of S, $\lambda \circ \lambda \leq \mu \;\Rightarrow\; \lambda \leq \mu$.*

4. *A fuzzy almost m-ideal (fuzzy generalized almost m-ideal) μ of S is called completely prime if*
$$\forall\; x \in S,\; \mu(x^2) \leq \mu\left(x\right).$$

5. *A fuzzy almost m-ideal (fuzzy generalized almost m-ideal) μ of S is called weakly completely semiprime if $\forall\; x \in S,\;\; \mu\left(x^2\right) = \mu\left(x\right)$.*

Remarks 6.1. *The following remarks follow from the above definitions.*

1. *A completely semiprime almost m-ideal (resp. generalized almost m-ideal) is a semiprime almost m-ideal, but the converse holds if the semigroup is commutative (Dutta and Biswas (1994)). Similar statements hold for the other classes of almost m-ideals.*

2. *A completely semiprime almost m-ideal (resp. generalized almost m-ideal) is weakly completely semiprime almost m-ideal (resp. generalized almost m-ideal), but the converse holds if the semigroup is commutative (Dutta and Biswas (1994)).*

Theorem 6.1. *The characteristic function, χ_A, on the subsemigroup(non-empty) A of S is a completely semiprime fuzzy almost m-ideal (completely semiprime fuzzy generalized almost m-ideal) of $S \;\Leftrightarrow\; A$ is a completely semiprime almost m-ideal (completely semiprime fuzzy almost m-ideal) of S.*

Proof. Let A be a completely semiprime almost (completely semiprime generalized almost m-ideal) of S, then χ_A is a fuzzy almost m-ideal(fuzzy almost generalized m-ideal) of S. We show that χ_A is completely semiprime. Let $x \in S$. There are two possibilities:

1. $x^2 \in A$. So, $x \in A$, by our hypothesis. This gives that $\chi_A(x) = 1$. Thus $\chi_A(x^2) \leq \chi_A(x)$.
2. $x^2 \notin A$. Then, $\chi_A(x^2) = 0 \leq \chi_A(x)$. Thus again $\chi_A(x^2) \leq \chi_A(x)$.

Cases 1 and 2 result in making χ_A a completely semiprime fuzzy almost m-ideal (completely semiprime fuzzy generalized almost m-ideal) of S.

For the reverse part, if χ_A is supposed a completely semiprime fuzzy almost m-ideal (completely semiprime fuzzy generalized almost m-ideal) of S, then A is an almost m-ideal (generalized almost m-ideal) of S due to the reason stated in Theorem 4.1. Let $x \in S$ such that $x^2 \in A$. Then $\chi_A(x^2) = 1$. By hypothesis, $\chi_A(x^2) \leq \chi_A(x)$. Since $\chi_A(x^2) = 1$, $\chi_A(x) = 1$. This brings $x \in A$, i.e., A is a semiprime almost m-ideal (semiprime generalized almost m-ideal) of S.

□

7. Maximality of Almost m-Ideals, Generalized Almost m-Ideals, Fuzzy Almost m-Ideals and Fuzzy Generalized Almost m-Ideals

Definition 7.1. *An almost m-ideal (generalized almost m-ideal) A of S is said to be maximal if* $\forall$ *almost m-ideals (generalized almost m-ideal)* $B(\neq S)$ *of S such that* $B \supseteq A$, *we have* $B = A$. *(Grillet (1969), Schwarz (1969)).*

Definition 7.2. *A fuzzy almost m-ideal (fuzzy generalized almost m-ideal)* λ *of S is said to be maximal if for all fuzzy almost m-ideals (fuzzy generalized almost m-ideal)* $\nu(\neq \chi_S)$ *of S such that* $\nu \geq \lambda$, *we have* $\mathrm{supp}(\nu) = \mathrm{supp}(\lambda)$. *(Grillet (1969), Schwarz (1969)).*

Remark 7.1. *The concepts of the minimal almost m-ideals (resp. generalized, fuzzy, fuzzy generalized) can be defined as is the practice for other classes of ideals.*

Theorem 7.1. *The characteristics function,* χ_A, *on a subsemigroup A of S is a maximal(minimal) fuzzy almost m-ideal* $\Leftrightarrow$ *A is a maximal (minimal) almost m-ideal of S.*

Proof. We prove this theorem for the case of the maximal almost m-ideals, the proof for the minimal almost m-ideals flows analogously. If A is assumed to *be a maximal almost m-ideal of S*, then χ_A becomes to be a fuzzy almost mi-deal of S by Theorem 4.1. To show χ_A is maximal, we take μ any fuzzy almost m-ideal of S such that $\mu \geq \chi_A$. Then $\text{supp}(\mu) \geq \text{supp}(\chi_A) = A$. Since, $\text{supp}(\mu)$ is an almost m-ideal of S, and A is maximal, $\text{supp}(\mu) = A = \text{supp}(\chi_A)$. Therefore, χ_A is maximal.

Conversely, suppose that χ_A is a maximal fuzzy almost m-ideal of S. We need to show that A is maximal. For this if G is any almost m-ideal of S with the presumption that $G \supseteq A$, we show that $G = A$ .Since χ_G is a fuzzy almost m-ideal of S by Theorem 4.1 such that $\chi_A \geq \chi_G$, so $G = \text{supp}(\chi_G) = \text{supp}(\chi_A) = A$. □

Corollary 7.1. *A semigroup S possesses no proper almost m-ideal (generalized almost m-ideals) if and only if for all fuzzy almost m-ideals (generalized almost m-ideals) μ of S,* $\text{supp}(\mu) = S$.

Proof. In the direct case the only almost m-ideals of S are $\{0\}$ and S. This implies that $\chi_{\{0\}}$ and χ_S are the only fuzzy almost m-ideals of S. $\text{supp}(\chi_{\{0\}}) =$ 0 or 1. But $\text{supp}(\chi_S) = S$.

Conversely if $\text{supp}(\chi_S) = S$, then since $\text{supp}(\chi_S)$ is almost m-ideal which is equal to S, it follows that S has no proper almost m-ideals.

□

Remark 7.2. *Theorem 7.1 and associated corollary equally hold for the case of the minimal almost m-ideals and related objects.*

Definition 7.3. *An almost m-ideal A of a semigroup S is known as an irreducible(strongly irreducible) almost m-ideal if $A_1 \cap A_2 = A$ ($A_1 \cap A_2 \subseteq A$) implies either $A_1 = A$ or $A_2 = A$ (either $A_1 \subseteq A$ or $A_2 \subseteq A$), for any two almost m-ideals A_1 and A_2 of S.*

Strongly irreducible almost m-ideal are irreducible almost m- ideal, but the converse is not true. The following theorem holds equally for all almost m-ideals and generalized almost m-ideals. However, we state and prove it for almost m-ideals; for other type, it holds similarly.

Theorem 7.2. *The given assertions are equivalent for a semigroup S. (Shabir and Kanwal (2007)).*

1. *The set of all almost m-ideals of S is totally ordered under inclusion of sets.*

2. *Every almost m-ideal of S is strongly irreducible almost m-ideal.*

3. *Every almost m-ideal of S is irreducible almost m-ideal.*

Proof. $1 \Rightarrow 2$: Suppose A is an almost m-ideal of S and for any almost m-ideals A_1 and A_2 of S, the statement $A_1 \cap A_2 \subseteq A$ holds. Since A is totally ordered under inclusion of sets, so $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. This implies, either $A_2 \cap A_1 = A_1$ or $A_1 \cap A_2 = A_2$. So, from our previous result either $A_1 \subseteq A$, or $A_2 \subseteq A$ resulting A into a strongly irreducible almost m-ideal of S.

$2 \Rightarrow 3$: Straightforward as strongly irreducible almost m-ideal of S is irreducible almost m-ideal.

$3 \Rightarrow 1$: For any two almost m-ideals of S, namely A_1 and A_2 we have the identity $A_1 \cap A_2 = A_2 \cap A_1$. Since each almost m-ideal of S is irreducible, $A_1 = A_2 \cap A_1$or $A_2 = A_2 \cap A_1$, which further implies, $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. That is, A_1 and A_2 are comparable making the collection of almost m-ideals of S a totally ordered set. □

8. Conclusions

In this article, we simultaneously presented the concepts of almost m-ideals, generalized almost m-ideals, fuzzy almost m-ideals and generalized fuzzy almost m-ideals in the realm of semigroup theory. We explored the interconnection and the dissimilarities among them through examples and theorems. Basic and important properties of these four related objects were explored. The conditions when a semigroup does not possess proper almost m-ideals and related objects were presented. The analogues of these results were also presented for the fuzzy almost m-ideals and fuzzy generalized almost m-ideals.

We showed that the union of arbitrary number of fuzzy almost m-ideals respectively related objects in semigroups is again a fuzzy almost m-ideal respectively related objects, whereas this property does not hold for their disjunction (intersection).

Some important characterizations of fuzzy almost m-ideals and related objects were given in terms of their characteristic functions.

The concepts of the prime, semiprime, completely prime, weakly completely prime were interpreted for these four classes of the ideals. Maximality of almost m-ideals, generalized almost m-ideals, fuzzy almost m-ideals and fuzzy generalized almost m-ideals were also discussed.

As a further extension of this work, several more properties of these ideals, and their applications can be searched out, and other classes of ideals like the ones in Anjum et al. (2020) can be generalized through positive integers in this way. Moreover, these concepts can be presented in other algebraic structures like semirings, near-rings, semi-near-rings etc. These can be introduced in non-associative structures (Kausar et al. (2020b,c)).

The concept of m-ideals will help explore more intrinsic properties of larger semigroups, thus paving their way for applications in more applied fields of mathematics, science and engineering.

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References

Akram, M., Yaqoob, N., and Khan, M. (2013). On (m,n)-ideals in la-semigroups. *Applied Mathematical Sciences*, 7(44):2187–2191.

Alam, S. E., Rao, S., and Davvaz, B. (2013). (m, n)-semirings and a generalized fault-tolerance algebra of systems. *Journal of Applied Mathematics*, 2013, Article ID 482391, 10 pages. https://doi.org/10.1155/2013/482391.

Anjum, R., Ullah, S., Chu, Y.-M., Munir, M., Kausar, N., and Kadry, S. (2020). Characterizations of ordered h-regular semirings by ordered h-ideals. *AIMS Mathematics*, 5(6):5768–5790.

Ansari, M. A. (2019). Roughness in generalized (m,n) bi-ideals in ordered la-semigroups. *Computer Science*, 14(2):371–386.

Ansari, M. A., Khan, M. R., and Kaushik, J. (2009). A note on (m,n) quasi-ideals in semigroups. *International Journal of Mathematical Analysis*, 3(38):1853–1858.

Bussaban, L. and Changphas, T. (2016). On (m,n)-ideals on (m,n)-regular ordered semigroups. *Songklanakarin Journal of Science and Technology*, 38(2):199–206.

Chinram, R. (2008). A note on (m,n)-quasi-ideals in semirings. *International Journal of Pure and Applied Mathematics*, 49(1):45–52.

Dutta, T. and Biswas, B. (1994). Fuzzy prime ideals of a semiring. *Bulletin of the Malaysian Mathematical Sciences Society*, 17:9–16.

Grillet, P. (1969). Intersections of maximal ideals in semigroups. *The American Mathematical Monthly*, 76(5):503–509.

Grimble, H. B. (1950). Prime ideals in semigroups. https://trace.tennessee.edu/utk_gradthes/1119.

Kausar, N. (2019). Direct product of finite intuitionistic anti fuzzy normal subrings over non-associative rings. *European Journal of Pure and Applied Mathematics*, 12(2):622–648.

Kausar, N., Alesemi, M., Salahuddin, and Munir, M. (2020a). Study on la-ring by their instuitionistic fuzzy ideals. *Mathematica Montisnigri*, 47(1):22–42.

Kausar, N., Munir, M., Islam, B., Alesemi, M., Salahuddin, and Gulzar, M. (2020b). Fuzzy *bi*-ideals in la-rings. *Italian Journal of Pure and Applied Mathematics*, (44):745–763.

Kausar, N., Munir, M., Islam, B., Alesemi, M., Salahuddin, and Gulzar, M. (2020c). Ideals in la-rings. *Italian Journal of Pure and Applied Mathematics*, (44):731–744.

Khan, N. M. and Mahboob, A. (2019). Left-m-filter, right-n-filter and (m,n)-filter on ordered semigroup. *Journal of Taibah University for Science*, 13(1):27–31.

Lajos, S. (1960). On (m,n)-ideals of semigroups, abstract of second hunger. In *Math. Congress I*, pages 42–44.

Lajos, S. (1963). Notes on (m,n)-ideals. i. *Proceedings of the Japan Academy*, 39(7):419–421.

Lajos, S. (1964). Notes on (m,n)-ideals ii. *Proceedings of the Japan Academy*, 40(8):631–632.

Lajos, S. (1965). Notes on (m,n)-ideals. iii. *Proceedings of the Japan Academy*, 41(5):383–385.

Lajos, S. (1969). On the *bi*-ideals in semigroups. *Proceedings of the Japan Academy*, 45(8):710–712.

Lajos, S. (1970). On (m,n)-ideals in regular duo semigroups. *ACTA Scientiarum Mathematicarum*, 31(1-2):179.

Mahboob, A., Khan, N. M., and Davvaz, B. (2020). (m,n)-hyperideals in ordered semihypergroups. *Categories and General Algebraic Structures with Applications*, 12(1):43–67.

Munir, M. (2018). On $m-bi$ ideals in semigroups. *Bulletin of the International Mathematical Virtual Institute*, 8(3):461–467.

Munir, M. and Ali, A. (2020). On generalization of quasi ideals in semirings. *Bulletin of the International Mathematical Virtual Institute*, 10(1):83–94.

Munir, M. and Habib, M. (2016). Characterizing semirings using their quasi and *bi*-ideals. *Proceedings of the Pakistan Academy of Sciences: A. Physical and Computational Sciences*, 53(4):469–475.

Munir, M., Kausar, N., Salahuddin, and Tehreem (2020). On the prime fuzzy $m-bi$ ideals in semigroups. *Journal of Mathematics and Computer Science*, 21(4):357–365.

Munir, M. and Shafiq, A. (2018). A generalization of bi ideals in semirings. *Bulletin of the International Mathematical Virtual Institute*, 8(1):123–133.

Nakkhasen, W. and Pibaljommee, B. (2019). On $m-bi$-hyperideals in semihyperrings. *Songklanakarin Journal of Science & Technology*, 41(6).

Park, Y. S. and Kim, J. P. (1992). Prime and semiprime ideals in semigroups. *Kyungpook Mathematical Journal*, 32(3):629–633.

Pibaljommee, B. and Nakkhasen, W. (2020). Connections of (m,n)-*bi*-quasi hyperideals in semihyperrings. *Thai Journal of Mathematics, Special Issue : Annual Meeting in Mathematics 2019*, pages 39–48.

Schwarz, Š. (1969). Prime ideals and maximal ideals in semigroups. *Czechoslovak Mathematical Journal*, 19(1):72–79.

Shabir, M. and Kanwal, N. (2007). Prime *bi*-ideals of semigroups. *Southeast Asian Bull. Math*, 31(4):757–764.

Shah, T. and Kausar, N. (2014). Characterizations of non-associative ordered semigroups by their fuzzy *bi*-ideals. *Theoretical Computer Science*, 529:96–110.

Suebsung, S., Wattanatripop, K., and Chinram, R. (2019). On almost (m,n)-ideals and fuzzy almost (m,n)-ideals in semigroups. *Journal of Taibah University for Science*, 13(1):897–902.