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A Class of Double Integrals Involving Gauss's Hypergeometric Function

Jun, S.[1], Kılıçman, A. *[2,3], Kim, I.[4], and Rathie, A. K.[5]

[1] *General Education Institute, Konkuk University, Republic of Korea*
[2] *Institute for Mathematical Research, Universiti Putra Malaysia, Malaysia*
[3] *Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, Malaysia*
[4] *Department of Mathematics Education, Wonkwang University, Republic of Korea*
[5] *Department of Mathematics, Vedant College of Engineering and Technology (Rajasthan Technical University), Bundi, Rajasthan, India*

E-mail: akilic@upm.edu.my
* *Corresponding author*

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ABSTRACT

One hundred interesting double integrals involving Gauss's hypergeometric function in the form of four general integrals (twenty five each) have been evaluated in terms of gamma function. More than two hundred special cases have also been given.

Keywords: Gauss's hypergeometric function, generalized hypergeometric function, Watson summation theorem, Edwards's double integral.

1. Introduction

In order to justify our doing, we must quote (see Sylvester (1973)):

" It seems to be expected of every pilgrim up the slope of the mathematical Parnassus, that he will at some point or other of his journey sit down and invent a definite integral or two towards the increase of the common stock" .

Very recently, a natural generalization of Edwards's double integral has been given by Kim et al. (2019) in the form

$$\begin{aligned} &\int_0^1 \int_0^1 x^{\mu-1} y^{\kappa+\mu-1} (1-x)^{\kappa-1} (1-y)^{\lambda-1} (1-xy)^{\nu-\kappa-\lambda} dxdy \\ &\quad = \frac{\Gamma(\kappa)\Gamma(\lambda)}{\Gamma(\kappa+\lambda)} \cdot \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \end{aligned} \tag{1}$$

provided $\mathrm{Re}(\kappa) > 0$, $\mathrm{Re}(\lambda) > 0$, $\mathrm{Re}(\mu) > 0$ and $\mathrm{Re}(\nu) > 0$.

For $\mu = \nu = 1$, (1) immediately reduces to Edwards's double integral (see Edwards (1954)).

In the same paper, Kim et al. (2019) have evaluated several double integrals with classical summation theorems, i.e. Watson, Dixon and Whipple theorems for the series ${}_3F_2$ of unit argument. We, however, in our present study, recall the following two integrals.

$$\begin{aligned} &\int_0^1 \int_0^1 x^{\mu-1} y^{\sigma+\mu-1} (1-x)^{\sigma-1} (1-y)^{\sigma-1} (1-xy)^{\nu-2\sigma} \\ &\qquad \times {}_2F_1 \left[\begin{matrix} \tau, \quad \rho \\ \frac{1}{2}(\tau+\rho+1) \end{matrix} ; \frac{y(1-x)}{1-xy} \right] dxdy \\ &= k\,\Omega \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\int_0^1 \int_0^1 x^{\mu-1} y^{\sigma+\mu-1} (1-x)^{\sigma-1} (1-y)^{\sigma-1} (1-xy)^{\nu-2\sigma} \\ &\qquad \times {}_2F_1 \left[\begin{matrix} \tau, \quad \rho \\ \frac{1}{2}(\tau+\rho+1) \end{matrix} ; \frac{1-y}{1-xy} \right] dxdy \\ &= k\,\Omega \end{aligned} \tag{3}$$

provided that $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\nu) > 0$ and $\mathrm{Re}(\sigma) > 0$.

In both the integrals k and Ω are given by,

$$k = \frac{\Gamma(\sigma)\Gamma(\sigma)}{\Gamma(2\sigma)} \cdot \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \tag{4}$$

and

$$\Omega = \frac{\Gamma(\frac{1}{2})\Gamma(\sigma+\frac{1}{2})\Gamma(\frac{1}{2}\tau+\frac{1}{2}\rho+\frac{1}{2})\Gamma(\sigma-\frac{1}{2}\tau-\frac{1}{2}\rho+\frac{1}{2})}{\Gamma(\frac{1}{2}\tau+\frac{1}{2})\Gamma(\frac{1}{2}\rho+\frac{1}{2})\Gamma(\sigma-\frac{1}{2}\tau+\frac{1}{2})\Gamma(\sigma-\frac{1}{2}\rho+\frac{1}{2})}. \tag{5}$$

Inspired by the double integrals (2) and (3), our aim, in this paper, is to evaluate the following four general double integrals

$$\int_0^1\int_0^1 x^{\mu-1}y^{\sigma+\eta+\mu-1}(1-x)^{\sigma+\eta-1}(1-y)^{\sigma-1}(1-xy)^{\nu-2\sigma-\eta} \times {}_2F_1\left[\begin{matrix}\tau, & \rho \\ \frac{1}{2}(\tau+\rho+\iota+1)\end{matrix}; \frac{y(1-x)}{1-xy}\right] dxdy, \tag{6}$$

$$\int_0^1\int_0^1 x^{\mu-1}y^{\sigma+\mu-1}(1-x)^{\sigma-1}(1-y)^{\sigma+\eta-1}(1-xy)^{\nu-2\sigma-\eta} \times {}_2F_1\left[\begin{matrix}\tau, & \rho \\ \frac{1}{2}(\tau+\rho+\iota+1)\end{matrix}; \frac{y(1-x)}{1-xy}\right] dxdy, \tag{7}$$

$$\int_0^1\int_0^1 x^{\mu-1}y^{\sigma+\mu+\eta-1}(1-x)^{\sigma+\eta-1}(1-y)^{\sigma-1}(1-xy)^{\nu-2\sigma-\eta} \times {}_2F_1\left[\begin{matrix}\tau, & \rho \\ \frac{1}{2}(\tau+\rho+\iota+1)\end{matrix}; \frac{1-y}{1-xy}\right] dxdy, \tag{8}$$

and

$$\int_0^1 \int_0^1 x^{\mu-1} y^{\sigma+\eta-1}(1-x)^{\sigma-1}(1-y)^{\sigma+\eta-1}(1-xy)^{\nu-2\sigma-\eta} \times {}_2F_1\left[\begin{matrix} \tau, & \rho \\ \frac{1}{2}(\tau+\rho+\iota+1) \end{matrix}; \frac{1-y}{1-xy}\right] dxdy, \tag{9}$$

each for $\iota, \eta = 0, \pm 1, \pm 2$.

In order to evaluate in all one hundred double integrals involving Gauss's hypergeometric function in the form of four master formulas, we need the following generalizations of classical Watson's summation theorem for the series ${}_3F_2$ which is obtained earlier by Lavoie et al. (1992) viz.

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} \tau, & \rho, & \sigma \\ \frac{1}{2}(\tau+\rho+\iota+1), & 2\sigma+\eta \end{matrix}; 1\right] \\ &= A_{\iota,\eta} \frac{2^{\tau+\rho+\iota-2}\Gamma(\frac{1}{2}(\tau+\rho+\iota+1))\Gamma(\sigma+\left[\frac{\eta}{2}\right]+\frac{1}{2})\Gamma(\sigma-\frac{1}{2}(\tau+\rho+|\iota+\eta|-\eta-1)}{\Gamma(\frac{1}{2})\Gamma(\tau)\Gamma(\rho)} \\ &\times \Big\{B_{\iota,\eta} \frac{\Gamma(\frac{1}{2}\tau+\frac{1}{4}(1-(-1)^{\iota}))\Gamma(\frac{1}{2}\rho)}{\Gamma(\sigma-\frac{1}{2}\tau+\frac{1}{2}+\left[\frac{\eta}{2}\right]-\frac{1}{4}(-1)^{\eta}(1-(-1)^{\iota}))\Gamma(\sigma-\frac{1}{2}\rho+\frac{1}{2}+\left[\frac{\eta}{2}\right])} \\ &+ C_{\iota,\eta} \frac{\Gamma(\frac{1}{2}\tau+\frac{1}{4}(1+(-1)^{\iota}))\Gamma(\frac{1}{2}\rho+\frac{1}{2})}{\Gamma(\sigma-\frac{1}{2}\tau+\left[\frac{\eta+1}{2}\right]+\frac{1}{4}(-1)^{\eta}(1-(-1)^{\iota}))\Gamma(\sigma-\frac{1}{2}\rho+\left[\frac{\eta+1}{2}\right])}\Big\} \\ &= \Omega_1 \end{aligned} \tag{10}$$

for $\iota, \eta = 0, \pm 1, \pm 2$.

Here, as usual, $[x]$ denotes the largest integer less than or equal to x and $|x|$ denotes its modulus. The coefficients $A_{\iota,\eta}$, $B_{\iota,\eta}$ and $C_{\iota,\eta}$ are given in Tables 1, 2 and 3 at the end of this paper.

Clearly, for $\iota = \eta = 0$, (10) reduces at once to the following classical Watson's summation theorem for the series ${}_3F_2$ (see Bailey (1935)) viz.

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} \tau, & \rho, & \sigma \\ \frac{1}{2}(\tau+\rho+1), & 2\sigma \end{matrix}; 1\right] \\ &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}(\tau+\rho+1))\Gamma(\sigma+\frac{1}{2})\Gamma(\sigma-\frac{1}{2}(\tau+\rho-1))}{\Gamma(\frac{1}{2}(\tau+1))\Gamma(\frac{1}{2}(\rho+1))\Gamma(\sigma-\frac{1}{2}\tau+\frac{1}{2}))\Gamma(\sigma-\frac{1}{2}\rho+\frac{1}{2}))} \end{aligned} \tag{11}$$

provided $\mathrm{Re}(2\sigma - \tau - \rho) > -1$.

2. Main Results

In order to calculate and compute several double integrals involving Gauss's hypergeometric function in the form of four general integrals which are given in the following theorems.

Theorem 2.1. *For $\iota, \eta = 0, \pm 1, \pm 2$, the following general result holds true.*

$$\int_0^1 \int_0^1 x^{\mu-1} y^{\sigma+\eta+\mu-1} (1-x)^{\sigma+\eta-1} (1-y)^{\sigma-1} (1-xy)^{\nu-2\sigma-\eta} \times {}_2F_1\left[\begin{matrix} \tau, & \rho \\ \frac{1}{2}(\tau+\rho+\iota+1) \end{matrix}; \frac{y(1-x)}{1-xy}\right] dxdy = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \cdot \frac{\Gamma(\sigma)\Gamma(\sigma+\eta)}{\Gamma(2\sigma+\eta)} \Omega_1 \tag{12}$$

provided $Re(\mu) > 0$, $Re(\nu) > 0$, $Re(\sigma) > 0$ for $\eta = 0, 1, 2$; $Re(\sigma + \eta) > 0$ for $\eta = 0, -1, -2$ and $Re(2\sigma - \tau - \rho + 2\eta + \iota + 1) > 0$ for $\iota, \eta = 0, \pm 1, \pm 2$. Also Ω_1 is the same as given in (10).

Theorem 2.2. *For $\iota, \eta = 0, \pm 1, \pm 2$, the following general result holds true.*

$$\int_0^1 \int_0^1 x^{\mu-1} y^{\sigma+\mu-1} (1-x)^{\sigma-1} (1-y)^{\sigma+\eta-1} (1-xy)^{\nu-2\sigma-\eta} \times {}_2F_1\left[\begin{matrix} \tau, & \rho \\ \frac{1}{2}(\tau+\rho+\iota+1) \end{matrix}; \frac{y(1-x)}{1-xy}\right] dxdy = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \cdot \frac{\Gamma(\sigma)\Gamma(\sigma+\eta)}{\Gamma(2\sigma+\eta)} \Omega_1 \tag{13}$$

provided $Re(\mu) > 0$, $Re(\nu) > 0$, $Re(\sigma) > 0$ for $\eta = 0, 1, 2$; $Re(\sigma + \eta) > 0$ for $\eta = 0, -1, -2$ and $Re(2\sigma - \tau - \rho + 2\eta + \iota + 1) > 0$ for $\iota, \eta = 0, \pm 1, \pm 2$. Also Ω_1 is the same as given in (10).

Theorem 2.3. *For $\iota, \eta = 0, \pm 1, \pm 2$, the following general result holds true.*

$$\int_0^1\int_0^1 x^{\mu-1}y^{\sigma+\eta+\mu-1}(1-x)^{\sigma+\eta+1}(1-y)^{\sigma-1}(1-xy)^{\nu-2\sigma-\eta}$$
$$\times {}_2F_1\left[\begin{matrix}\tau, & \rho \\ \frac{1}{2}(\tau+\rho+\iota+1)\end{matrix}; \frac{1-y}{1-xy}\right]dxdy$$
$$=\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\cdot\frac{\Gamma(\sigma)\Gamma(\sigma+\eta)}{\Gamma(2\sigma+\eta)}\,\Omega_1 \tag{14}$$

provided $Re(\mu) > 0$, $Re(\nu) > 0$, $Re(\sigma) > 0$ *for* $\eta = 0, 1, 2$; $Re(\sigma+\eta) > 0$ *for* $\eta = 0, -1, -2$ *and* $Re(2\sigma-\tau-\rho+2\eta+\iota+1) > 0$ *for* $\iota, \eta = 0, \pm1, \pm2$. *Also* Ω_1 *is the same as given in* (10).

Theorem 2.4. *For* $\iota, \eta = 0, \pm1, \pm2$, *the following general result holds true.*

$$\int_0^1\int_0^1 x^{\mu-1}y^{\sigma+\mu-1}(1-x)^{\sigma-1}(1-y)^{\sigma+\eta-1}(1-xy)^{\nu-2\sigma-\eta}$$
$$\times {}_2F_1\left[\begin{matrix}\tau, & \rho \\ \frac{1}{2}(\tau+\rho+\iota+1)\end{matrix}; \frac{1-y}{1-xy}\right]dxdy$$
$$=\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\cdot\frac{\Gamma(\sigma)\Gamma(\sigma+\eta)}{\Gamma(2\sigma+\eta)}\,\Omega_1 \tag{15}$$

provided $Re(\mu) > 0$, $Re(\nu) > 0$, $Re(\sigma) > 0$ *for* $\eta = 0, 1, 2$; $Re(\sigma+\eta) > 0$ *for* $\eta = 0, -1, -2$ *and* $Re(2\sigma-\tau-\rho+2\eta+\iota+1) > 0$ *for* $\iota, \eta = 0, \pm1, \pm2$. *Also* Ω_1 *is the same as given in* (10).

Proof. First, denoting the left-hand side of (12) by I, we have

$$I=\int_0^1\int_0^1 x^{\mu-1}y^{\sigma+\eta+\mu-1}(1-x)^{\sigma+\eta-1}(1-y)^{\sigma-1}(1-xy)^{\nu-2\sigma-\eta}$$
$$\times\, {}_2F_1\left[\begin{matrix}\tau, & \rho \\ \frac{1}{2}(\tau+\rho+\iota+1)\end{matrix}; \frac{y(1-x)}{1-xy}\right]dxdy.$$

Next, expressing ${}_2F_1$ as a series, then we have

$$I = \int_0^1 \int_0^1 x^{\mu-1} y^{\sigma+\eta+\mu-1}(1-x)^{\sigma+\eta-1}(1-y)^{\sigma-1}(1-xy)^{\nu-2\sigma-\eta} \times \sum_{n=0}^{\infty} \frac{(\tau)_n(\rho)_n}{(\frac{1}{2}(\tau+\rho+\iota+1))_n\, n!} \frac{y^n(1-x)^n}{(1-xy)^n} dxdy.$$

Now, changing the order of summation and integration which is justified. Then, we have

$$I = \sum_{n=0}^{\infty} \frac{(\tau)_n(\rho)_n}{(\frac{1}{2}(\tau+\rho+\iota+1))_n n!} \times \int_0^1 \int_0^1 x^{\mu-1} y^{\sigma+\eta+\mu+n-1}(1-x)^{\sigma+\eta+n-1}(1-y)^{\sigma-1}(1-xy)^{\nu-2\sigma-\eta-n} dxdy.$$

Evaluating the resulting integral with help of the known results (1), then we obtain

$$I = \sum_{n=0}^{\infty} \frac{(\tau)_n(\rho)_n}{(\frac{1}{2}(\tau+\rho+\iota+1))_n\, n!} \times \frac{\Gamma(\sigma+n)\Gamma(\sigma+\eta)}{\Gamma(2\sigma+\eta+n)} \cdot \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}.$$

Using the relation

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(n)},$$

we have

$$I = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \cdot \frac{\Gamma(\sigma)\Gamma(\sigma+\eta)}{\Gamma(2\sigma+\eta)} \times \sum_{n=0}^{\infty} \frac{(\tau)_n(\rho)_n(\sigma)_n}{(\frac{1}{2}(\tau+\rho+\iota+1))_n(2\sigma+\eta)_n\, n!}.$$

Summing up the series, we have

$$I = \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \cdot \frac{\Gamma(\sigma)\Gamma(\sigma+\eta)}{\Gamma(2\sigma+\eta)} \times {}_3F_2\left[\begin{matrix}\tau, & \rho, & \sigma \\ \frac{1}{2}(\tau+\rho+\iota+1), & 2\sigma+\eta\end{matrix};1\right].$$

Now for ${}_3F_2$, we use (10) and we get the result (12). By using similar method, the other general results (13) to (15) given in the Theorems 2 to 4 can also be proven.

□

3. Special Cases

Here, we shall mention more than two hundred interesting special cases of our main findings.

(1) In (12), if we let $\rho = -2n$ and put $\tau + 2n$ for τ, or $\rho = -2n-1$ and put $\tau+2n+1$ for τ, where n is zero or a positive integer, then one of the two terms on the right-hand side of (12) will vanish. Thus, we get following fifty special cases given in two corollaries.

Corollary 3.1. *For $\iota, \eta = 0, \pm1, \pm2$, the following general result (containing 25 results) holds true.*

$$\begin{aligned}&\int_0^1\int_0^1 x^{\mu-1}y^{\sigma+\eta+\mu-1}(1-x)^{\sigma+\eta-1}(1-y)^{\sigma-1}(1-xy)^{\nu-2\sigma-\eta}\\ &\qquad\times {}_2F_1\left[\begin{matrix}-2n, & \tau+2n \\ \frac{1}{2}(\tau+\iota+1)\end{matrix};\frac{y(1-x)}{1-xy}\right]dxdy\\ &= D_{\iota,\eta}\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\cdot\frac{\Gamma(\sigma)\Gamma(\sigma+\eta)}{\Gamma(2\sigma+\eta)}\\ &\qquad\times\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\tau-\sigma+\frac{3}{4}-\frac{(-1)^\iota}{4}\left[\frac{1}{2}\eta+\frac{1}{4}(1+(-1)^\iota)\right]\right)_n}{\left(\sigma+\frac{1}{2}+\left[\frac{\eta}{2}\right]\right)_n\left(\frac{1}{2}\tau+\frac{1}{4}(1+(-1)^\iota)\right)_n}\\ &= \Omega_2,\end{aligned} \tag{16}$$

where the coefficients $D_{\iota,\eta}$ are given in Table 4 at the end of this paper.

Corollary 3.2. *For $\iota, \eta = 0, \pm1, \pm2$, the following general result (containing 25 results) holds true.*

$$\int_0^1\int_0^1 x^{\mu-1}y^{\sigma+\eta+\mu-1}(1-x)^{\sigma+\eta-1}(1-y)^{\sigma-1}(1-xy)^{\nu-2\sigma-\eta}$$
$$\times\, {}_2F_1\left[\begin{matrix}-2n-1, & \tau+2n+1\\ \frac{1}{2}(\tau+\iota+1)\end{matrix}; \frac{y(1-x)}{1-xy}\right] dxdy$$
$$= E_{\iota,\eta}\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\cdot\frac{\Gamma(\sigma)\Gamma(\sigma+\eta)}{\Gamma(2\sigma+\eta)}$$
$$\times\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}\tau-\sigma+\frac{5}{4}-\frac{(-1)^{\iota}}{4}-\left[\frac{1}{2}\eta+\frac{1}{4}(1+(-1)^{\iota}\right]\right)_n}{\left(\sigma+\frac{1}{2}+\left[\frac{\eta+1}{2}\right]\right)_n\left(\frac{1}{2}\tau+\frac{1}{4}(3-(-1)^{\iota})\right)_n} \tag{17}$$
$$=\Omega_3,$$

where the coefficients $E_{\iota,\eta}$ are given in Table 5 at the end of this paper.

(2) In (13) to (15), if we apply the same conditions mentioned in special case **(1)**, we get six general result containing 150 results given in the following six corollaries.

Corollary 3.3. *For $\iota,\eta=0,\pm1,\pm2$, the following general result (containing 25 results) holds true.*

$$\int_0^1\int_0^1 x^{\mu-1}y^{\sigma+\mu-1}(1-x)^{\sigma-1}(1-y)^{\sigma+\eta-1}(1-xy)^{\nu-2\sigma-\eta}$$
$$\times\, {}_2F_1\left[\begin{matrix}-2n, & \tau+2n\\ \frac{1}{2}(\tau+\iota+1)\end{matrix}; \frac{1-y}{1-xy}\right] dxdy \tag{18}$$
$$=\Omega_2,$$

where Ω_2 is the same as in (16).

Corollary 3.4. *For $\iota,\eta=0,\pm1,\pm2$, the following general result (containing 25 results) holds true.*

$$\int_0^1\int_0^1 x^{\mu-1}y^{\sigma+\mu-1}(1-x)^{\sigma-1}(1-y)^{\sigma+\eta-1}(1-xy)^{\nu-2\sigma-\eta}$$
$$\times\, {}_2F_1\left[\begin{matrix}-2n-1, & \tau+2n+1\\ \frac{1}{2}(\tau+\iota+1)\end{matrix}; \frac{1-y}{1-xy}\right] dxdy \tag{19}$$
$$=\Omega_3,$$

where Ω_3 *is the same as in* (17).

Corollary 3.5. *For* $\iota, \eta = 0, \pm 1, \pm 2$, *the following general result (containing 25 results) holds true.*

$$\int_0^1 \int_0^1 x^{\mu-1} y^{\sigma+\eta+\mu-1} (1-x)^{\sigma+\eta-1} (1-y)^{\sigma-1} (1-xy)^{\nu-2\sigma-\eta} \times {}_2F_1 \left[\begin{matrix} -2n, \quad a+2n \\ \frac{1}{2}(a+\iota+1) \end{matrix} ; \frac{1-y}{1-xy} \right] dxdy = \Omega_{2,} \tag{20}$$

where Ω_2 *is the same as in* (16).

Corollary 3.6. *For* $\iota, \eta = 0, \pm 1, \pm 2$, *the following general result (containing 25 results) holds true.*

$$\int_0^1 \int_0^1 x^{\mu-1} y^{\sigma+\eta+\mu-1} (1-x)^{\sigma+\eta-1} (1-y)^{\sigma-1} (1-xy)^{\nu-2\sigma-\eta} \times {}_2F_1 \left[\begin{matrix} -2n-1, \quad \tau+2n+1 \\ \frac{1}{2}(\tau+\iota+1) \end{matrix} ; \frac{1-y}{1-xy} \right] dxdy = \Omega_3, \tag{21}$$

where Ω_3 *is the same as in* (17).

Corollary 3.7. *For* $\iota, \eta = 0, \pm 1, \pm 2$, *the following general result (containing 25 results) holds true.*

$$\int_0^1 \int_0^1 x^{\mu-1} y^{\sigma+\mu-1} (1-x)^{\sigma-1} (1-y)^{\sigma+\eta-1} (1-xy)^{\nu-2\sigma-\eta} \times {}_2F_1 \left[\begin{matrix} -2n, \quad \tau+2n \\ \frac{1}{2}(\tau+\iota+1) \end{matrix} ; \frac{1-y}{1-xy} \right] dxdy = \Omega_{2,} \tag{22}$$

where Ω_2 *is the same as in* (16).

Corollary 3.8. *For* $\iota, \eta = 0, \pm 1, \pm 2$, *the following general result (containing 25 results) holds true.*

$$\begin{aligned}&\int_0^1\int_0^1 x^{\mu-1}y^{\sigma+\mu-1}(1-x)^{\sigma-1}(1-y)^{\sigma+\eta-1}(1-xy)^{\nu-2\sigma-\eta}\\&\qquad\times {}_2F_1\left[\begin{matrix}-2n-1, & \tau+2n+1\\ \multicolumn{2}{c}{\frac{1}{2}(\tau+\iota+1)}\end{matrix};\frac{1-y}{1-xy}\right]dxdy\\&=\Omega_3,\end{aligned}\tag{23}$$

where Ω_3 *is the same as in* (17).

(3) In (12), if we set $\iota,\eta=0$, we get

$$\begin{aligned}&\int_0^1\int_0^1 x^{\mu-1}y^{\sigma+\mu-1}(1-x)^{\sigma-1}(1-y)^{\sigma-1}(1-xy)^{\nu-2\sigma}\\&\qquad\times{}_2F_1\left[\begin{matrix}\tau, & \rho\\ \multicolumn{2}{c}{\frac{1}{2}(\tau+\rho+1)}\end{matrix};\frac{y(1-x)}{1-xy}\right]dxdy\\&=\quad\pi 2^{1-2\sigma}\,\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\cdot\frac{\Gamma(\sigma)\Gamma(\frac{1}{2}\tau+\frac{1}{2}\rho+\frac{1}{2})\Gamma(\sigma-\frac{1}{2}\tau-\frac{1}{2}\rho+\frac{1}{2})}{\Gamma(\sigma-\frac{1}{2}\tau+\frac{1}{2})\Gamma(\sigma-\frac{1}{2}\rho+\frac{1}{2})}\end{aligned}\tag{24}$$

provided $\mathrm{Re}(\mu)>0$, $\mathrm{Re}(\nu)>0$, $\mathrm{Re}(\sigma)>0$ and $\mathrm{Re}(2\sigma-\tau-\rho)>-1$.

Let us consider some very interesting special cases of (24).

(a) In (24), if we set $\tau=\rho=\frac{1}{2}$ and use the result [Prudnikov et al. (1990)]

$${}_2F_1\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}\\ \multicolumn{2}{c}{1}\end{matrix};z\right]=\frac{2}{\pi}K(\sqrt{z}),$$

where $K(k)$ is the complete elliptic integral of the first kind defined by

$$K(k)=\int_0^{\frac{\pi}{2}}\frac{dt}{\sqrt{1-k^2\sin^2 t}},$$

we get

$$\int_0^1 \int_0^1 x^{\mu-1} y^{\sigma+\mu-1} (1-x)^{\sigma-1} (1-y)^{\sigma-1} (1-xy)^{\nu-2\sigma} \times K\left(\sqrt{\frac{y(1-x)}{1-xy}}\right) dxdy = \pi^2\, 2^{-2\sigma}\, \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \cdot \frac{\Gamma(\sigma)\Gamma(\sigma)}{\Gamma(\sigma+\frac{1}{4})\Gamma(\sigma+\frac{1}{4})} \tag{25}$$

provided $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\nu) > 0$ and $\mathrm{Re}(\sigma) > 0$.

(b) In (24), if we set $\tau = \rho = 1$ and use the result [Prudnikov et al. (1990)]

$$ {}_2F_1\left[\begin{matrix} 1, & 1 \\ & \frac{3}{2} \end{matrix}; z\right] = \frac{\sin^{-1}(\sqrt{z})}{\sqrt{z(1-z)}},$$

we get

$$\int_0^1 \int_0^1 x^{\mu-1} y^{\sigma+\mu-1} (1-x)^{\sigma-\frac{3}{2}} (1-y)^{\sigma-\frac{3}{2}} (1-xy)^{\nu-2\sigma+1} \times \sin^{-1}\left(\sqrt{\frac{y(1-x)}{1-xy}}\right) dxdy = \pi^{\frac{3}{2}}\, 2^{-2\sigma}\, \frac{\Gamma(\mu)\Gamma(\nu)\Gamma(\sigma-\frac{1}{2})}{\Gamma(\mu+\nu)\Gamma(\sigma)} \tag{26}$$

provided $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\nu) > 0$ and $\mathrm{Re}(\sigma) > \frac{1}{2}$.

(c) In (24), if we take $\rho = -\tau$ and use the result [Prudnikov et al. (1990)].

$$ {}_2F_1\left[\begin{matrix} \tau, & -\tau \\ & \frac{1}{2} \end{matrix}; z\right] = \cos(2\tau \sin^{-1}(\sqrt{z})),$$

we get

$$\int_0^1 \int_0^1 x^{\mu-1} y^{\sigma+\mu-1} (1-x)^{\sigma-1} (1-y)^{\sigma-1} (1-xy)^{\nu-2\sigma} \times \cos\left[2\tau \sin^{-1}\sqrt{\frac{y(1-x)}{1-xy}}\right] dxdy = \pi^{\frac{3}{2}}\, 2^{1-2\sigma}\, \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \cdot \frac{\Gamma(\sigma)\Gamma(\sigma+\frac{1}{2})}{\Gamma(\sigma-\frac{1}{2}\tau+\frac{1}{2})\Gamma(\sigma-\frac{1}{2}\tau+\frac{1}{2})} \tag{27}$$

provided $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\nu) > 0$ and $\mathrm{Re}(\sigma) > 0$.

We conclude this section by giving a remark that similar results can also be obtained from the results (17) to (23). We omit the details.

Remark 1. In (13), if we take $\mu = \nu = 1$, we get a known result earlier obatined by Rathie et al. (1999).

Remark 2. For other integrals of this type and related single integrals, we refer to Rathie et al. (1999)], Jun and Kılıçman (2018) and Kim et al. (2020, 2018).

Table 1: Table for $A_{\iota,\eta}$

$\iota \diagdown \eta$	-2	-1	0	1	2
2	$\frac{1}{2(\sigma-1)(\tau-\rho-1)(\tau-\rho+1)}$	$\frac{1}{2(\tau-\rho-1)(\tau-\rho+1)}$	$\frac{1}{4(\tau-\rho-1)(\tau-\rho+1)}$	$\frac{1}{4(\tau-\rho-1)(\tau-\rho+1)}$	$\frac{1}{8(\sigma+1)(\tau-\rho-1)(\tau-\rho+1)}$
1	$\frac{1}{(\sigma-1)(\tau-\rho)}$	$\frac{1}{(\tau-\rho)}$	$\frac{1}{(\tau-\rho)}$	$\frac{1}{2(\tau-\rho)}$	$\frac{1}{2(\sigma+1)(\tau-\rho)}$
0	$\frac{1}{2(\sigma-1)}$	1	1	1	$\frac{1}{2(\sigma+1)}$
-1	$\frac{1}{(\sigma-1)}$	1	2	2	$\frac{2}{(\sigma+1)}$
-2	$\frac{1}{2(\sigma-1)}$	1	1	2	$\frac{2}{(\sigma+1)}$

Table 2: Table for $B_{\iota,\eta}$

$\iota \diagdown \eta$	-2	-1	0	1	2
2	$\sigma(\tau+\rho-1)-$ $(\tau+1)(\rho+1)+2$	$\tau+\rho-1$	$\tau(2\sigma-\tau)+$ $\rho(2\sigma-\rho)-2\sigma+1$	$2\sigma(\tau+\rho-1)-$ $(\tau-\rho)^2+1$	$B_{2,2}$
1	$\sigma-\rho-1$	1	1	$2\sigma-\tau+\rho$	$2\sigma(\sigma+1)-$ $(\tau-\rho)(\sigma-\rho+1)$
0	$(\sigma-\tau-1)(\sigma-\rho-1)+$ $(\sigma-1)(\sigma-2)$	1	1	1	$(\sigma-\tau+1)(\sigma-\rho+1)+$ $\sigma(\sigma+1)$
-1	$2(\sigma-1)(\sigma-2)-$ $(\tau-\rho)(\sigma-\rho-1)$	$2\sigma-\tau+\rho-2$	1	1	$\sigma-b+1$
-2	$B_{-2,-2}$	$B_{-2,-1}$	$\tau(2\sigma-\tau)+$ $\rho(2\sigma-\rho)-2\sigma+1$	$\tau+\rho-1$	$\sigma(\tau+\rho-1)-$ $(\tau-1)(\rho-1)$

$$B_{-2,-1} = 2(\sigma-1)(\tau+\rho-1)-(\tau-\rho)^2+1$$

$$B_{2,2} = 2\sigma(\sigma+1)[(2\sigma+1)(\tau+\rho-1)-\tau(\tau-1)-\rho(\rho-1)]-(\tau-\rho-1)(\tau-\rho+1)[(\sigma+1)(2\sigma-\tau-\rho+1)+\tau\rho]$$

$$B_{-2,-2} = 2(\sigma-1)(\sigma-2)[(2\sigma-1)(\tau+\rho-1)-\tau(\tau+1)-\rho(\rho+1)+2]-(\tau-\rho-1)(\tau-\rho+1)[(\sigma-1)(2\sigma-\tau-\rho-3)+\tau\rho]$$

Table 3: Table for $C_{\iota,\eta}$

$\iota \diagdown \eta$	-2	-1	0	1	2
2	-4	$-(4\sigma-\tau-\rho-3)$	-8	$-[8\sigma^2-2\sigma(\tau+\rho-1)-(\tau-\rho)^2+1]$	$-4(2\sigma+\tau-\rho+1)(2\sigma-\tau+\rho+1)$
1	$-(\sigma-\tau-1)$	-1	-1	$-(2\sigma+\tau-\rho)$	$-[2\sigma(\sigma+1)+(\tau-\rho)(\sigma-\tau+1)]$
0	4	1	0	-1	-4
-1	$2(\sigma-1)(\sigma-2)+(\tau-\rho)(\sigma-\tau-1)$	$2\sigma+\tau-\rho-2$	1	1	$\sigma-\tau+1$
-2	$4(2\sigma-\tau+\rho-3)(2\sigma+\tau-\rho-3)$	$C_{-2,-1}$	8	$4\sigma-\tau-\rho+1$	4

$$C_{-2,-1} = 8\sigma^2 - 2(\sigma-1)(\tau+\rho+7) - (\tau-\rho)^2 - 7$$

Table 4: Table for $D_{\iota,\eta}$

$\iota \diagdown \eta$	-2	-1	0	1
2	$\frac{(\tau+1)[(\sigma-1)(\tau-1)+2n(\tau+2n)]}{(\sigma-1)(\tau+4n-1)(\tau+4n+1)}$	$\frac{(\tau+1)(\tau-1)}{(\tau+4n+1)(\tau+4n-1)}$	$\frac{(\tau+1)[(\tau-1)(2\sigma-\tau-1)-4n(\tau+2n)]}{(2\sigma-\tau-1)(\tau+4n+1)(\tau+4n-1)}$	$\frac{(\tau+1)[(\tau-1)(2\sigma-\tau-1)-8n(\tau+2n)]}{(2\sigma-\tau-1)(\tau+4n+1)(\tau+4n-1)}$
1	$\frac{\tau(\sigma+2n-1)}{(\sigma-1)(\tau+4n)}$	$\frac{\tau}{\tau+4n}$	$\frac{\tau}{\tau+4n}$	$\frac{\tau(2\sigma-\tau-4n)}{(2\sigma-\tau)(\tau+4n)}$
0	$1-\frac{2n(\tau+2n)}{(\sigma-1)(2\sigma-\tau-3)}$	1	1	1
-1	$1-\frac{2n(2\sigma+\tau+4n-2)}{(\sigma-1)(2\sigma-\tau-4)}$	$1-\frac{4n}{(2\sigma-\tau-2)}$	1	1
-2	$D_{-2,-2}$	$1-\frac{8n(\tau+2n)}{(\tau-1)(2\sigma-\tau-3)}$	$1-\frac{4n(\tau+2n)}{(\tau-1)(2\sigma-\tau-1)}$	1

$\eta \diagdown \iota$	2	1	0	-1	-2
2	$D_{2,2}$	$\frac{\tau[(\sigma+1)(2\sigma-\tau)-2n(2\sigma+\tau+4n+2)]}{(\sigma+1)(2\sigma-\tau)(\tau+4n)}$	$1-\frac{2n(\tau+2n)}{(\sigma+1)(2\sigma-\tau+1)}$	$1+\frac{2n}{(\sigma+1)}$	$1+\frac{2n(\tau+2n)}{(\sigma+1)(\tau-1)}$

$$D_{2,2} = \frac{(\tau+1)[(\tau-1)(\sigma+1)(2\sigma-\tau+1)(2\sigma-\tau-1) - 2\tau n(6\sigma+\tau+5)(2\sigma-\tau+1) + 4n^2(5\tau^2+4\tau-5-4\sigma(3\sigma-\tau+4)) + 64n^3(\tau+n)]}{(\sigma+1)(2\sigma-\tau+1)(2\sigma-\tau-1)(\tau+4n+1)(\tau+4n-1)}$$

$$D_{-2,-2} = 1 - \frac{2\tau n(6\sigma+\tau-7)(2\sigma-\tau-3) - 4n^2[5\tau^2-4\tau-21-4\sigma(3\sigma-\tau-8)] - 64n^3(\tau+n)}{(\sigma-1)(\tau-1)(2\sigma-\tau-3)(2\sigma-\tau-5)}$$

Table 5: Table for $E_{\iota,\eta}$

$\iota \diagdown \eta$	-2	-1	0	1	2
2	$\frac{(\tau+1)(2\sigma-\tau-3)}{(\sigma-1)(\tau+4n+1)(\tau+4n+3)}$	$\frac{(\tau+1)(4\sigma-\tau-3)}{(\tau+4n+1)(\tau+4n+3)(2\sigma-1)}$	$\frac{2(\tau+1)}{(\tau+4n+1)(\tau+4n+3)}$	$E_{2,1}$	$\frac{(\tau+1)(2\sigma+\tau+4n+3)(2\sigma-\tau-4n-1)}{(\sigma+1)(2\sigma-\tau-1)(\tau+4n+1)(\tau+4n+3)}$
1	$\frac{(\sigma-\tau-2n-2)}{(\sigma-1)(\tau+4n+2)}$	$\frac{2\sigma-\tau-2}{(\tau+4n+2)(2\sigma-1)}$	$\frac{1}{\tau+4n+2}$	$\frac{(2\sigma+\tau+4n+2)}{(2\sigma+1)(\tau+4n+2)}$	$\frac{(\sigma+\tau+2)(2\sigma-\tau)-2n(3\tau-2\sigma+4n+2)}{(\sigma+1)(2\sigma-\tau)(\tau+4n+2)}$
0	$\frac{-1}{(\sigma-1)}$	$\frac{-1}{(2\sigma-1)}$	0	$\frac{1}{(2\sigma+1)}$	$\frac{1}{(\sigma+1)}$
-1	$E_{-1,-2}$	$\frac{-(2\sigma+\tau+4n)}{\tau(2\sigma-1)}$	$\frac{-1}{\tau}$	$\frac{-(2\sigma-\tau)}{\tau(2\sigma+1)}$	$\frac{-(\sigma-\tau-2n)}{\tau(\sigma+1)}$
-2	$\frac{-(2\sigma+\tau+4n-1)(2\sigma-\tau-4n-5)}{(\tau-1)(\sigma-1)(2\sigma-\tau-5)}$	$E_{-2,-1}$	$\frac{-2}{(\tau-1)}$	$\frac{-(4\sigma-\tau+1)}{(\tau-1)(2\sigma+1)}$	$\frac{-(2\sigma-\tau+1)}{(\tau-1)(\sigma+1)}$

$$E_{2,1} = \frac{(\tau+1)[(4\sigma+\tau+3)(2\sigma-\tau-1) - 8n(\tau+2n+2)]}{(\tau+4n+1)(\tau+4n+3)(2\sigma+1)(2\sigma-\tau-1)} \ ; \ E_{-2,-1} = -\ \frac{[(4\sigma+\tau-1)(2\sigma-\tau-3) - 8n(\tau+2n+2)]}{(\tau-1)(2\sigma-1)(2\sigma-\tau-3)}$$

$$E_{-1,-2} = -\ \frac{[(\sigma+\tau)(2\sigma-\tau-4) - 2n(3\tau-2\sigma+4n+6)]}{\tau(\sigma-1)(2\sigma-\tau-4)}$$

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