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Solving Neutral Delay Differential Equation of Pantograph Type

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ABSTRACT

Neutral Delay Differential Equation (NDDE) of pantograph type has been solved by developing a fifth order explicit multistep block method. NDDE has become of great interest among researchers in its industrial applications. In finding the solution for the problem, a two-point explicit multistep block method has been modelled by applying Taylor Series interpolation polynomial. The proposed method will solve pantograph NDDE at two points concurrently with the strategy of consistent step-size. The implementation is based on multistep method Adam Bashforth formula in predictor mode. In handling pantograph delay, Lagrange interpolation polynomial needs to be applied to find the solution of delay terms that are larger than the initial value given. The delay derivatives are estimated using divided difference formula. The order and convergence have been determined to ensure the reliability of the proposed explicit block method. The stability analysis has been constructed using test equation for NDDE. Numerical results obtained have shown that the suggested method is suitable and applicable for solving pantograph equation.

Keywords: Explicit Multistep Block Method, Initial-value problem, Neutral Delay Differential Equation, Pantograph delay.

1. Introduction

Recently, Neutral Delay Differential Equation has been increasingly applied as the appropriate model in time delay problem especially in industry related to engineering. Pantograph delay is one of the time delay system except that it is in a proportional condition and the name is taken after the pantograph on train which is an apparatus mounted on the roof of an electric train. Time delays frequently emerge in feedback loops involving sensors and actuators. They constantly present in a structural testing technique named as real-time dynamic substructuring. A feedback loop happens when an output of a system are directed back or become the input in the next cycle that forms a circuit or loop. These transmission systems are often occured in communication technologies where the system includes a command device which the control command is detected by a sensor and sends the informations to an actuator that executes them by converting the signal's energy into mechanical motion. The problem that has been faced by most engineers is when the time for the signal to be received by controller has been delayed. This time delay problem should already be taken into consideration at the early design stage as it is one of the factors that influence the dynamic. The most particular remote-control system associated with feedback loop is the one provided by human operator where the control command will be issued to the control action and guided the system properly. This operation can be found in remote control of cars, remote operation of large construction cranes, trains, trams and cockpit control of an airplane's engines and control surfaces. With the speedy evolution of communication technologies, the transmission system of estimated signals to a remote control center is becoming simpler and has given more apportunities for researchers to suggest more solution for the problem.

2. Development of method

A series of numerical solutions have been introduced by Jackiewicz in the early 1980's. Jackiewicz (1982), Jackiewicz (1984), Jackiewicz (1986) and Jackiewicz (1987) have proposed both one-step and multistep methods based on predictor-corrector scheme for the numerical approach of NDDE. In 2010, Chen and Wang (2010) have applied a variational iteration method (VIM) which is the analytical solution for NDDE of pantograph delay. Continued by Biazar and Ghanbari (2012), where they have presented the use of homotopy pertubation method (HPM) to solve NDDE with proportional delay (pantograph equation). Both method have been compared with previous two-stage order-one Runge-Kutta method by Wang et al. (2009) and one-leg θ-method which has been discussed by Wang and Li (2007) previously. The VIM has then being modified by Ghaneai et al. (2012) to prove the effectiveness of the method for solving proportional delay of NDDE. In order to control the convergence region, they have introduced a new parameter for the approximate solution. The modified VIM has provide a simple way in adjusting the convergence region. In Lv and

Gao (2013), the applicability of reproducing kernel Hilbert space method (RKHSM) to solve NDDE with proportional delay has been proven where the performance of RKHSM has been compared with the two-stage order-one Runge-Kutta method Wang et al. (2009), one-leg θ-method Wang and Li (2007) and VIM Chen and Wang (2010). Later, an implicit two-point one-block method for the numerical solution of NDDE with pantograph type has been developed by Ishak et al. (2013) and the stability properties of the method has been analysed by Ishak et al. (2014). Numerical results obtained have achieved the desired accuracy. Ishak and Ramli (2015) have extended the method in Ishak et al. (2013) into an implicit three-point one-block method using variable step-size technique. The method's performance has been compared with Ishak et al. (2013) and has shown to be accurate. After that, Seong and Majid (2015) have implemented the use predictor-corrector method into a new fully implicit two-step block method of order four. The method is suitable to solve first order NDDE with pantograph type problem. An analytical exact solution of NDDE is then being introduced by Ahmad and Fatima (2016) known as Differential Transform method (DTM). DTM has shown to be efficient in restricting the convergence region and producing estimate solution with only a few hand computation. Finally in 2017, Sakar (2017) has improved a homotopy analysis method (HAM) for solving NDDE problem. High-accuracy approximate solutions have been obtained after being compared with previous analytical methods. The new idea of this research is to develop an explicit block multistep method for the solution of NDDE since none of the researchers have applied an explicit method in solving pantograph equation. Thus, an explicit two-point multistep block method will be formulated and considered in this research as an approach for solving NDDE with pantograph type.

3. Methodology

In this section, the analyses of fifth order two-point explicit multistep block method (2PEBM5) for solving pantograph NDDE are discussed. The formulation, order, convergence and stability properties of 2PEBM5 are being explained thoroughly. A first order linear NDDE with pantograph type is given as shown below:

$$\begin{aligned} y'(x) &= f(x, y(x), y(qx), y'(qx)) \\ y(x) &= \phi(x). \end{aligned} \quad [1]$$

$0 < q < 1$ is the restricted ratio for proportional delay while $y(x) = \phi(x)$ is its initial function. qx is the delays terms while $y(qx)$ and $y'(qx)$ are the expressions of delay solutions. The development of 2PEBM5 has been adapted from Majid and Suleiman (2011) in Adam-Bashforth predictor mode. Lagrange interpolation polynomial and forward divided difference will be applied to estimate the delay solutions, $y(qx)$ and $y'(qx)$ respectively. 2PEBM5 will approximate the solution for NDDE problem at two points concurrently.

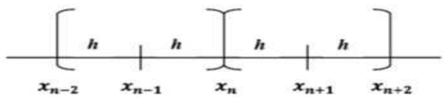

Figure 1: Two-point multistep block method

From Figure 1, the interval from x_{n-2} until x_n is the first block while the second block contains interval from x_n to x_{n+2}. The evaluated solutions of first block will be applied as the initial values for the second block and the procedure will keep repeating for the next iteration in another block.

3.1 Formulation of method

Based on Lambert (1973), a linear multistep method is given by:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}. \qquad [2]$$

A linear difference operator associated with [2]:

$$L[y(x):h] = \sum_{j=0}^{k} \Big[\alpha_j y(x+jh) - h\beta_j y'(x+jh)\Big]. \qquad [3]$$

In order to evaluate $y(x_{n+1})$ and $y(x_{n+2})$, [3] will be expanded by using Taylor series interpolation polynomial. Collecting all terms will produce:

$$L[y(x):h] = C_0 y(x) + C_1 y^{(1)}(x) + ... + C_p h^p y^{(p)}(x). \qquad [4]$$

Following [2] to [4], the derivation of 2PEBM5 becomes:

$$\begin{aligned} y_{n+k} + \alpha_0 y_{n+(k-1)} &= h \sum_{i=0}^{k+3} \beta_i y' \Big[x + (i-(k+3))h\Big] \\ y_{n+(k+1)} + \alpha_0 y_{n+(k-1)} &= h \sum_{i=1}^{k+4} \beta_i y' \Big[x + (i-(k+4))h\Big] \end{aligned} \qquad [5]$$

where the value of k is 1. After letting $\alpha_0 = -1$ and expanding individual terms of $y(x)$ and $y'(x)$ in [5] by Taylor series will produce:

$$\begin{aligned} &\Big[y(x) + 5hy'(x) + \frac{25}{2}h^2y''(x) + \frac{125}{6}h^3y'''(x) + \frac{625}{24}h^4y^{iv}(x) + \frac{625}{24}h^5y^{v}(x)\Big] = \\ &\Big[y(x) + 4hy'(x) + 8h^2y''(x) + \frac{32}{3}h^3y'''(x) + \frac{32}{3}h^4y^{iv}(x) + \frac{128}{15}h^5y^{v}(x)\Big] + h\beta_0 \\ &\Big[y'(x)\Big] + h\beta_1\Big[y'(x) + hy''(x) + \frac{1}{2}h^2y'''(x) + \frac{1}{6}h^3y^{iv}(x) + \frac{1}{24}h^4y^{v}(x)\Big] + h\beta_2 \\ &\Big[y'(x) + 2hy''(x) + 2h^2y'''(x) + \frac{4}{3}h^3y^{iv}(x) + \frac{2}{3}h^4y^{v}(x)\Big] + h\beta_3 \\ &\Big[y'(x) + 3hy''(x) + \frac{9}{2}h^2y'''(x) + \frac{9}{2}h^3y^{iv}(x) + \frac{27}{8}h^4y^{v}(x)\Big] + h\beta_4 \\ &\Big[y'(x) + 4hy''(x) + 8h^2y'''(x) + \frac{32}{3}h^3y^{iv}(x) + \frac{32}{3}h^4y^{v}(x)\Big] \end{aligned} \qquad [6]$$

and

$$\begin{aligned}
&\Big[y(x_{n+1}) + 5hy'(x_{n+1}) + \frac{25}{2}h^2y''(x_{n+1}) + \frac{125}{6}h^3y'''(x_{n+1}) + \frac{625}{24}h^4y^{iv}(x_{n+1})\\
&+ \frac{625}{24}h^5y^{v}(x_{n+1})\Big] = \Big[y(x_{n+1}) + 3hy'(x_{n+1}) + \frac{9}{2}h^2y''(x_{n+1}) + \frac{9}{2}h^3y'''(x_{n+1})\\
&+ \frac{27}{8}h^4y^{iv}(x_{n+1}) + \frac{81}{40}h^5y^{v}(x_{n+1})\Big] + h\beta_1\Big[y'(x_{n+1})\Big] + h\beta_2\Big[y'(x_{n+1})\\
&+ hy''(x_{n+1}) + \frac{1}{2}h^2y'''(x_{n+1}) + \frac{1}{6}h^3y^{iv}(x_{n+1}) + \frac{1}{24}h^4y^{v}(x_{n+1})\Big]\\
&+ h\beta_3\Big[y'(x_{n+1}) + 2hy''(x_{n+1}) + 2h^2y'''(x_{n+1}) + \frac{4}{3}h^3y^{iv}(x_{n+1})\\
&+ \frac{2}{3}h^4y^{v}(x_{n+1})\Big] + h\beta_4\Big[y'(x_{n+1}) + 3hy''(x_{n+1}) + \frac{9}{2}h^2y'''(x_{n+1})\\
&+ \frac{9}{2}h^3y^{iv}(x_{n+1}) + \frac{27}{8}h^4y^{v}(x_{n+1})\Big] + h\beta_5\Big[y'(x_{n+1}) + 4hy''(x_{n+1})\\
&+ 8h^2y'''(x_{n+1}) + \frac{32}{3}h^3y^{iv}(x_{n+1}) + \frac{32}{3}h^4y^{v}(x_{n+1})\Big].
\end{aligned} \qquad [7]$$

Collecting all terms in [6] and [7] yields to the following block:

$$\begin{aligned}
y_{n+1} &= y_n + \frac{h}{720}\Big[1901f_n - 2774f_{n-1} + 2616f_{n-2} - 1274f_{n-3} + 251f_{n-4}\Big]\\
y_{n+2} &= y_n + \frac{h}{90}\Big[269f_{n+1} - 266f_n + 294f_{n-1} - 146f_{n-2} + 29f_{n-3}\Big].
\end{aligned} \qquad [8]$$

The formulation produced above is a two-point explicit multistep block method (2PEBM5) in predictor Adam-Bashforth formula. 2PEBM5 will be applied to solve NDDE with pantograph type.

3.2 Order and error constant

According to Lambert (1973), the order and error constant of 2PEBM can be obtained by applying:

$$C_p = \sum_{j=0}^{k}\Big[\frac{j^p\alpha_j}{p!} - \frac{j^{p-1}\beta_j}{(p-1)!}\Big] \qquad [9]$$

where p is the order of 2PEBM as long as $C_0 = C_1 = \cdots = C_p = 0$ and if $C_{p+1} \neq 0$, then it is called as an error constant. Applying [9]:

$$C_0 = \sum_{j=0}^{k} \alpha_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad C_3 = \sum_{j=0}^{k} (\frac{j^3 \alpha_j}{3!} - \frac{j^2 \beta_j}{2!}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_1 = \sum_{j=0}^{k} (j\alpha_j - \beta_j) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad C_4 = \sum_{j=0}^{k} (\frac{j^4 \alpha_j}{4!} - \frac{j^3 \beta_j}{3!}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$C_2 = \sum_{j=0}^{k} (\frac{j^2 \alpha_j}{2!} - j\beta_j) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad C_5 = \sum_{j=0}^{k} (\frac{j^5 \alpha_j}{5!} - \frac{j^4 \beta_j}{4!}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus, 2PEBM is of order 5 with an error constant of:

$$C_6 = \sum_{j=0}^{k} (\frac{j^6 \alpha_j}{6!} - \frac{j^5 \beta_j}{5!}) = \begin{bmatrix} \frac{95}{288} \\ \frac{14}{45} \end{bmatrix}.$$

3.3 Order of convergence

The aproximate solution for 2PEBM5 has been denoted in equation [8] while the exact solution is:

$$\begin{aligned} Y_{n+1} =& y_n + \frac{h}{720}\Big[1901 f_n - 2774 f_{n-1} + 2616 f_{n-2} - 1274 f_{n-3} + 251 f_{n-4}\Big] \\ &+ \frac{95}{288} h^6 Y^{(6)}(\xi_n) + R_6, \\ Y_{n+2} =& y_n + \frac{h}{90}\Big[269 f_{n+1} - 266 f_n + 294 f_{n-1} - 146 f_{n-2} + 29 f_{n-3}\Big] \\ &+ \frac{14}{45} h^6 Y^{(6)}(\xi_n) + R_6. \end{aligned} \qquad [10]$$

The difference between both exact and approximate solutions will be measured and $Y_{n+1} - y_{n+1}$ will be let as d_{n+1} while $Y_{n+2} - y_{n+2}$ and $Y_n - y_n$ as d_{n+2} and d_n respectively. After assuming the existence of boundary B for $|Y^6(\xi_n)|$:

$$\begin{aligned} |d_{n+1}| \leq& \Big(1 + \frac{1901}{720} hL\Big)|d_n| - \frac{2774}{720} hL|d_{n-1}| + \frac{2616}{720} hL|d_{n-2}| - \frac{1274}{720} hL|d_{n-3}| \\ &+ \frac{251}{720} hL|d_{n-4}| + \frac{95}{288} h^6 B + O(h^7), \\ |d_{n+2}| \leq& \frac{269}{90} hL|d_{n+1}| + \Big(1 - \frac{266}{90} hL\Big)|d_n| + \frac{294}{90} hL|d_{n-1}| - \frac{146}{90} hL|d_{n-2}| \\ &+ \frac{29}{90} hL|d_{n-3}| + \frac{14}{45} h^6 B + O(h^7). \end{aligned} \qquad [11]$$

where R_6 is the remaider term,

$$R_{p+1} = C_{p+2} h^{p+2} Y^{p+2}(\xi) = O(h^7). \qquad [12]$$

Here $p + 2 \geq 1$ is called the order of convergence. Thus, from [11], the order of convergence is seven.

3.4 Consistency and zero-stability

Formula stated in [2] is converged when it is consistent and zero-stable. The conditions for 2PEBM5 to be consistent are, 2PEBM5 needs to have order $p \geq 1$ and follows two characteristics shown below:

$$\sum_{j=0}^{k} \alpha_j = 0 \quad and \quad \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j. \qquad [13]$$

2PEBM5 is proved to have order $5 = p \geq 1$. Rewritten [8]:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} + h \begin{bmatrix} 0 & \frac{251}{720} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-5} \\ f_{n-4} \end{bmatrix} + h$$
$$\begin{bmatrix} -\frac{1274}{720} & \frac{2616}{720} \\ \frac{29}{90} & -\frac{146}{90} \end{bmatrix} \begin{bmatrix} f_{n-3} \\ f_{n-2} \end{bmatrix} + h \begin{bmatrix} -\frac{2774}{720} & \frac{1901}{720} \\ \frac{294}{90} & -\frac{266}{90} \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} + h \qquad [14]$$
$$\begin{bmatrix} 0 & 0 \\ \frac{269}{90} & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix}$$

where [14] is equivalent to:

$$A_3 Y_{N+3} = A_2 Y_{N+2} + h \sum_{j=0}^{3} B_j F_{N+j}. \qquad [15]$$

Conditions in [13] are satisfied when:

$$\sum_{j=0}^{k} \alpha_j = \sum_{j=0}^{7} \alpha_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

followed by:

$$\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{7} j\alpha_j = \sum_{j=0}^{k} \beta_j = \sum_{j=0}^{5} \beta_j = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then, 2PEBM5 is zero-stable as there is no root of:

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j = 0 \qquad [16]$$

has modulus greater than one:

$$\rho(\xi) = \sum_{0}^{7} \alpha_j \xi^j = \begin{bmatrix} \xi^5(-1+\xi) \\ \xi^5(-1+\xi^2) \end{bmatrix}.$$

Hence, 2PEBM5 is proved to be converged since it satisfied both properties of consistent and zero-stable.

3.5 Stability of method

Linear test equation for NDDE is given by:

$$\sum_{j=0}^{3} A_j Y_{N+j} = h \sum_{j=0}^{3} B_j \Big(aY_{N+j} + bY_{N+j-m} + cY'_{N+j-m} \Big). \qquad [17]$$

Letting $H_1 = ha$ and $H_2 = hb$, [17] will become:

$$\begin{aligned} &(A_0 - H_1B_0)Y_N + (A_1 - H_1B_1)Y_{N+1} + (A_2 - H_1B_2)Y_{N+2} + (A_3 - H_1B_3) \\ &Y_{N+3} - (H_2B_0 + cB_0)Y_{N-m} - (H_2B_1 + cB_1)Y_{N+1-m} - (H_2B_2 + cB_2) \\ &Y_{N+2-m} - (H_2B_3 + cB_3)Y_{N+3-m} = 0. \end{aligned} \qquad [18]$$

Choosing $m = 1$, then the characteristic polynomial of NP-stability 2PEBM5:

$$\begin{aligned} \pi(t) = det|&(A_0 - B_0H_1)t^m + (A_1 - B_1H_1)t^{1+m} + (A_2 - B_2H_1)t^{2+m} \\ &+ (A_3 - B_3H_1)t^{3+m} - (B_0H_2 + cB_0)t^0 - (B_1H_2 + cB_1)t^1 \\ &- (B_2H_2 + cB_2)t^2 - (B_3H_2 + cB_3)t^3| \\ &= 0. \end{aligned} \qquad [19]$$

The NP-stability region for 2PEBM5 shown in Figure 2 lie inside the close region.

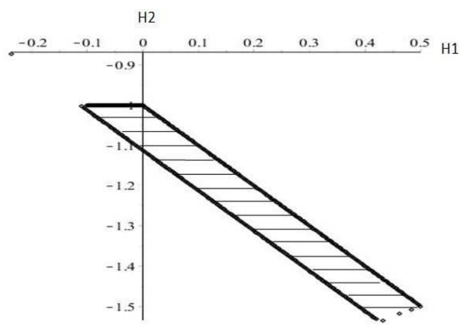

Figure 2: NP-stability for 2PEBM5

As stated by Aziz (2015),

Definition 3.1. *For the step-size h, if a, b and c are complex, the region R_{NP} in the $(H_1, H_2)-$plane is called the NP-stability region if for any $(H_1, H_2) \in R_{NP}$.*

NP-stability term originally introduced by Bellen et al. (1988) for neutral delay differential equation with analogous stability properties with ordinary differential equation. Since the set of all roots in stability polynomial obtained for the method are $|t| < 1$, therefore, 2PEBM5 is said to be absolute stable.

4. Implementation of method

2PEBM5 has been proposed to solved NDDE with pantograph type in [1]. As known by many researchers, multistep method need to have previous values for it to be applied in solving differential equations. Thus, before applying 2PEBM5, five initial values need to be approximated first. In this research, Runge-Kutta order 4 (RK4) has been chosen to evaluate the initial solutions for NDDE. As the delay terms and its derivative also need to be considered, a Lagrange interpolation polynomial and difference formula have been used in estimating both of the delay solutions, $y(qx)$ and $y'(qx)$ respectively. On the first few iterations, backward divided difference shown below is applied to solve the delay derivative as the estimated values are not enough to complete the calculations:

$$y'(qx) = \frac{y(qx) - (y(qx) - h)}{h}. \qquad [20]$$

As the number of iterations are complete to be taken as the delay derivative's calculation, a forward divided difference is the being applied:

$$y'(qx) = \frac{(y(qx) + h) - y(qx)}{h}. \qquad [21]$$

For delay terms without the presence of its derivative, the application of Lagrange interpolation polynomial will be used if none of the delay solutions have been calculated in previous iteration. The formula of Lagrange formula is as follows:

$$\begin{aligned} P(x) &= L_{n,0}(x)f(x_0) + \ldots + L_{n,n}(x)f(x_n) \\ &= \sum_{k=0}^{n} f(x_k)L_{n,k}(x) \end{aligned} \qquad [22]$$

where,

$$L_{n,k}(x) = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}$$

$$k = 0, 1, \ldots, n.$$

The forward and backward divided difference formulas given above are of order 2. Any order of divided difference formula is suitable to be applied in differential equations supposedly. However, it is hard to handle a pantograph equation as the delay in proportional condition is sensitive to be evaluated using higher order formula. Thus, a new initiative in using lower order formula is introduced in this research. The algorithm shown below have been bulit in C programme with constant step-size technique.

Algorithm

Step 1: Input the given initial values of $a = x_0$, $b = x_n$, h and $N = \frac{b-a}{h}$.
Step 2: RK4 is applied for approximating the starting values.
Step 3: For $i = 1, 3, \ldots$
The approximate value y_{n+i} is evaluated by using:
$$y_{n+i} = y_n + \frac{h}{720}\Big[1901 f_{n+i-1} - 2774 f_{n+i-2} + 2616 f_{n+i-3} - 1274 f_{n+i-4} + 251 f_{n+i-5}\Big].$$
Step 4: For $i = 2, 4, \ldots$
The approximate value y_{n+i} is evaluated by using:
$$y_{n+i} = y_n + \frac{h}{90}\Big[269 f_{n+i-1} - 266 f_{n+i-2} + 294 f_{n+i-3} - 146_{n+i-4} + 29 f_{n+i-5}\Big].$$
Step 5: If none the delay terms have been calculated in previous iteration, then $y(qx)$ need to be solved by using Lagrange interpolation polynomial:
$$P(x) = L_{n,0}(x) f(x_0) + \ldots + L_{n,n}(x) f(x_n) = \sum_{k=0}^{n} f(x_k) L_{n,k}(x).$$
Step 6: $y'(x - \tau_i)$ is obtained by applying backward difference formula for the first few iteration and forward difference formula as the number of delays are adequate to be taken as approximate value for the next iteration.
Step 7: The maximum and average errors, total steps, function evaluations and time taken are estimated.
Step 8: End.

5. Numerical results and discussions

Three examples of pantograph delay with exact solutions have been solved in this section by using 2PEBM5. The accuracy and competency of 2PEBM5 has also been proved. Example 1 and 3 have been taken from Ishak et al. (2014) while Example 2 is taken from Seong and Majid (2015). 2PEBM5 is being compared with a two-point one-block method of order 5 (2P1B5) and a two-point multistep block method of order 4 (2PBM4). Both methods have been compared in terms of total steps, average and maximum errors that have been computed. The notations below are used in Table 1 - Table 3.

h	Step size
MTD	Method
FCN	Total function calls
TS	Total Step
TIME	Time Taken
AVERE	Average Error
MAXE	Maximum Error
2PEBM5	Two-point Explicit Multistep Block Method (Order 5)
2P1B5	Two-point One-block Method from Ishak et al. (2014) (Order 5)
2PBM4	Two-point Multistep Block Method from Seong and Majid (2015) (Order 4)
5e-7	5×10^{-7}

Example 5.1. *Ishak et al. (2014)*

$$y'(x) = -y(x) + \frac{1}{2}y\Big(\frac{x}{2}\Big) + \frac{1}{2}y'\Big(\frac{x}{2}\Big), \qquad x \in [0,1]$$

$$y(0) = 1$$

Exact solution: $y(x) = e^{-x}$.

Table 1: Numerical results for Example 5.1.

MTD	h	FCN	TS	MAXE	AVERE	TIME(s)
2PEBM5	0.1	8	7	2.9778e-04	2.1363e-04	0.172
	0.01	53	52	2.4802e-07	3.9767e-07	0.188
	0.001	503	502	2.4552e-10	4.1900e-10	0.203
ABM5	0.1	11	10	1.9747e-04	2.0830e-04	0.255
	0.01	101	100	1.8527e-07	3.0378e-07	0.265
	0.001	1001	1000	1.8407e-10	3.1482e-10	0.298
RK4	0.1	41	10	1.9899e-02	4.3734e-02	0.267
	0.01	401	100	3.2283e-03	7.3136e-03	0.274
	0.001	4001	1000	1.0010e-03	1.0785e-03	0.341

MTD	TOL	TS	AVERE	MAXE
2P1B5	10^{-2}	20	7.3136e-04	1.8351e-03
	10^{-4}	27	1.8858e-05	2.9229e-05
	10^{-6}	71	9.0282e-06	2.1266e-05
	10^{-8}	166	1.1494e-06	2.1357e-06
	10^{-10}	236	4.5605e-08	5.2514e-08

Example 5.2. *Seong and Majid (2015)*

$$y'(x) = \sin(x)y(x) + cos\left(\frac{x}{2}\right)y\left(\frac{x}{2}\right) - sin\left(\frac{x}{2}\right)y'\left(\frac{x}{2}\right) + \cos(x) - \sin^2(x), \qquad x \in [0,1]$$

$$y(0) = 0$$

Exact solution: $y(x) = \sin(x)$.

Table 2: Numerical results for Example 5.2.

MTD	h	FCN	TS	MAXE	AVERE	TIME(s)
2PEBM5	0.1	8	7	1.3642e-03	4.5641e-03	0.182
	0.01	53	52	1.1020e-04	4.4479e-0	0.193
	0.001	503	502	1.0871e-05	4.4689e-05	0.208
ABM5	0.1	11	10	3.8358e-03	1.4755e-03	0.260
	0.01	101	100	4.4968e-04	1.1619e-04	0.266
	0.001	1001	1000	4.4753e-05	1.0938e-05	0.297
RK4	0.1	41	10	3.4118e-02	1.8734e-02	0.266
	0.01	401	100	2.7846e-03	1.3602e-03	0.304
	0.001	4001	1000	2.7147e-04	1.3067e-04	0.325

MTD	h	FCN	AVERE	MAXE
2PBM4	0.1	16	6.9355e-04	7.6850e-04
	0.01	148	1.3081e-05	1.5352e-05
	0.001	1498	1.7775e-07	2.2373e-07

Example 5.3. *Ishak et al. (2014)*

$$y'(x) = -y(x) + \frac{1}{10}y\left(\frac{4}{5}x\right) + \frac{1}{2}y'\left(\frac{4}{5}x\right) + \left(\frac{8}{25}x - \frac{1}{2}\right)e^{-\frac{4}{5}x} + e^{-x}, \qquad x \in [0,1]$$

$$y(0) = 0$$

Exact solution: $y(x) = xe^{-x}$.

Table 3: Numerical results for Example 5.3.

MTD	h	FCN	TS	MAXE	AVERE	TIME(s)
2PEBM5	0.1	8	7	2.8366e-04	4.0025e-04	0.187
	0.01	53	52	5.9000e-07	3.7166e-07	0.198
	0.001	503	502	6.2781e-10	3.6825e-10	0.213
ABM5	0.1	11	10	2.7316e-04	2.6386e-04	0.268
	0.01	101	100	2.7581e-07	4.4952e-07	0.286
	0.001	1001	1000	2.7590e-10	4.7159e-10	0.306
RK4	0.1	41	10	1.1889e-02	4.0247e-02	0.285
	0.01	401	100	1.9778e-02	2.5258e-02	0.292
	0.001	4001	1000	2.1540e-02	2.3759e-02	0.337

MTD	TOL	TS	AVERE	MAXE
2P1B5	10^{-2}	70	1.2275e-02	4.5486e-02
	10^{-4}	97	3.9287e-04	1.1403e-03
	10^{-6}	118	1.6162e-06	4.8664e-06
	10^{-8}	173	2.7266e-07	4.8730e-07
	10^{-10}	300	1.7216e-08	3.9765e-08

Numerical results for Example 1 - 3 have been tabulated in Table 1 - Table 3 respectively. From the obtained results in Example 1, 2PEBM5 has produced average and maximum errors at 10^{-7} even when the number of steps are 52. As for 2P1B5 in variable step size strategy, the average and maximum errors are 10^{-6} at a total step of 166 which is doubled from 2PEBM5. The total time estimated for 2PEBM5 is also lesser than ABM5 and RK4 since the proposed method is a block method which produced less computational effort. In Example 2, the function evaluation for 2PEBM5 is lesser than 2PBM4 even though 2PEBM5 is of order five and 2PBM4 is of order 4. A method of lower order supposedly has lower function evaluation than the higher order method. But, the theory does not applied for 2PEBM5 which has shown to outperform fourth order method in terms of function calls. The same condition occurred in Example 3 where both maximum and average errors are 10^{-4} even at total step 7 while for 2P1B5, the average error is 10^{-4} and maximum error 10^{-3} at total step 97. The maximum and average error for 2PEBM5 have also comparable to ABM5 eventhough 2PEBM5 is an explicit method. The advantages of applying 2PEBM5 are, it has reduced the total step taken, function evaluation called and the computational time consumed. Explicit method has also provided simpler calculation than an implicit method as it is an independent method which does not depend on other values and only a single formula is needed to complete an iteration. Besides, many authors have neglected the uses and advantages of explicit method, thus the application of 2PEBM5 in this research is an approach to prove that explicit method is performing well with other method. Implicit method has been said theoretically and proved numerically to have better and accurate results compared to explicit method, but in NDDE case, explicit method seems to cope well with the delay terms. The delay is known to be slow while explicit method is said to be less accurate than any implicit method. Thus, they are suitable to be pair up as they already share the characteristic. In terms of accuracy, 2PEBM5 has produced comparable results compared to the implicit 2P1B5 and 2PBM4. The total step and function evaluation between those methods have also shown a big difference. These analyses and parameters have highlighted the benefits of implementing explicit block method for the numerical solutions of NDDE with pantograph type.

6. Conclusion

In this article, 2PEBM5 has been implemented in solving first order linear NDDE with pantograph type by producing two approximate solutions in a single step with constant step-size technique. Numerical results obtained have shown that 2PEBM5 is suitable and applicable to be implemented as it gives better results than ABM5, RK4, 2P1B5 and 2PBM4. Additionally, 2PEBM5 has reduced the step taken, function called and time consumed as it produces faster computational calculation.

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