

Chromatic Equivalence Class of the Join of Certain Tripartite Graphs

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ABSTRACT

For a simple graph G , let $P(G;\lambda)$ be the chromatic polynomial of G . Two graphs G and H are said to be chromatically equivalent, denoted $G \sim H$ if $P(G;\lambda) = P(H;\lambda)$. A graph G is said to be chromatically unique, if $H \sim G$ implies that $H \cong G$. Chia [4] determined the chromatic equivalence class of the graph consisting of the join of p copies of the path each of length 3. In this paper, we determined the chromatic equivalence class of the graph consisting of the join of p copies of the complete tripartite graph $K_{1,2,3}$.
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INTRODUCTION

All graphs considered in this paper are finite, undirected, simple and loopless. For a graph G , we denote by $P(G;\lambda)$ (or $P(G)$), the chromatic polynomial of G . Two graphs G and H are said to be *chromatically equivalent*, or *χ -equivalent*, denoted $G \sim H$ if $P(G) = P(H)$. It is clear that the relation " \sim " is an equivalence relation on the family of graphs. We denote by $[G]$ the equivalence class determined by G under " \sim ". A graph G is said to be *chromatically unique*, or *χ -unique*, if $[G] = \{G\}$, i.e., $H \sim G$ implies that $H \cong G$. Many families of *χ -unique* graphs are known (see [8, 9]), relatively fewer results concerning the chromatic equivalence class of graphs are known (see [2, 3, 4]). In this paper, our main purpose is to determine the chromatic equivalence class of the graph consisting of the join of p copies of the complete tripartite graph $K_{1,2,3}$.

In what follows, we let K_n denote the complete graph on n vertices, K_{p_1, p_2, \dots, p_t} the complete t -partite graph having n_i vertices in the i -th partite set, P_n and C_n the path and cycle on n vertices, respectively and $\chi(G)$ the chromatic number of G . Let W_n denote the wheel of order n and U_n the graph obtained from W_n by deleting a spoke of W_n . Also let $n(A, G)$ denote the number of subgraph A in G and $i(A, G)$ the number of induced subgraph A in G .

The join of two graphs G and H , denoted $G + H$, is the graph obtained from the union of G and H by joining every vertex of G to every vertex of H .

Let F be a graph and let $G = F + F + \dots + F$ or pF denote the join of p (≥ 2) copies of F . We wish to determine $[G]$. Let $J_p(F)$ denote the set of all graphs H which are of the form $H = H_1 + H_2 + \dots + H_p$, where $H_i \in [F]$, $i = 1, 2, \dots, p$.

In [4], Chia posed the following problem

Problem: *What are those graphs F for which $J_p(F) = [G]$?*

and solve the problem for the case $F = P_q$. In this paper, by making very minor modification to the technique used in [4], we solve the above problem for the case $F = K_{1,2,3}$.

PRELIMINARY RESULTS AND NOTATIONS

A spanning subgraph is called a *clique cover* if its connected components are complete graphs. Let G be a graph on n vertices. Let $s_k(G)$ denote the number of clique cover of G with k connected components, $k = 1, 2, \dots, n$. If the chromatic polynomial of G is

$P(G, \lambda) = \sum_{k=1}^n s_k(\overline{G})(\lambda)_k$ where $(\lambda)_k = \lambda(\lambda - 1) \cdots (\lambda - k + 1)$, then the polynomial

$\sigma(G, k) = \sum_{k=1}^n s_k(\overline{G})x^k$ is called the σ -polynomial of G (see Brenti(1992)). It is easy

to see that $\sigma(G, x) = x^n$ if and only if $G = K_n$ since $s_k(G) = 0$ for $k < \chi(G) = n$. Also note that $s_n(G) = 1$ and $s_{n-1}(G) = m$ if G has m edges. Clearly, $P(G, \lambda) = P(H, \lambda)$ if and only if $\sigma(G, x) = \sigma(H, x)$ and $s_k(G) = s_k(H)$ for $k = 1, 2, \dots$.

If $\sigma(G, x) = xf(x)$ for some irreducible polynomial $f(x)$ over the rational number field, then $\sigma(G, x)$ is said to be irreducible.

Lemma 2.1. (Farrell (1980)) Let G and H be two graphs such that $G \sim H$. Then G and H have the same number of vertices, edges and triangles. If both G and H has no K_4 as subgraph, then $i(C_p, G) = i(C_p, H)$. Moreover,

$$\begin{aligned} & -i(C_5, G) + i(K_{2,3}, G) + 2i(U_5, G) + 3i(W_5, G) \\ & = -i(C_5, H) + i(K_{2,3}, H) + 2i(U_5, H) + 3i(W_5, H). \end{aligned}$$

Lemma 2.2. (Brenti (1992)) Let G and H be two disjoint graphs. Then

$$\sigma(G + H, x) = \sigma(G, x)\sigma(H, x).$$

In particular,

$$\sigma(K_{n_1, n_2, \dots, n_t}, x) = \prod_{i=1}^t \sigma(O_{n_i}, x).$$

Lemma 2.3. (Liu (1992)) Let G be a connected graph with n vertices and m edges. Assume that G is not the complete graph K_n . Then

$$s_{n-2}(G) \leq \binom{m-1}{2}$$

and equality holds if and only if G is the path P_{m+1} .

A CHROMATIC EQUIVALENCE CLASS

We first have the following lemma which follows readily from Lemma 2.1.

Lemma 3.1. $[K_{1,2,3}] = \{K_{1,2,3}, K_{2,2,2} - e\}$ where e is an edge of $K_{2,2,2}$.

We now have our main theorem as follow.

Theorem 3.1. Let $G = K_{1,2,3} + K_{1,2,3} + \dots + K_{1,2,3}$ be the join of p copies of $K_{1,2,3}$. Then $[G] = J_p(K_{1,2,3})$.

Proof. Let $H \sim G$, we will show that $H \in J_p(K_{1,2,3})$. Since $P(G) = P(H)$ implies that $\sigma(G) = \sigma(H)$, it is more convenient to look at $\sigma(G)$ and $\sigma(H)$. First note that $\sigma(K_{1,3}) = x(x^3 + 3x^2 + x) = \sigma(K_{2,2} - e)$ with $[K_{1,3}] = \{K_{1,3}, K_{2,2} - e\}$, and $\sigma(K_{1,2,3}) = x(x^2 + x)(x^3 + 3x^2 + x) = P(K_{2,2,2} - e)$. So, $\sigma(G) = [x(x^2 + x)(x^3 + 3x^2 + x)]^p = [(x^2 + x)(x^4 + 3x^3 + x^2)]^p$, having p irreducible factors of x , $x^2 + x$ and $x^3 + 3x^2 + x$ respectively.

Let n and m denote the number of vertices and edges in H respectively. Then $n = 6p$ and $m = 36 \binom{p}{2} + 11p = 18p^2 - 7p$ so that $\sigma(H) = \sigma(G) = \sum_{i=1}^{6p} s_i(\overline{G})x^i$. Moreover, H is uniquely $3p$ -colorable as G is so.

Let V_1, V_2, \dots, V_{3p} be the color classes of the unique $3p$ -coloring of H . Let V_{ij} denote the subgraph induced by $V_1 \cup V_j, i \neq j$. Call V_{ij} a 2-color subgraph of H .

Case (i): Every V_i has exactly two vertices.

In this case, V_{ij} is either a path P_4 or else a cycle C_4 because, by Theorem 12.16 of [6], V_{ij} is connected for $i \neq j$. Note that the number of 2-color subgraphs in H is $\binom{3p}{2} = \frac{1}{2}(9p^2 - 5p) + p$. By looking at the number of edges in H , we see that exactly p

of the 2-color subgraphs V_{ij} are P_4 and the rest of the 2-color subgraphs are C_4 . This means that \overline{H} has only P_4 and K_2 as subgraph so that $H = s\overline{P_4} + r\overline{K_2}$ ($s, r \geq 0$). Consequently,

$$\sigma(H) = [(x^4 + 3x^3 + x^2)^s (x^2 + x)^r] = [x(x^2 + x)(x^3 + 3x^2 + x)]^p = \sigma(G).$$

Obviously, $s, r \geq 1$ so that $\sigma(H) = (x^4 + 3x^3 + x^2)(x^2 + x)\sigma(H_1)$ and that by Lemma 3.1, $H = (K_{2,2,2} - e) + H_1$ for some graph H_1 . Since $\sigma(H_1) = [x(x^2 + x)(x^3 + 3x^2 + x)]^{p-1}$, by induction on p , we have $H_1 \in J_{p-1}(K_{1,2,3})$. This implies that $H \in J_p(K_{1,2,3})$.

Case (ii): Not every V_i has exactly two vertices.

Then there is a j such that $|V_j| = 1$. Without loss of generality, let $|V_j| = i$ for $j = 1, \dots, r, r \geq 1$. Then $H = K_r + H_*$ for some graph H_* . Let F_1, F_2, \dots, F_t be the connected components of $\overline{H_*}$. Then $H = K_r + \overline{F_1} + \dots + \overline{F_t}$ with $H_* = \overline{F_1} + \dots + \overline{F_t}$.

If for some $i, F_i = K_3$, then \overline{H} contains a subgraph $K_1 \cup K_3$. This means that $H = K_{1,3} + H'$ for some graph H' and so

$$\sigma(H) = (x^4 + 3x^3 + x^2)\sigma(H') = [x(x^2 + x)(x^3 + 3x^2 + x)]^p = \sigma(G).$$

Clearly, $\sigma(H')$ must contain a factor $(x^2 + x)$ so that $\sigma(H) = (x^4 + 3x^3 + x^2)\sigma(H'')\sigma(H_1)$ (where $\sigma(H'') = x^2 + x$) for some graph H_1 . Obviously, $\overline{H''} = K_2$. Hence, $H = K_{1,2,3} + H_1$ with $\sigma(H_1) = [x(x^2 + x)(x^3 + 3x^2 + x)]^{p-1}$. Again, by induction on p , we have $H \in J_p(K_{1,2,3})$.

If for some $i, F_i = K_2$, then $H = K_2 + H'$. By the similar argument as above, $\sigma(H')$ must contain a factor $(x^3 + 3x^2 + x)$ so that $H = K_{1,2,3} + H_1$ or $(K_{2,2,2} - e) + H_1$ with $\sigma(H_1) = [x(x^2 + x)(x^3 + 3x^2 + x)]^{p-1}$. Again, by induction on p , we have $H \in J_p(K_{1,2,3})$.

If for some $i, F_i = P_4 (= K_{2,2} - e)$, then $H = P_4 + H'$. By the similar argument as above, $\sigma(H')$ must contain a factor $(x^2 + x)$ so that $H = (K_{2,2,2} - e) + H_1$ with $\sigma(H_1) = [x(x^2 + x)(x^3 + 3x^2 + x)]^{p-1}$. Again, by induction on p , we have $H \in J_p(K_{1,2,3})$.

So, assume that F_i is not K_2, K_3 or P_4 for any $i = 1, \dots, t$. Let n_i and m_i denote the number of vertices and edges in F_i respectively. Then $\sum_{i=1}^t m_i = 4p$, the number of edges in \overline{H} .

If $n_i \leq 3$, then $F_i = P_3$. However, this is impossible because $\sigma(G)$ does not contain

$(x^3 + 2x^2)$ as a factor. Hence, $n_i \geq 4$. This implies that $6p = |V(G)| = r + \sum_{i=1}^t n_i \geq r + 4t$ so that $t < 3p/2$ because $r \geq 1$.

Since $H = K_r + H_*$, we have $\sigma(H) = x^r \cdot \sigma(H_*)$. It follows that $s_{n-2}(\overline{H}) = s_{n_*-2}(\overline{H_*})$ where n_* is the number of vertices in H_* . Note that

$$\sigma(H_*) = \sum_{j=1}^{n_*} s_j(\overline{H_*})x^j = \prod_{i=1}^t \sigma(\overline{F_i})$$

where

$$\sigma(\overline{F_i}) = \sum_{k=1}^{n_i} s_k(F_i)x^k = x^{n_i} + m_i x^{n_i-1} + s_{n_i-2}(F_i)x^{n_i-2} + \dots,$$

$i = 1, \dots, t$.

By multiplying all the terms in $\prod_{i=1}^t \sigma(\overline{F_i})$ and by equating the coefficient of x^{n_*-2} , we have by Lemma 2.3,

$$\begin{aligned} s_{n_*-2}(\overline{H_*}) &= \sum_{1 \leq i \leq j \leq t} m_i m_j + \sum_{i=1}^t s_{n_i-2}(F_i) \\ &\leq \sum_{1 \leq i \leq j} m_i m_j + \sum_{i=1}^t \binom{m_i - 1}{2}. \end{aligned}$$

Consequently,
$$\begin{aligned} s_{n_*-2}(\overline{H_*}) &\leq \frac{\sum_{1 \leq i \leq j \leq t} 2m_i m_j + \sum_{i=1}^t (m_i^2 - 3m_i + 2)}{2} \\ &= \frac{\left(\sum_{i=1}^t m_i\right)^2 - 3\sum_{i=1}^t m_i + 2t}{2} \\ &= \frac{16p^2 - 12p + 2t}{2} \\ &< \frac{16p^2 - 9p}{2} \end{aligned}$$

because $t < 3p/2$. However, this is a contradiction because $s_{n-2}(\overline{H}) = s_{6p-2}(\overline{G}) =$

$$4p + 16 \binom{p}{2} = (16p^2 - 8p) / 2 > s_{n_*-2}(\overline{H_*}).$$

This completes the proof.

Remark: Note that for even p , our main result is a special case of Theorem 5.1 in (Ho, (2004)).

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