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Some Topics on Weighted Generalized Inverse and Kronecker Product of Matrices

Zeyad Abdel Aziz Al Zhour & Adem Kilicman
Department of Mathematics
Universiti Malaysia Terengganu, 21030 Kuala Terengganu, Terengganu.
E-mail: zeyad1968@yahoo.com , akilicman@umt.edu.my

ABSTRACT

In this paper, we present two general representations for the weighted generalized inverse $A_{d,W}$, which extends earlier results on the Drazin inverse A_d, group inverse A_g and usual inverse A^{-1}. The first one concerns with the matrix expression involving Moore-Penrose inverse A^+. The second one holds on the Kronecker products of two and several matrices. Furthermore, some necessary and sufficient conditions for Drazin and weighted Drazin inverses are given for the reverse order law $(AB)_d = B_d A_d$ and $(AB)_{d,Z} = B_{d,R} A_{d,W}$ to hold. Finally, we apply our result to present the solution of restricted singular matrix equations.

Keywords: Kronecker Product, Weighted Drazin Inverses, General algebraic structures, Index. Nilpotent matrix.
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INTRODUCTION AND PRELIMINARY RESULT

One of the important types of generalized inverses of matrices is the weighted Drazin inverse, which has several important applications such as, applications in singular differential, difference equations, Markov chains, statistical problems, control system analysis, curve fitting, numerical analysis and Kronecker product systems [e.g., 4, 7, 9, 13, 15, 16, 17]. Here we use the following notations. Let $M_{m,n}$ be the set of all matrices over the complex number field $\mathbb{C}$ and when $m = n$, we write M_n instead of $M_{n,n}$. For matrix $A \in M_{m,n}$, let A^* be the conjugate transpose of A and rank(A) be the rank of A. If $A \in M_m$ is a given matrix, then the smallest non-negative integer k such that

$$rank(A^{k+1}) = rank(A^k) \tag{1}$$

is called the *index of A* and is denoted by $Ind(A)=k$.

It is well known that the Drazin inverse *(DI)* of $A \in M_m$ with $Ind(A)=k$ is defined to be the unique solution $X \in M_m$ satisfying the following three matrix equations:

$$A^k XA = A^k, \quad XAX = X, \quad AX = XA \tag{2}$$

and is often denoted by $X = A_d$. Note that the first equation in (2) can be written as $A^{k+1}X = A^k$. In particular, when $Ind(A)=1$, the Drazin inverse of $A \in M_m$ is called the *group inverse* of A, and is often denoted by A_g, but when $Ind(A)=0$ and $A \in M_m$ is a non-singular matrix, then $A_d = A^{-1}$.

Wang [13] gave that for $A \in M_m$ with $Ind(A) = k$,

$$A^k (A_d)^k A^k = A^k, \ (A_d)^k A^k (A_d)^k = (A_d)^k, \ A^k (A_d)^k = (A_d)^k A^k \tag{3}$$

By the uniqueness of the DI, we have

$$(A^k)_d = (A_d)^k \tag{4}$$

For more properties concerning Drazin inverses, see [e.g., 3, 4, 10, 14].

Cline and Greville [5] extended the Drazin inverse of square matrix to rectangular matrix and called it as the weighted Drazin inverse *(WDI)*. The *WDI* of $A \in M_{m,n}$ with respect to the matrix $W \in M_{n,m}$ is defined to be the unique solution $X \in M_{m,n}$ of the following three matrix equations:

$$(AW)^{k+1}XW = (AW)^k, \quad XWAWX = X, \quad AWX = XWA \tag{5}$$

where

$$k = \max \{Ind(AW), Ind(WA)\} \tag{6}$$

and is often denoted by $X = A_{d,W}$. In particular, when $A \in M_m$ and $W = I_m$, then $A_{d,w}$ reduce to A_d, i.e., $A_d = A_{d,Im}$. If $A \in M_m$ is non-singular square matrix and $W = I_m$, it is easily seen that $Ind(A) = 0$ and $A_{d,w} = A_d = A^{-1}$ satisfies the matrix equations (5).

The properties of *WDI* can be found in [e.g., 8, 18, 19]. Some notable properties are: If $A \in M_{m,n}$ with respect to the matrix $W \in M_{m,n}$ and $k = \max\{Ind(AW), Ind(WA)\}$, then:

i. $$A_{d,W} = A\{(WA)_d\}^2 = \{(AW)_d\}^2 A \tag{7}$$

ii $A_{d,w}W = (AW)_d$, $WA_{d,W} = (WA)_d$ (8)

iii $WAWA_{d,W} = WA(WA)_d$, $A_{d,W}WAW = (WA)_d AW$ (9)

iv One closed-form solution of $A_{d,w}$ for a rectangular matrix $A \in M_{m,n}$

$$A_{d,W} = \begin{cases} \lim\limits_{\alpha \to 0} & (AW)^{l+2} + \alpha^2 I)^{-1} (AW)^l A \\ \lim\limits_{\alpha \to 0} & A(WA)^l ((WA)^{l+2} + \alpha^2 I)^{-1} \end{cases} \begin{matrix} if & l \geq k \\ if & l \geq k \end{matrix} \tag{10}$$

The *Moore-Penrose inverse* (MPI) is a generalization of the inverse of non-singular matrix to the inverse of a singular and rectangular matrix. The MPI of a matrix $A \in M_{m,n}$ is defined to be the unique solution $X \in M_{n,m}$ of the following four Penrose equations:

$$AXA = A,\ XAX = A,\ (AX)^* = XA,\ (XA)^* = XA \tag{11}$$

and is often denoted by $X = A^+$.

Note that if $A \in M_m$ is non-singular matrix, then $A+ = A^{-1}$. Regarding various basic properties concerning MPI, see [e.g., 2, 3, 4, 10].

The *general algebraic structures* (GAS) of the matrices $A \in M_{m,n}$ $W \in M_{n,m}$ A^+, W^+, and $A_{d,W} \in M_{n,m}$ with $k = max\{Ind(AW), Ind(WA)\}$ are (see [e.g., 4,19,20,21]):

$$A = L\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} Q^{-1},\quad W = Q\begin{bmatrix} W_{11} & 0 \\ 0 & W_{22} \end{bmatrix} L^{-1},\quad A^+ = Q\begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} L^{-1}, \tag{12}$$

$$W^+ = L\begin{bmatrix} W_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1},\quad A_{d,W} = L\begin{bmatrix} (W_{11}A_{11}W_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \tag{13}$$

where L, Q, A_{11}, W_{11} are non-singular matrices, and A_{22}, W_{22}, $A_{22}W_{22}$, $W_{22}A_{22}$ are nilpotent matrices (A matrix $A \in M_n$ is called nilpotent if $A^k = 0$ for some positive integer k).

In particular, when $A \in M_n$ with $Ind(A) = k$, $W = I_m$ and $L = Q$, then we have

$$A = L\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} L^{-1}, \quad A_d = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} L^{-1} \tag{14}$$

where L and A_{11} are non-singular matrices, and A_{22} is a nilpotent matrix.

Greville [6] first studied the reverse order law for the Drazin inverse of the product of two matrices A and $B \in M_n$. He proved that $(AB)_d = B_d A_d$ holds under the condition $AB = BA$.

Tian [12] gave a necessary and sufficient condition for the reverse order law $(AB)_d = B_d A_d$ by using a rank identity. A similar result for reverse order law for Drazin inverse of general multiple matrix product was presented by Wang [14] as follows: Let A and $B \in M_n$ be given with k=max $\{Ind(A), Ind(B), Inx(AB)\}$. Then

$$(AB)_d = B_d A_d \tag{15}$$

if and only if

$$rank\begin{bmatrix} (-1)^n (AB)^{2k+1} & 0 & 0 & (AB)^k \\ 0 & 0 & A^{2k+1} & A^k \\ 0 & B^{2k+1} & B^k A^k & 0 \\ (AB)^k & B^k & 0 & 0 \end{bmatrix} = rank(A^k) + rank(B^k) + rank((AB)^k) \tag{16}$$

Finally, the Kronecker product of $A = \left[a_{ij}\right] \in M_{m,n}$ and $B = \left[b_{kl}\right] \in M_{p,q}$ is given by

$$A \otimes B = \left[a_{ij} B\right]_{ij} \in M_{mp,nq} \tag{17}$$

where $a_{ij} B \in M_{p,q}$ is the ij - th block .

For any compatible matrices A, B, C and D; and any real number r , we shall make frequent use of the following properties of the Kronecker product (see [e.g.,1,2,7,11,22]):

i. If AC and BD are well defined, then.

$$(A\otimes B)(C\otimes D) = AC\otimes BD \tag{18}$$

ii. If A and B are square positive (semi) definite matrices, then

$$(A\otimes B)^r = A^r \otimes B^r \tag{19}$$

iii. $rank\ (A\otimes B) = rank\ (A) rank(B)$ (20)

iv. $Vec\ (AXB^T) = (B\otimes A)\ Vec\ (X)$ (21)

where

$$Vec(X) = \left[x_{11}\ldots x_{m1}\ x_{12}\ldots x_{m2}\ldots x_{1m}\ldots x_{mn}\right]^T \tag{22}$$

denotes vectorization by columns of arbitrary matrix $X \in M_{m,n}$

v. If A and B are nilpotent matrices,then $A\otimes B$ is nilpotent matrix

vi. If A and B are unitary matrices, then $A\otimes B$ is unitary matrix.

In this paper, some new matrix expressions involving the three kinds of generalized inverses of the Kronecker products matrices are established. In addition, by using the general algebraic structures of matrices (*GAS*), the necessary and sufficient conditions for Drazin and weighted Drazin inverses are also given for the reverse order laws $(AB)_d = B_d A_d$ and $(AB)_{d,Z} = B_{d,R} A_{d,W}$ to hold. Finally, we apply our result to present the solution of restricted singular matrix equations $(WAW)\ X\ (RBR)^T = C$.

MAIN RESULT

Observe that, in general, if A and $B \in M_n$ are nilpotent matrices, then AB need not be nilpotent. As an example, let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

It is easy to verify that A and B are nilpotent matrices, but $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not nilpotent, because $(AB)^k = AB \neq 0$ for all positive integer k . This observation is important to give a nilpotent condition when we use the GAS of matrices under usual product as follows:

Theorem 1 : Let

$$A = L\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} L^{-1},\quad B = L\begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} L^{-1},\quad W = L\begin{bmatrix} W_{11} & 0 \\ 0 & W_{22} \end{bmatrix} L^{-1}, \tag{23}$$

$$R = L\begin{bmatrix} R_{11} & 0 \\ 0 & R_{22} \end{bmatrix} L^{-1} \quad , Z = L\begin{bmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{bmatrix} L^{-1} \tag{24}$$

be the general algebraic structures, respectively, of *A, B, W, R* and $Z \in M_n$ with $k = \max\{Ind(AW), Ind(WA), Ind(BR), Ind(RB), Ind(ABZ), Ind(ZAB)\}$. Then

$$(AB)_{d,Z} = B_{d,R}\, A_{d,w} \tag{25}$$

if and only if $A_{22}B_{22}$ is a nilpotent matrix and

$$(R_{11}B_{11}R_{11})^{-1}\left(W_{11}A_{11}W_{11}\right)^{-1} = \left(Z_{11}A_{11}B_{11}Z_{11}\right)^{-1} \tag{26}$$

Or equivalently

$$R_{11}^{-1}B_{11}^{-1}R_{11}^{-1}W_{11}^{-1}A_{11}^{-1}W_{11}^{-1} = Z_{11}^{-1}B_{11}^{-1}A_{11}^{-1}Z_{11}^{-1} \tag{27}$$

Proof : The *GAS* of *A,B,W, R* and *Z* in the assumptions assure that A_{11}, B_{11}, W_{11}, R_{11} Z_{11} and *L* are non-singular matrices, and A_{22}, B_{22}, W_{22}, R_{22} and Z_{22} are nilpotent. Then it is well known that the *GAS* of $A_{d,W}$ and $B_{d,R}$ are given by

$$A_{d,W} = L\begin{bmatrix} \left(W_{11}A_{11}W_{11}\right)^{-1} & 0 \\ 0 & 0 \end{bmatrix} L^{-1},\quad B_{d,R} = L\begin{bmatrix} \left(R_{11}B_{11}R_{11}\right)^{-1} & 0 \\ 0 & 0 \end{bmatrix} L^{-1} \tag{28}$$

Computation shows that

$$B_{d,R}A_{d,W} = L\begin{bmatrix} \left(R_{11}B_{11}R_{11}\right)^{-1}\left(W_{11}A_{11}W_{11}\right)^{-1} & 0 \\ 0 & 0 \end{bmatrix} L^{-1} \tag{29}$$

and

$$(AB)_{d,Z} = \left(L \begin{bmatrix} A_{11}B_{11} & 0 \\ 0 & A_{22}B_{22} \end{bmatrix} L^{-1} \right)_{d,Z} = L \begin{bmatrix} (Z_{11}A_{11}B_{11}Z_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix} L^{-1} \tag{30}$$

if and only if $A_{22}B_{22}$ is a nilpotent matrix. It is clear from (29) and (30) that

$$(AB)_{d,Z} = B_{d,R}A_{d,w}$$

if and only if $A_{22}B_{22}$ is a nilpotent matrix and

$$\left(R_{11}B_{11}R_{11}\right)^{-1}\left(W_{11}A_{11}W_{11}\right)^{-1} = \left(Z_{11}A_{11}B_{11}Z_{11}\right)^{-1}$$

This completes the proof of Theorem 1. ■

If we set $W = R = Z = I_n$ in Theorem 1, we obtain the sufficient and necessary condition for the reverse order of Drazin inverse as follows.

Corollary 1 Let

$$A = L\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} L^{-1}, B = L\begin{bmatrix} B_{11} & 0 \\ 0 & B_{22} \end{bmatrix} L^{-1}$$

be the *GAS* of A and $B \in M_n$, respectively, with k = max *{Ind(A), Ind(B), Ind(AB)}*. Then

$$(AB)_d = B_d A_d$$

if and only if $A_{22}B_{22}$ is a nilpotent matrix.

Now, we can also apply the GAS in order to find a new representation of *WDI* as follows:

Theorem 2: Let $A \in M_{m,n}$ and $W \in M_{n,m}$ such that $A_{22}W_{22}$ and $W_{22}A_{22}$ are nilpotent matrices of index k in *GAS* form. Then the *WDI* of A with respect to the matrix W can be written as matrix expression involving *MPI* by

$$A_{d,W} = \left\{ (AW)^k \left[(AW)^{2k+1} \right]^+ (AW)^k \right\} W^+ \tag{31}$$

where k= max *{ind(AW), Ind(WA)}*.

Proof: Due to the *GAS* of A, A^+W, W^+ and $A_{d,W}$ there exists non-singular matrices L, A_{11} and W_{11} , and nilpotent matrices A_{22} and W_{22} such that

$$A = L\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} Q^{-1},\ W = Q\begin{bmatrix} W_{11} & 0 \\ 0 & W_{22} \end{bmatrix} L^{-1},\ W^+ = Q\begin{bmatrix} W_0^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1},$$

Since $A_{22}W_{22}$ and $W_{22}A_{22}$ are nilpotent matrices of index k, then $(A_{22}W_{22})^k=0$, and it is easy to show that

$$(AW)^k = L\begin{bmatrix}(A_{11}W_{11})^k & 0\\ 0 & 0\end{bmatrix}L^{-1},\quad \left[(AW)^{2k+1}\right]^+ = L\begin{bmatrix}(A_{11}W_{11})^{-2k-1} & 0\\ 0 & 0\end{bmatrix}$$

Computation shows that

$$\begin{aligned}
&\left\{(AW)^k\left[(AW)^{2k+1}\right]^+(AW)^k\right\}W^+\\
&= L\begin{bmatrix}(A_{11}W_{11})^k & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}(A_{11}W_{11})^{-2k-1} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}(A_{11}W_{11})^k & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}W_{11}^{-1} & 0\\ 0 & 0\end{bmatrix}Q^{-1}\\
&= L\begin{bmatrix}(A_{11}W_{11})^k(A_{11}W_{11})^{-2k-1}(A_{11}W_{11})^kW_{11}^{-1} & 0\\ 0 & 0\end{bmatrix}Q^{-1}\\
&= L\begin{bmatrix}(A_{11}W_{11})^{-1}W_{11}^{-1} & 0\\ 0 & 0\end{bmatrix}Q^{-1}\\
&= L\begin{bmatrix}(W_{11}A_{11}W_{11})^{-1} & 0\\ 0 & 0\end{bmatrix}Q^{-1}\\
&= A_{d,w}
\end{aligned}$$

This completes the proof of Theorem 2. ■

If A is a square matrix with $Ind(A) = k$ and set $W = I_n$ in Theorem 2, we obtain the following corollary which is given by Wang [14]:

Corollary 2 Let $A\in M_n$ with $Ind(A)=k$, then

$$A_d = A^k\left(A^{2k+1}\right)^+A^k \tag{32}$$

Theorem 3 Let $A\in M_{m,n,}$ $W\in M_{n,m,}$ $B\in M_{p,q}$ and $R\in M_{q,p}$ be matrices with

$$k_1=\max\left\{Ind(AW), Ind(WA)\right\}, k_2=\max\left\{Ind(BR), Ind(RB)\right\}$$

Also, let $Z=W\otimes R$ and k = max $\{k_1,k_2\}$. Then

i. $Ind\left\{(A\otimes B)Z\right\}=k$ (33)

ii. $(A\otimes B)_{d,z}=A_{d,w}\otimes B_{d,R}$ (34)

Proof: i. By assumptions, we have

$$rank(AW)^{k_1}=rank(AW)^{k_1+1}, rank(BR)^{k_2}=rank(BR)^{k_2+1}$$

From properties of Kronecker products, we have

$$rank\left\{(A\otimes B)Z\right\}^r = rank\left\{(A\otimes B)(W\otimes R)\right\}^r = rank\left\{AW\otimes BR\right\}^r$$

$$= rank\left\{AW\right\}^r \; rank\left\{BR\right\}^r .$$

Similarly,

$$rank\left\{(A\otimes B)Z\right\}^{r+1} = rank\left\{AW\right\}^{r+1} \; rank\left\{BR\right\}^{r+1} .$$

It is obvious that the smallest non-negative integer such that

$$rank\left\{(A\otimes B)Z\right\}^{r+1} = rank\left\{(A\otimes B)Z\right\}^{r} .$$

is $k = \max\{k_1, k_2\}$. Hence (33) is true.

ii. Let $X=A_{d,w}\otimes B_{d,R}$ and $Z=W\otimes R$. From properties of the Kronecker product and (5) we have

$$\left((A\otimes B)Z\right)^{k+1}XZ=\left((A\otimes B)(W\otimes R)\right)^{k+1}\left(A_{d,w}\otimes B_{d,R}\right)(W\otimes R)$$

$$=\left((AW)^{k+1}A_{d,w}W\right)\otimes\left((BR)^{k+1}B_{d,R}R\right)$$

$$=(AW)^k\otimes(BR)^k=(AW\otimes BR)^k=\left((A\otimes B)(W\otimes R)\right)^k$$

$$=\left\{(A\otimes B)Z\right\}^k \tag{35}$$

$$XZ(A\otimes B)ZX=\left(A_{d,w}\otimes B_{d,R}\right)(W\otimes R)(A\otimes B)(W\otimes R)\left(A_{d,w}\otimes B_{d,R}\right)$$

$$=\left(A_{d,w}WAWA_{d,w}\right)\otimes\left(B_{d,R}RBRB_{d,R}\right)=A_{d,W}\otimes B_{d,R} \tag{36}$$

$$= X.$$

$$(A\otimes B)ZX = (A\otimes B)(W\otimes R)\left(A_{d,w}\otimes B_{d,R}\right)$$

$$= AWA_{d,w}\otimes BRB_{d,R}=A_{d,w}WA\otimes B_{d,R}RB$$

$$= \left(A_{d,W}\otimes B_{d,R}\right)(W\otimes R)(A\otimes B)$$

$$= XZ\,(A\otimes B) \tag{37}$$

From (35)-(37) we can obtain (34) immediately. ■

If A and B are square matrices with $Ind(A) = k_1$ and $Ind(B) = k_2$, respectively, and set $W = I_m$ and $R = I_n$ in Theorem 3 we obtain the following corollary which is given by Wang [13]:

Corollary: *3* Let $A \in M_m$ and $B \in M_n$ with $Ind(A) = k_1$ and $Ind(B) = k_2$, respectively. Then

$$Ind\left\{\left(A \otimes B\right)\right\} = max\{k_1, k_2\} \tag{38}$$

and

$$\left(A \otimes B\right)_d = A_d \otimes B_d \tag{39}$$

More particularly, if $Ind(A) = Ind(B) = 1$, then we have

.

$$\left(A \otimes B\right)_g = A_g \otimes B_g \tag{40}$$

Corollary: 4 Let $A_i \in M_{m(i),n(i)}$ and $W_i \in M_{n(i),m(i)}\left(1 \le i \le r, r \ge 2\right)$ be matrices with

$$k_i = \max\left\{Ind\left(A_iW_i\right), Ind\left(W_iA_i\right)\right\}, i = 1, 2, ..., r.$$

Then

$$Ind\left\{\left(\prod_{i=1}^{r}\otimes A_i\right)Z\right\} = k, \tag{41}$$

and
$$\left(\prod_{i=1}^{r}\otimes A_i\right)_{d,z} = \prod_{i=1}^{r}\otimes\left(A_i\right)_{d,w_i} \tag{42}$$

where $k = max\left\{k_1, k_2, ..., k_r\right\}$ and $Z = \Pi_{i=1}^{r} \otimes W_i$. In particular,

i. if $A_i \in M_{m(i)}$ and $W_i = I_{n_i}\left(1 \le i \le k, k \ge 2\right)$, we then have

$$Ind\left(\prod_{i=1}^{r}\otimes A_i\right) = k \tag{43}$$

and

$$\left(\prod_{i=1}^{r}\otimes A_i\right)_d = \prod_{i=1}^{r}\otimes\left(A_i\right)_d \tag{44}$$

where $k = max\ \{Ind\ (A_i),\ i = 1,2...,r\}$

ii. if $Ind(A_1)=Ind(A_2)=...=Ind(A_r)=1$, we then have

$$\left(\prod_{i=1}^{r}\otimes A_i\right)_g = \prod_{i=1}^{r}\otimes\left(A_i\right)_g \tag{45}$$

Proof

The proof of (41) is by induction on r . The base case (when $r = 2$) has been established in (33) of Theorem 3. In the induction hypothesis, we assume that

$$Ind\left\{\left(\prod_{i=1}^{r-1}\otimes A_i\right)\left(\prod_{i=1}^{r-1}\otimes W_i\right)\right\} = Ind\left\{\prod_{i=1}^{r-1}\otimes A_iW_i\right\}$$

$$= \gamma = \max\left\{k_1, k_{2,\dots,} k_{r-1}\right\}$$

Now

$$Ind\left\{\left(\prod_{i=1}^{r}\otimes A_i\right)Z\right\} = Ind\left\{\left(\prod_{i=1}^{r}\otimes A_i\right)\left(\prod_{i=1}^{r}\otimes W_i\right)\right\}$$

$$= Ind\left\{\prod_{i=1}^{r}\otimes A_iW_i\right\}$$

$$= Ind\left\{\left(\prod_{i=1}^{r-1}\otimes A_iW_i\right)\otimes\left(A_rW_r\right)\right\}$$

$$= \max\left\{\gamma, k_r\right\} = \max\left\{k_1, k_2, \dots, k_r\right\} = k$$

The proof of (42) is also by induction on r. The base case (when r=2) has been established in (34) of Theorem 3. In the induction hypothesis, we assume that

$$\left(\prod_{i=1}^{r-1}\otimes A_i\right)_{d,\Pi_{i=1}^{r-1}\otimes W_i} = \prod_{i=1}^{r-1}\otimes\left(A_i\right)_{d,W_i}$$

Now

$$\left(\prod_{i=1}^{r}\otimes A_i\right)_{d,Z} = \left[\left(\prod_{i=1}^{r-1}\otimes A_i\right)_{d,\Pi_{i=1}^{r-1}\otimes W_i}\right]\otimes\left(A_r\right)_{d,W_r}$$

$$= \left(\prod_{i=1}^{r-1}\otimes\left(A_i\right)_{d,W_i}\right)\otimes\left(A_r\right)_{d,W_r}$$

$$= \prod_{i=1}^{r}\otimes\left(A_i\right)_{d,W_i}$$

The proof of three special cases in (43)-(45) are straightforward. ■

One of the important application of Theorem 3 is that the weighted Drazin inverse of Kronecker product arise naturally in solving the so-called restricted singular matrix equations *(RSME)* as follows.

Theorem 4: Let $A \in M_{m,n}, W \in M_{n,m}, B \in M_{p,q}, R \in M_{q,p}$ and $C \in M_{n,q}$ be given constant matrices and $X \in M_{m,p}$ be an unknown matrix to be solved. Also, let

$$L = R \otimes W, k_1 = Ind\left(\left(B \otimes A\right)L\right), k_2 = Ind\left(L\left(B \otimes A\right)\right) \tag{46}$$

such that

$$r\left(\left(B \otimes A\right)L\right)^{k_1} = r\left(L\left(B \otimes A\right)\right)^{k_2}, VecC \in R\left(L\left(B \otimes A\right)\right)^{k_2}, VecX \in R\left(\left(B \otimes A\right)L\right)^{k_1} \tag{47}$$

Then the unique solution of the following *RSME*

$$\left(WAW\right) X \left(RBR\right)^T = C \tag{48}$$

is given by

$$X = A_{d,W} C B_{d,R}^T \tag{49}$$

Proof

Using identity (21) it is not difficult to transform (48) into the vector form as:

$$\left(L\left(B \otimes A\right)L\right) VecX = VecC \tag{50}$$

It is easy to verify under conditions (47) that the unique solution of (50) is

$$VecX = \left(B \otimes A\right)_{d,L} VecC = \left(B_{d,R} \otimes A_{d,W}\right) VecC$$

$$= Vec\left(A_{d,W} C B_{d,R}^T\right)$$

which is the required result. ■

An important particular case of Theorem 4 is that when $m = n$, $p = q$, $W = I_m$ and $R = I_p$, we obtain the following corollary:

Corollary 5 : Let $A \in M_m, B \in M_p$ and $C \in M_{m,p}$ be given constant matrices and $X \in M_{m,p}$ be an unknown matrix to be solved. Then the unique solution of the following *RSME*

$$AXB^T = C : VecC, \; VecX \in R\left(B \otimes A\right)^k, k = Ind\left(B \otimes A\right) \tag{51}$$

is given by

$$X = A_d C B_d^T \tag{52}$$

CONCLUSION

In this paper, we have presented two general representations for weighted Drazin inverse related to Moore-Penrose inverse and Kronecker product of two and several matrices. These representations are viewed as a generalization of Wang's results in [13, Lemma 1.1, and 14, Theorem 2.2]. Furthermore, some necessary and sufficient conditions for Drazin and weighted Drazin inverses are given for the reverse order law $(AB)_d = B_d A_d$ and $\left(AB\right)_{d,zm} = B_{d,R} A_{d,w}$ to hold. Although the results are applied to solve the restricted singular matrix equations, the idea adopted can be easily extended to solve the coupled restricted singular matrix equations. It is natural to ask if we can extend our results to the Minkowski inverse in Minkowski space. This will be part of future research.

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