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Numerical Evaluation for Cauchy Type Singular Integrals Using Modification of Discrete Vortex Method and Spline Approximation

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ABSTRACT

This paper deals with the construction of an efficient quadrature formula for singular integrals (SI) of Cauchy type based on modification of discrete vortex method (MDV) and interpolation linear spline. The estimations of errors are obtained in the classes of H^{α}(K, [-1,1]) and C^1([-1,1]). Numerical analysis are also given

Key words: singular integral, quadrature formula, canonic partition, discrete vortex method, approximation, spline.
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INTRODUCTION

This paper is concerned with Cauchy type singular integrals of the form

$$\int_{-1}^{1} \frac{f(t)}{t-x} dt, \quad x \in (-1,1) \tag{1}$$

where $f(t) \in H^{\alpha}(K,[-1,1])$

Numerical evaluation of (1) has been considered in [2], where the convergence of quadrature formula was provided in the interval $x \in [t_j + \delta, t_{j+1} - \delta]$, where $\{t_j\}$ form an equal partition of [-1,1] and $\delta \in [h/4, 3h/4]$, $h = \frac{2}{N+1}, t_{j+1} = t_j + h$. This paper discusses the convergence of quadrature formula for any singular point x over the closed interval $[t_j, t_{j+1}]$. Moreover, the rate of convergence of quadrature formula is identified in the classes of functions $H^{\alpha}([-1,1],K)$ and C^1([-1,1]) and numerical examples are given to verify the validity of quadrature formula. These examples are also compared with the quadrature sum obtained by discrete vortex method [1].

CONSTRUCTION OF THE QUADRATURE FORMULA (QF)

Let us examine *SI* (1). We divide the interval [-1,1] into $N+1$ equal subintervals $[t_k, t_{k+1}]$, $k=0,...,N]$ where $t_k=-1+kh$, $k=0,...,N+1$. The set $E=\{t_k, k=1,...,N\}$ is called a canonic partition of the interval [-1,1] (see [1]). Let $Q(j)=\{j-1, j, j+1, j=1,...,N\}$, where it is assumed that $\{j\}$ is a fixed integer. We consider two cases:

i. Singular point x does not coincide with the knots points $\{-1=t_0<t_1<t_2<...<t_N<t_{N+1}=1\}$, that is $x=t_j+\varepsilon, j=1,...,N,$ where $\varepsilon\in(0,h)$.

ii. Singular point x coincides with the knot points, that is $x=t_j$, $j=1,...,N$.

Now let $s_{1,v}(t)$ and $s_{2,v}(t)$ be a linear interpolation spline [6] such that

$$s_{1,v}(t)=\frac{1}{h}\left[\left(t_{v+1}-t\right)f\left(t_v\right)+\left(t-t_v\right)f\left(t_{v+1}\right)\right],\ v=0,...,N,\ t\in\left[t_v,t_{v+1}\right] \qquad (2)$$

$$s_{2,v}(t)=\frac{1}{2h}\left[\left(t_{v+1}-t\right)f\left(t_{v-1}\right)+\left(t-t_{v-1}\right)f\left(t_{v+1}\right)\right],\ v=1,...,N,\ t\in\left[t_{v-1},t_{v+1}\right] \qquad (3)$$

which has the following properties:

a. if $f(t)=c$ then $s_{1,v}(t)=s_{2,v}(t)=c$

b. if $f(t)=at+b$ then $s_{1,v}(t)=s_{2,v}(t)=at+b$.

In the first case, we use (2) to construct QF for SI (1), giving

$$\int_{-1}^{1}\frac{f(t)\,dt}{t-\left(t_j+\varepsilon\right)}=\sum_{k=0,k\notin Q(j)}^{N}\int_{t_k}^{t_{k+1}}\frac{f(t)\,dt}{t-\left(t_j+\varepsilon\right)}+\sum_{v=j-1}^{j+1}\int_{t_v}^{t_{v+1}}\frac{f(t)\,dt}{t-\left(t_j+\varepsilon\right)} \qquad (4)$$

$$=\sum_{k=0,k\notin Q(j)\cup j+2}^{N+1}A_k\left(t_j+\varepsilon\right)f\left(t_k\right)+\sum_{v=j-1}^{j+1}\int_{t_v}^{t_{v+1}}\frac{s_{1,v}(t)\,dt}{t-\left(t_j+\varepsilon\right)}+R\left(t_j+\varepsilon\right),$$

where,

$$A_k\left(t_j+\varepsilon\right)=\frac{h}{t_k-\left(t_j+\varepsilon\right)},\ k=1,...,j-2,\ j+3,...,N \qquad (5)$$

The coefficients A_k are similar to the coefficients obtained by MDV [1]. The quadrature formula (4) is exact for the linear function $f(t)$, and due to the properties of $s_{1,v}(t)$ one can find that A_0 and A_{N+1} are given by

$$\left.\begin{aligned}
A_0\left(t_j+\varepsilon\right)&=\frac{1}{2}\left[\int_{-1}^{1}\frac{(1-t)\,dt}{t-\left(t_j+\varepsilon\right)}\right.\\
&\left.-\sum_{k=1,k\notin Q(j)\cup j+2}^{N}\frac{\left(1-t_k\right)h}{t_k-\left(t_j+\varepsilon\right)}-\sum_{v=j-1}^{j+1}\int_{t_v}^{t_{nu+1}}\frac{(1-t)\,dt}{t-\left(t_j+\varepsilon\right)}\right],\\
A_{N+1}\left(t_j+\varepsilon\right)&=\frac{1}{2}\left[\int_{-1}^{1}\frac{(1+t)\,dt}{t-\left(t_j+\varepsilon\right)}\right.\\
&\left.-\sum_{k=1,k\notin Q(j)\cup j+2}^{N}\frac{\left(1+t_k\right)h}{t_k-\left(t_j+\varepsilon\right)}-\sum_{v=j-1}^{j+1}\int_{t_v}^{t_{v+1}}\frac{(1+t)\,dt}{t-\left(t_j+\varepsilon\right)}\right].
\end{aligned}\right\} \qquad (6)$$

Substituting (5) and (6) into (4), we obtain

$$\int_{-1}^{1}\frac{\varphi(t)\,dt}{t-(t_j+\varepsilon)}=\sum_{k=1,k\neq j+2,k\notin Q(j)}^{N}\frac{\varphi(t_k)h}{t_k-(t_j+\varepsilon)}$$

$$+\sum_{v=j-1}^{j+1}\int_{t_v}^{t_{v+1}}\frac{s_{1,v}^{*}(t)\,dt}{t-(t_j+\varepsilon)}+R_N(t_j+\varepsilon), \tag{7}$$

where

$$\varphi(t)=f(t)-\frac{1}{2}\Big[(1-t)f(-1)+(1+t)f(1)\Big], \tag{8}$$

$$s_{1,v}^{*}(t)=s_1(t)-\frac{1}{2}\Big[(1-t)f(-1)+(1+t)f(1)\Big]. \tag{9}$$

Evaluating the corresponding integral in quadrature formula (7), one has

$$\int_{-1}^{1}\frac{f(t)\,dt}{t-(t_j+\varepsilon)}=\sum_{k=0}^{N+1}A_k(t_j+\varepsilon)f(t_k)+R_N(t_j+\varepsilon), \tag{10}$$

where

$$A_k(t_j+\varepsilon)=\frac{h}{t_k-(t_j+\varepsilon)},k=1,...j-2,j+3,...,N,$$

$$A_0(t_j+\varepsilon)=\frac{1}{2}\Big[\big(1-(t_j+\varepsilon)\big)T_N(t_j+\varepsilon)-2h\Big],$$

$$A_{N+1}(t_j+\varepsilon)=\frac{1}{2}\Big[\big(1+(t_j+\varepsilon)\big)T_N(t_j+\varepsilon)+2h\Big],$$

$$A_{j-1}(t_j+\varepsilon)=-\left(1+\frac{\varepsilon}{h}\ln\frac{\varepsilon}{h+\varepsilon}\right),$$

$$A_j(t_j+\varepsilon)=\ln\frac{h-\varepsilon}{h+\varepsilon}+\varepsilon\ln\frac{\varepsilon^2}{h^2-\varepsilon^2},$$

$$A_{j+1}(t_j+\varepsilon)=2\ln\frac{2h-\varepsilon}{h-\varepsilon}+\varepsilon\ln\frac{(h-\varepsilon)^2}{\varepsilon(2h-\varepsilon)},$$

$$A_{j+2}(t_j+\varepsilon)=1-\frac{h-\varepsilon}{h}\ln\frac{2h-\varepsilon}{h-\varepsilon},$$

$$T_N(t_j+\varepsilon)=\ln\left|\frac{1-(t_j+\varepsilon)}{1+(t_j+\varepsilon)}\frac{h+\varepsilon}{2h-\varepsilon}\right|-\sum_{k=1,k\notin Q(j)\cup j+2}^{N}\frac{h}{t_k-(t_j+\varepsilon)}.$$

For the second case, we use the spline function (3) to give

$$\int_{-1}^{1}\frac{f(t)dt}{t-t_j}=\sum_{k=0,k\neq\{j-1,j\}}^{N}\int_{t_k}^{t_{k+1}}\frac{f(t)dt}{t-t_j}+\int_{t_{j-1}}^{t_{j+1}}\frac{f(t)dt}{t-t_j}$$

$$=\sum_{k=0,k\notin Q(j)}^{N+1}B_k\left(t_j\right)f\left(t_k\right)+\int_{t_{j-1}}^{t_{j+1}}\frac{s_{2,j}\left(t\right)dt}{t-t_j}+R\left(t_j\right), \tag{11}$$

where

$$B_k(t_j)=\frac{h}{t_k-t_j}\quad,\quad k=1,....,j-2,j+2,...,N \tag{12}$$

The QF (11) is exact for linear function *f(t)*, hence we obtain

$$\left.\begin{aligned}B_0(t_j)&=\frac{1}{2}\left[\int_{-1}^{1}\frac{(1-t)dt}{t-t_j}-\int_{t_{j-1}}^{t_j}\frac{(1-t)dt}{t-t_j}-\sum_{k=1,k\in Q(j)}^{N}\frac{(1-t_k)}{t_k-t_j}\right],\\ B_{N+1(t_j)}&=\frac{1}{2}\left[\int_{-1}^{1}\frac{(1+)dt}{t-t_j}-\int_{t_{j-1}}^{t_{j+1}}\frac{(1-t)dt}{t-t_j}-\sum_{k=1,k\notin Q(j)}^{N}\frac{(1+t_k)h}{t_k-t_j}\right].\end{aligned}\right\} \tag{13}$$

Substituting (12) and (13) into (11) yields

$$\int_{-1}^{1}\frac{\varphi(t)dt}{t-t_j}=\sum_{k=1,k\notin Q(j)}^{N}\frac{\varphi\left(t_k\right)h}{t_k-t_j}+\int_{t_{j-1}}^{t_{j+1}}\frac{S*_{2,V}\ (t)dt}{t-t_j}+R\left(t_j\right) \tag{14}$$

where $\varphi(t)$ is defined by (8) and

$$s_{2v}^{*}\left(t\right)=s_{2,v}\left(t\right)-\tfrac{1}{2}\left[\left(1-t\right)f\left(-1\right)+\left(1+t\right)f\left(1\right)\right]. \tag{15}$$

Evaluating the integral on the right side of equation (14) and taking into account formulas (8) and (15), we obtain

$$\int_{-1}^{1}\frac{f(t)dt}{t-t_j}=\sum_{k=0}^{N+1}B_k\left(t_j\right)f\left(t_k\right)+R\left(t_j\right) \tag{16}$$

where

$$B_k(t_j)=\frac{h}{t_k-t_j},\quad k=1,...j-2,j+2,...,N,$$

$$B_0\left(t_j\right)=\frac{1}{2}\left[\left(1-t_j\right)S_N\left(t_j\right)-2h\right],$$

$$B_{N+1}\left(t_j\right) = \frac{1}{2}\left[\left(1+t_j\right)S_N\left(t_j\right)+2h\right],$$

$$B_{j-1}\left(t_j\right) = -1$$

$$B_j\left(t_j\right) = 0$$

$$B_{j+1}\left(t_j\right) = 2\text{In}2$$

$$S_N\left(t_j\right) = \ln\left|\frac{1-t_j}{2(1+t_j)}\right| - \sum_{k=1,k\notin Q(j)}^{N} \frac{h}{t_k - t_j}.$$

ESTIMATION OF ERRORS

Let $H^{\alpha}\left(K,[-1,1]\right)$ and $C^1\left([-1,1]\right)$ be classes of functions at which satisfy Hölder condition and continously differentiable function of the first order, respectively.

Theorem 1: Let $f\left(t\right)\in H^{\alpha}\left([-1,1],K\right)$ and E be a set of canonic partition of the interval [-1,1]. Then for the error of quadrature formula (10) the following estimation of error is valid

$$\left|R_N\left(t_j+\varepsilon\right)\right| \leq \begin{Bmatrix} L_1 h^{\alpha} en(N=1), \varepsilon = h/2, or \varepsilon = 0 \\ L_2 h^{\alpha} en(N+1) + L_2 h^{\delta}, \varepsilon \neq h/2 and \varepsilon \in (0,h) \end{Bmatrix}$$

where

$$0 < \delta \leq \log_h \frac{|2\varepsilon - h|h}{(2h-\varepsilon)(h+\varepsilon)},$$

$$L_1 = 8K\left(1+\frac{1.506}{\alpha\ln(N+1)}\right), L_2 = K\left(0.068h^{5-\delta} + 0.567h^{2-\delta} + 0.516\right),$$

Theorem 2: Let $f\left(t\right)\in C^1\left([-1,1]\right)$ and E be a set of canonic partition of the interval [-1,1]. Then the error of quadrature formula (10) can be written as

$$\left|R_N\left(t_j+\varepsilon\right)\right| \leq \begin{cases} L_1^* h In(N+1), \varepsilon = h/2 or \varepsilon = 0, \\ L_1^* In(N+1) + L_2^* h^{\delta}, \varepsilon \neq h/2, and \varepsilon \in (0,h), \end{cases}$$

where

$$L_1^* = 8M_1\left(1+\frac{0.78}{\ln(N+1)}\right),\ L_2^* = M_1\left(0.068h^{5-\delta}+0.567h^{2-\delta}+0.516\right),$$

$$M_1 = \max_{-1\le\theta\le 1}\left|f^{'}(\theta)\right|$$

Theorem 3: (Euler-Makleron theorem) Let *f(x)* be defined on [*a,b*]. If *f(x)* is a continuously differentiable function up to 2*k*, then the following formula is true

$$\int_a^b f(x)dx = \frac{h}{2}\left(f(a)+f(b)\right)+\sum_{j=1}^{n-1} f(a+jh)h + R_{2k}(f), \tag{17}$$

where

$$R_{2k}(f) = -\frac{h^{2k}}{(2k)!}(b-a)B_{2k}f^{2k}(\xi),\xi\in[a,b],h=\frac{b-a}{n},$$

and B_{2k} is Bernoulli number. In addition, if for any $x\in[a,b]$ the following inequalities

$$f^{(2k)}(x)\ge 0 \text{ and } f^{(2k+2)}(x)\ge 0 \text{ (or } f^{(2k)}(x)\le 0 \text{ and } f^{(2k)}(x)\ge 0\text{)}$$

are true, then

$$R_{2k}(f) = -\frac{h^{2k}B_{2k}}{(2k)!}\left[f^{2k-1}(b)-f^{2k-1}(a)\right].$$

For the proof see [3].

The proof of theorems 1 and 2 are based on the following lemmas.

Lemma 1: Let *f(t)* be a continuous function on [-1,1].

If $f(t)\in H^\alpha([-1,1],K)$ then for any $t',t'',t\in[-1,1]$, the following estimations are true

a. $|\varphi(t'')-\varphi(t')|\le 2K|t''-t'|^\alpha$

b. $|\varphi(t)|\le K(1-t^2)^\alpha$

If $f(t)\in C^1([-1,1])$ then for any $t',t'',t\in[-1,1]$, the following estimations are valid

c. $|\varphi(t'')-\varphi(t')|\le 2M_1|t''-t'|$,

d. $|\varphi(t)|\le M_1(1-t^2)$.

where $M_1 = \max_{\xi\in[-1,1]}\left|f^{'}(\xi)\right|$

Lemma 1 is proved in [4].

Lemma 2: Let x be any singular point in the interval $[t_j, t_{j+1}]$, $j=1,...,N-1$. Then the following statement is true

$$\int_{-1}^{t_{j-1}} \frac{dt}{|t-x|} + \int_{t_{j+2}}^{1} \frac{dt}{|t-x|} \le 2ln(N+1).$$

Proof: Since $x \in \left[t_j, t_{j+1}\right]$, one has

$$\int_{-1}^{t_{j-1}} \frac{dt}{|t-x|} + \int_{t_{j+2}}^{1} \frac{dt}{|t-x|} = ln\left|\frac{1-x^2}{\left(x-t_{j-1}\right)\left(t_{j+2}-x\right)}\right|$$

$$\le \ln \le \left|\frac{1-t_j^2}{\left(t_j-t_{j-1}\right)\left(t_{j+2}-t_j\right)}\right| \le 2\ln\left(N+1\right).$$

Lemma 3: Let $[x_1, x_2]$ be any interval on R. If $a \in \left(x_1, x_2\right)$ and $0 \le \beta \le 1$, then the following statements are valid

a. $r_1^{\beta} + r_2^{\beta} \le 2^{1-\beta}\left(r_1 + r_2\right)^{\beta}$,

b. $\left|r_1^{\beta} - r_2^{\beta}\right| \le \left|r_1 - r_2\right|^{\beta}$,

where $r_1 = \left|x_1 - a\right|, r_2 = \left|x_2 - a\right|$.

For the proof, see [5].

Let $g_{1,v}(x)$ be given by

$$g_{1,v}(t)=f(t)-s_{1,v}(t), \qquad v=j-1,j+1, \tag{18}$$

where $t \in [t_v, t_{v+1}]$.

Lemma 4: Let $f(x)$ be a continuous function on [-1,1], and x be any point on the interval $[t_j, t_{j+1}]$. Then the following estimations are true

a. If $f\left(t\right) \in H^{\alpha}\left(K, \left[t_j, t_{j+1}\right]\right)$, then $\left|g_{1,j}\left(x\right)\right| \le 2^{\alpha} K\left(\left|x-t_j\right|\left|x-_{j+1}\right|\right)^{\alpha} h^{-\alpha}$.

b. If $f\left(t\right) \in C^1\left(\left[t_j, t_{j+1}\right]\right)$, then $\left|g_{1,j}\left(x\right)\right| \le M_1\left(\left|x-t_j\right|\left|x-t_{j+1}\right|\right)h^{-1}$,

where $M_1 = \max\left|f'\left(c_1\right) - f'\left(c_2\right)\right|$, $c_1 \in (t_j, x)$, $c_2 \in (x, t_{t+1})$,

Proof of Lemma 4a: Since $x \in \left[t_j, t_{j+1}\right]$, from (2) and (18) it follows that

$$\left|g_{1,j}\left(x\right)\right| = \frac{1}{h}\left|\left(t_{j+1}-x\right)\left(f\left(x\right)-f\left(t_j\right)\right)+\left(x-t_j\right)\left(f\left(x\right)-f\left(t_{j+1}\right)\right)\right|$$

$$\leq \frac{K}{h}\left(\left|x-t_j\right|\left|x-t_{j+1}\right|\right)^{\alpha}\left(\left|x-t_{j+1}\right|^{1-\alpha}+\left|x-t_j\right|^{1-\alpha}\right)$$

$$\leq 2^{\alpha} K\left(\left|x-t_j\right|\left|x-t_{j+1}\right|\right)^{\alpha} h^{-\alpha}.$$

Proof of Lemma 4b: Due to (2), (18) and the Mean Value Theorem

$$\left|g_{1,j}(x)\right|=\frac{1}{h}\left|\left(t_{j+1}-x\right)\left(f(x)-f\left(t_j\right)\right)+\left(x-t_j\right)\left(f(x)-f\left(t_{j+1}\right)\right)\right|$$

$$=\frac{1}{h}\left|\left(t_{j+1}-x\right)f'(c_1)\left(x-t_j\right)+\left(x-t_j\right)f'(c_2)\left(x-t_{j+1}\right)\right|$$

$$\leq M_1\left(\left|x-t_j\right|\left|x-t_{j+1}\right|\right)h^{-1}$$

Lemma 5: Let *f(t)* be a continuous function. For any $x\in\left[t_j,t_{j+1}\right]$, the following inequalities are valid

a. If $f(t)\in H^{\alpha}\left(K,\left[t_{j-1},t_{j+2}\right]\right)$, then

$$\left|s_{1,v}(t)-s_{1,j}(x)\right|\leq Kh^{\alpha-1}\left|t-x\right|, \text{ for all } v=j-1,j,j+1$$

b. If $f(t)\in C^1\left(\left[t_{j-1},t_{j+2}\right]\right)$, then

$$\left|s_{1,v}(t)-s_{1,j}(x)\right|\leq M_1\left|t-x\right|, \text{for all } v=j-1,j,j+1$$

where $M_1=\max\limits_{t_{j-1}\leq\theta\leq t_{j+2}}\left|f'(\theta)\right|.$

Proof of Lemma 5a: For the case $v=j-1$, using (2) and applying Lemma 3a, one has

$$\left|S_{1,j-1}(t)-S_{1,j}(x)\right|=\frac{1}{h}\left|(t-t_j)(f(t_j)-f(t_{j-1}))+(x-t_j)(f(t_{j+1})-f(t_{j+1}))\right|$$

$$\leq Kh^{\alpha-1}\left[\left|t-t_j\right|+\left|x-t_j\right|\right]\leq Kh^{\alpha-1}\left|t-x\right|.$$

The case $í=j+1$ is proved similarly. For the case $í=$ j, points *x* and *t* belong to $[t_j,t_{j+1}]$. Then in view of (2), one gets

$$\left|s_{1,j}(t)-s_{1,j}(x)\right| = \frac{1}{h}\left|(t-x)\left(f\left(t_{j+1}-f\left(t_j\right)\right)\right|$$

$$\leq Kh^{\alpha-1}\left|t-x\right|$$

Lemma 5b is proved analogously.

Lemma 6: Let $f(t)$ be a continuous function on [-1,1] and $s_{1,v}(t)$, $g_{1,v}(t)$ be defined in (2), (18) respectively. Let x be any point of the interval $[t_j, t_{j+1}]$

a. If $f(t) \in H^{\alpha}\left(K, \left[t_{j-1}, t_{j+1}\right]\right)$, then

i. $\left|g_{1,v}(t) - g_{1,j}(x)\right| \le 3K\left|t-x\right|^{\alpha}$, for $v=j-1,\ j+1$

ii. $\left|g_{1,v}(t) - g_{1,j}(x)\right| \le 2K\left|t-x\right|^{\alpha}$ for $v = j$.

b. If $f(t) \in C^{1}\left(\left[-1,1\right]\right)$, then

$\left|g_{1,v}(t) - g_{1,j}(x)\right| \le 2M_1\left|t-x\right|$, for all $v = j-1, j, j+1$

where $M_1 = \max\limits_{\xi \in \left[t_{j-1}, t_{j+2}\right]} \left|f'(\xi)\right|$

Proof of Lemma 6a:

i. Since $x \in \left[t_j, t_{j+1}\right], t \in \left[t_{j-1}, t_j\right]$ and $v = j - 1$, due to Lemma 5a, we obtain

$$\left|g_{1,j-1}(t) - g_{1,j}(x)\right| = \left|f(t) - f(x) - \left(s_{1,j-1}(t) - s_{1,j}(x)\right)\right|$$
$$\le K\left|t-x\right|^{\alpha} + Kh^{\alpha-1}\left|t-x\right| \le 3K\left|t-x\right|^{\alpha}.$$

The case $v = j + 1$ is asserted in a similar way as above.

ii. In the case $t, x \in \left[t_j, t_{j+1}\right]$ and $v = j$, using Lemma 5a, one has

$$\left|g_{1,j}(t) - g_{1,j}(x)\right| = \left|f(t) - f(x) - \left(s_{1,j}(t) - s_{1,j}(x)\right)\right|$$
$$\le K\left|t-x\right|^{\alpha} + Kh^{\alpha-1}\left|t-x\right| \le 2K\left|t-x\right|^{\alpha}.$$

Proof of Lemma 6b: For the case $v = j - 1$ the point t is contained in the interval $[t_{j-1}, t_j]$ Then due to Lemma 5a, for any $x \in \left[t_j, t_{j+1}\right]$ we obtain

$$\left|g_{1,j-1}(t) - g_{1,j}(x)\right| = \left|f(t) - f(x) - \left(s_{1,j-1}(t) - s_{1,j}(x)\right)\right|$$
$$\le \left|f'(c_1)\right|\left|t-x\right| + M_1\left|t-x\right| \le 2M_1\left|t-x\right|.$$

Let

$$R^*\left(t_j+\varepsilon\right)=\left(1-\left(t_j+\varepsilon\right)^2\right)^{\alpha}\left|\int_{-1}^{t_{j-1}}\frac{dt}{t-\left(t_j+\varepsilon\right)}+\int_{t_{j+2}}^{1}\frac{dt}{t-\left(t_j+\varepsilon\right)}\right. \tag{19}$$

$$\left.-\sum_{k=1,k\notin Q(j)\cup j+2}^{N}\frac{h}{t_k-\left(t_j+\varepsilon\right)}\right|$$

For $\varepsilon=0$ we have

$$R^*\left(t_j\right)=\left(1-t_j^2\right)^{\alpha}\left|\int_{-1}^{t_{j-1}}\frac{dt}{t-t_j}+\int_{t_{j+1}}^{1}\frac{dt}{t-t_j}-\sum_{k=1,k\notin Q(j)}^{N}\frac{h}{t_k-t_j}\right|$$

Lemma 7: For any singular point $\left(t_j+\varepsilon\right), j=1,...,N$ where $t_j\in E$ the following statement is valid

$$\left|R^*\left(t_j+\varepsilon\right)\right|\le\begin{cases}C_1h^{\alpha},\ when\ \varepsilon=h/2\ or\varepsilon=0\\ C_2h^{\delta}\ \ when\ \varepsilon\ne h/2\ and\,\varepsilon\in(0,h)\end{cases}$$

where

$C_1 = 0.5 + 0.067h,$

$C_2 = 0.0068h^{5-\delta} + 0.567h^{2-\delta} + 0.516.$

Proof of Lemma 7: Let us examine the following function

$$f\left(t\right)=\frac{1}{t-(t_j+\varepsilon)}, \tag{20}$$

where t runs either $[-1,t_{j-1}]$ or $[t_{j+2},1]$. It is obvious that the second and fourth derivatives of the function (20) are negative at $t\in\left[-1,t_{j-1}\right]$ and both positive on $[t_{j+2},1]$. Applying Euler-Makleron formula to the integral in (19), we obtain

$$\int_{-1}^{t_{j-1}}\frac{dt}{t-\left(t_j+\varepsilon\right)}=\sum_{k=1}^{j-2}\frac{h}{t_k-\left(t_j+\varepsilon\right)}-\frac{h}{2}\left(\frac{1}{1+\left(t_j+\varepsilon\right)}+\frac{1}{h+\varepsilon}\right)+$$

$$+\frac{h^4}{120}\left[-\frac{1}{\left(h+\varepsilon\right)^4}+\frac{1}{\left(1+\left(t_j+\varepsilon\right)\right)^4}\right]$$

The second integral of (19) can be represented as

$$\int_{t_{j+2}}^{1}\frac{dt}{t-(t_j+\varepsilon)}=\sum_{k=j+3}^{N}\frac{h}{t_k-(t_j+\varepsilon)}+\frac{h}{2}\left(\frac{1}{2h-\varepsilon}-\frac{1}{1-(t_j+\varepsilon)}\right)+$$

$$+\frac{h^4}{120}\left[-\frac{1}{\left(1-(t_j+\varepsilon)\right)^4}+\frac{1}{(2h-\varepsilon)^4}\right]$$

Substituting these into (19), yields

$$R^*(t_j+\varepsilon)=\left(1-(t_j+\varepsilon)^2\right)^{\alpha}\left|\frac{h}{2}\left(\frac{1}{1-(t_j+\varepsilon)}+\frac{1}{2h-\varepsilon}-\frac{1}{h-\varepsilon}\right)\right.$$

$$\left.+\frac{h^4}{120}\left(\frac{1}{\left(1-(t_j+\varepsilon)\right)^4}-\frac{1}{\left(1+(t_j+\varepsilon)\right)^4}+\frac{1}{(2h-\varepsilon)^4}-\frac{1}{(h+\varepsilon)^4}\right)\right|$$

$$\le\left(1-(t_j+\varepsilon)^2\right)^{\alpha}\left[\frac{|t_j+\varepsilon|h}{2\left(1-(t_j+\varepsilon)^2\right)}+\frac{|t_j+\varepsilon|h^4}{15\left(1-(t_j+\varepsilon)^2\right)^3}\right.$$

$$\left.+\frac{|2\varepsilon-h|h}{2(2h-\varepsilon)(h+\varepsilon)}\left(1+\frac{h^4\left(5h^2-2h\varepsilon+2\varepsilon^2\right)}{20\left[(2h-\varepsilon)(h+\varepsilon)\right]^3}\right)\right].$$

Setting $A(\varepsilon)=\frac{|2\varepsilon-h|h}{2(2h-\varepsilon)(h+\varepsilon)}$, gives

$$R^*(t_j+\varepsilon)\le\left(1-(t_j+\varepsilon)^2\right)^{\alpha}\left[\frac{|t_j+\varepsilon|h}{2\left(1-(t_j+\varepsilon)^2\right)}+\frac{|t_j+\varepsilon|h^4}{15\left(1-(t_j+\varepsilon)^2\right)^3}\right.$$

$$\left.+A(\varepsilon)\left(0.5+\frac{h^4\left(5h^2-2h\varepsilon+2\varepsilon^2\right)}{40\left[(2h-\varepsilon)(h+\varepsilon)\right]^3}\right)\right]. \tag{21}$$

If ε = *h*/2 , then $A(\varepsilon)\equiv 0$ and so

$$R^*(t_{0j})\le\left(1-t_{0j}^2\right)^{\alpha}\left[\frac{|t_{0j}|h}{2\left(1-t_{0j}^2\right)}+\frac{1}{15}\frac{|t_{0j}|h^4}{\left(1-t_{0j}^2\right)^3}\right].$$

The expressions on the right hand side of lattar expressions attains its maximum value at $j = N$. We therefore have

$$R^*\left(t_{0N}\right) \le \left(1-t_{0N}^2\right)^{\alpha}\left[\frac{\left|t_{0N}\right| h}{2\left(1-t_{0N}^2\right)}+\frac{\left|t_{0N}\right| h^4}{15\left(1-t_{0N}^2\right)^3}\right] \le C_1 h^{\alpha} .$$

Let $\varepsilon \neq h/2$, then on the right hand side of (21) reaches its maximum value at $t_j = 0$. Thus

$$R^*(\varepsilon) \le \left(1-\varepsilon^2\right)^{\alpha}\left[\frac{\varepsilon h}{2\left(1-\varepsilon^2\right)}+\frac{\varepsilon h^4}{15\left(1-\varepsilon^2\right)^3}+A(\varepsilon)\left(0.5+\frac{h^4\left(5h^2-2h\varepsilon+2\varepsilon^2\right)}{40\left[(2h-\varepsilon)(h+\varepsilon)\right]^3}\right)\right]$$

$$\le 0.567h^2+0.068h^5+0.516\frac{\left|2\varepsilon-h\right| h}{(2h-\varepsilon)(h+\varepsilon)} .$$

Since $0<\varepsilon<h$ there is a δ such that

$$\frac{\left|2\varepsilon-h\right| h}{(2h-\varepsilon)(h+\varepsilon)} < h^{\delta} .$$

Thus, we have the following error for $R^*(\varepsilon)$

$$R^*(\varepsilon) \le C_2 h^{\delta} .$$

Proof of Theorem 1: Using (7), (8) and (9) we obtain

$$\left|R_N\left(t_j+\varepsilon\right)\right| = \left|\int_{-1}^{1}\frac{\varphi(t)\,dt}{t-\left(t_j+\varepsilon\right)}-\sum_{k=1,k\notin Q(j)\cup j+2}^{N}\frac{\varphi\left(t_k\right)h}{t_k-\left(t_j+\varepsilon\right)}-\sum_{v=j-1}^{j+1}\int_{t_v}^{t_{v+1}}\frac{s^*(t)\,dt}{t-\left(t_j+\varepsilon\right)}\right| \quad (22)$$

$$\le R_1\left(t_j+\varepsilon\right)+R_2\left(t_j+\varepsilon\right)+R_3\left(t_j+\varepsilon\right)+R_4\left(t_j+\varepsilon\right)+R_5\left(t_j+\varepsilon\right),$$

where

$$R_1\left(t_j+\varepsilon\right)=\left|\int_{-1}^{t_1}\frac{\varphi(t)-\varphi\left(t_j+\varepsilon\right)}{t-\left(t_j+\varepsilon\right)}dt\right|,$$

$$R_2\left(t_j+\varepsilon\right)=\left|\frac{\varphi\left(t_{j+2}\right)-\varphi\left(t_j+\varepsilon\right)}{t_{j+2}-\left(t_j+\varepsilon\right)}h\right|,$$

$$R_3\left(t_j+\varepsilon\right)=\left|\sum_{\substack{k=1\\k\notin Q(j)}}^{N}\int_{t_k}^{t_{k+1}}\left(\frac{\varphi(t)-\varphi(t_j+\varepsilon)}{t-(t_j+\varepsilon)}-\frac{\varphi(t_k)-\varphi(t_j+\varepsilon)}{t_k-(t_j+\varepsilon)}dt\right)\right|,$$

$$R_4\left(t_j+\varepsilon\right)=\left|\varphi\left(t_j+\varepsilon\right)\right|\left|\left(\int_{-1}^{t_{j-1}}+\int_{t_{j+2}}^{1}\right)\frac{dt}{t-\left(t_j+\varepsilon\right)}-\sum_{k=1,k\notin Q(j)\cup j+2}^{N}\frac{h}{t_k-\left(t_j+\varepsilon\right)}\right|,$$

$$R_5\left(t_j+\varepsilon\right)=\left|\sum_{v=j-1}^{j+1}\int_{t_v}^{t_{v+1}}\frac{f\left(t\right)-s_{1v}\left(t\right)}{t-\left(t_j+\varepsilon\right)}dt\right|.$$

Due to Lemma 1a, one gets

$$R_1\left(t_j+\varepsilon\right)\le 2K\int_{-1}^{t_1}\frac{dt}{\left|t-\left(t_j+\varepsilon\right)\right|^{1-\alpha}}\le\frac{2K}{\alpha}h^{\alpha},$$

$$R_2\left(t_j+\varepsilon\right)\le 2K\frac{h}{\left|t_{j+2}-\left(t_j+\varepsilon\right)\right|^{1-\alpha}}\le 2Kh^{\alpha}.$$

Appling Lemmas 1a & 2, and changing the view of expressions R_3 yields

$$R_3(t_j+\varepsilon)=\left|\sum_{k=1,k\notin Q(j)}^{N}\int_{t_k}^{t_{k+1}}\frac{\varphi(t)-\varphi(t_k)}{t-(t_j+\varepsilon)}\right.$$

$$\left.+\sum_{k=1,k\notin Q(j)}^{N}\int_{t_k}^{t_{k+1}}\frac{\varphi(t_k)-\varphi(t_j+\varepsilon)t_k-t}{t_k-(t_j+\varepsilon)t-(t_{j+\varepsilon})}dt\right|\le 8kh^{\alpha}\ln(N+1).$$

R_4 depends on ε , and due to lemmas 1b & 7, one has

$$\left|R_4\left(t_j+\varepsilon\right)\right|\le\begin{cases}K(0.5+0.067h)h^{\alpha} & when\ \varepsilon=h/2 or \varepsilon=0\\ K(0.567h^{2-\delta}+0.068h^{5-\delta}+0.516)h^{\alpha} & when\ \varepsilon\ne h/2 and \varepsilon\in(0,h)\end{cases}$$

To obtain the error of R_5 we use the formula (18) and Lemmas 3a, 3 b, 4a, 6a which gives

$$R_5\left(t_j+\varepsilon\right)\le\left|\sum_{v=j-1}^{j+1}\int_{t_v}^{t_{v+1}}\frac{g_{1,v}(t)-g_{1,j}\left(t_j+\varepsilon\right)}{t-\left(t_j+\varepsilon\right)}dt\right|+\left|\sum_{v=j-1}^{j+1}g_{1,j}\left(t_j+\varepsilon\right)\int_{t_v}^{t_{v+1}}\frac{dt}{t-\left(t_j+\varepsilon\right)}dt\right|$$

$$\le\left|\int_{t_{j-1}}^{t_j}\frac{g_{1,j-1}(t)-g_{1,j}\left(t_j+\varepsilon\right)}{t-\left(t_j+\varepsilon\right)}dt\right|+\left|\int_{t_j}^{t_{j+1}}\frac{g_{1,j}(t)-g_{1,j}\left(t_j+\varepsilon\right)}{t-\left(t_j+\varepsilon\right)}dt\right|$$

$$+\left|\int_{t_{j+1}}^{t_{j+2}}\frac{g_{1,j+1}(t)}{t-\left(t_j+\varepsilon\right)}dt\right|+\left|g_{1,j}\left(t_j+\varepsilon\right)\right|\left|\int_{t_{j-1}}^{t_{j+2}}\frac{dt}{t-\left(t_j+\varepsilon\right)}dt\right|$$

$$\le 3K\int_{t_{j-1}}^{t_j}\frac{dt}{\left|t-\left(t_j+\varepsilon\right)\right|^{1-\alpha}}+2K\int_{t_j}^{t_{j+1}}\frac{dt}{\left|t-\left(t_j+\varepsilon\right)\right|^{1-\alpha}}$$

$$+3K\int_{t_{j+1}}^{t_{j+2}}\frac{dt}{\left|t-\left(t_j+\varepsilon\right)\right|^{1-\alpha}}+2^{\alpha}Kh^{-\alpha}\left(\varepsilon\left(h-\varepsilon\right)\right)^{\alpha}\left|\ln\frac{t_{j+2}-\left(t_j+\varepsilon\right)}{t_{j-1}-\left(t_j+\varepsilon\right)}\right|$$

$$\le\frac{10K}{\alpha}h^{\alpha}+0.3365\cdot 2^{-3\alpha}Kh^{\alpha}\le\frac{10.012K}{\alpha}h^{\alpha}$$

Substituting R_1 - R_5 into (22) proves Theorem 1. Case $\varepsilon = 0$ is proved with the help of (14), Euler-Makleron formula (17) and corresponding Lemmas 1-7. Theorem 2 is proved in the same manner.

NUMERICAL EXPERIMENTS

Let $f\left(t\right)=\sqrt{2+x}$. Then the exact solution for the singular integral (1) is of the form

$$J\left(x\right)=2\left(\sqrt{3}-1\right)-\sqrt{2+x}\,\ln\left|\frac{\sqrt{2+x}+\sqrt{3}}{\sqrt{2+x}-\sqrt{3}}\cdot\frac{\sqrt{2+x}-1}{\sqrt{2+x}-1}\right| \tag{23}$$

Discrete vortex method (MDV) is computed as follows

$$R\left(t_j+\varepsilon\right)=\int_{-1}^{1}\frac{f(t)\,dt}{t-\left(t_j+\varepsilon\right)}-\sum_{k=1}^{N}\frac{f\left(t_k\right)h}{t_k-\left(t_j+\varepsilon\right)} \tag{24}$$

In the following Tables 1 and 2, we give some numerical results for QF which is computed by (10) in comparison with discrete vortex method (24).

Table 1: N=19, h=0.1, ε=h/2=0.05

J	x	Exact	QF(10)	MDV(23)	Error QF(10)	Error MDV(23)
1	-0.85	3.50945297	3.50004671	3.83400106	0.00940626	0.32454809
2	-0.75	2.97589515	1.96481899	3.14085145	0.00777002	0.16495631
5	-0.45	2.97589515	1.96481899	2.00241576	0.00343272	0.03416404
9	-0.05	0.85937671	0.86021137	0.82981108	0.00083465	0.02956564
10	0.05	0.56722871	0.56893212	0.52349425	0.00170341	0.04373446
11	0.15	0.25864346	0.26115329	0.1995157	0.00250982	0.05912777
15	0.55	-1.30398789	-1.29877653	-1.47012596	0.00521136	0.16613806
17	0.75	-2.5695796	-2.56332642	-2.90870964	0.006255318	0.33913004
18	0.85	-3.59033634	-3.58371227	-4.20038185	0.00662407	0.61004551

Table 2: N=19, h=0.1, e=3h/10=0.006 x=tj+e

J	x	Exact	QF(10)	MDV(23)	Error (10)	Error (23)
1	-0.974	5.22136546	5.21921387	6.69926564	0.032918	1.47790018
2	-0.954	4.66652574	4.66458	5.9558151	0.00194574	1.28928936
10	-0.794	3.18357856	3.18343851	4.368207	0.00014005	1.18462844
30	-0.394	1.181062184	1.81330641	301385954	0.00268458	1.32797356
49	-0.014	0.7557534	0.75943646	2.22260855	0.00368306	1.46685515
50	0.006	0.69747387	0.70117494	2.17125343	0.00370108	1.47377957
51	0.026	0.6386345	0.64235055	2.11929467	0.00371605	1.48066018
60	0.206	0.07639575	0.08011742	1.61685395	0.00372167	1.61685395
80	0.606	-1.60130914	-1.59829472	0.050805299	0.00301443	1.65211443
90	0.806	-3.08347374	-3.08112854	-1.41439163	0.0023452	1.41439163
97	0.946	-5.50750977	-5.50576759	-4.04466021	0.00174218	1.46284956
98	0.966	-6.34391529	-6.34228745	-5.092454	0.00162784	1.25146129

CONCLUSION

In Table 1 where the singular point x is located in the middle of the subinterval $[t_j, t_{j+1}]$, both methods show good convergence. Nevertheless quadrature formula (10) is better than (23) in terms of errors. Whereas Table 2 shows that when singular point x is not located in the middle of the interval, MDV (23) is not good fit but QF(10) still provides good convergence. Hence our QF provides convergence for any singular point in the interval (-1,1).

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