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Chromatically Unique Bipartite Graphs With Certain 3-independent Partition Numbers III

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ABSTRACT

For integers *p, q, s* with $p \geq q \geq 2$ and $s \geq 0$, let $K_2^{-s}(p,q)$ denote the set of 2-connected bipartite graphs which can be obtained from *K(p,q)* by deleting a set of s edges. In this paper, we prove that for any graph $G \in K_2^{-s}(p,q)$ with $p \geq q \geq 3$ and $1 \leq s \leq q - 1$ if the number of 3-independent partitions of G is $2^{p-1} + 2^{q-1} + s + 4$, then *G* is chromatically unique. This result extends both a theorem by Dong et al. [2]; and results in [4] and [5].

Keywords : Chromatic polynomial, Chromatically equivalence, Chromatically unique graphs.

INTRODUCTION

All graphs considered here are simple graphs. For a graph G, let *V(G)*, *E(G)*, *δ(G)*, *Δ(G)* and *P(G,λ)* be the vertex set, edge set, minimum degree, maximum degree and the chromatic polynomial of *G*, respectively.

Two graphs *G* and *H* are said to be *chromatically equivalent* (or simply χ - *equivalent*), symbolically $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. The equivalence class determined by *G* under ~ is denoted by [G]. A graph *G* is *chromatically unique* (or simply χ-unique) if $H \cong G$ whenever $H \sim G$, i.e, $[G] = \{G\}$ up to isomorphism. For a set ς of graphs, if $[G] \subseteq \varsigma$ for every $G \in \varsigma$, then ς is said to be $\chi - closed$. For two sets ς_1 and ς_2 of graphs, if $P(G_1, \lambda) \neq P(G_2, \lambda)$ for every $G_1 \in \varsigma_1$ and $G_2 \in \varsigma_2$, then ς_1 and ς_2 are said to be *chromatically disjoint*, or simply ÷-*disjoint* .

For integers *p, q, s* with $p \geq q \geq 2$ and $s \geq 0$, let K_2^{-s} *(p,q)* denote the set of connected (resp. 2-connected) bipartite graphs which can be obtained from $K_{p,q}$ by deleting a set of *s* edges.

For a bipartite graph $G = (A, B; E)$ with bipartition A and B and edge set E, let be $G'=(A',B';E')$ the bipartite graph induced by the edge set where $E' = \{xy \mid xy \notin E, x \in A, y \in B\}$ $A' \subseteq A$ and $B' \subseteq B$. We write $G' = K(p,q) - G$ where $p = |A|$ and $q = |B|$.

In [1], Dong et al. proved the following result.

Theorem 1.1 **:** For integers p, q, s with $p \geq q \geq 2$ and $0 \leq s \leq q - 1$, $\kappa_2^{-s}(p,q)$ is ÷-*closed.*

Throughout this paper, we fix the following conditions for p, q and s:

$$p \geq q \geq 3 \text{ and } 1 \leq s \leq q - 1 .$$

For a graph G and a positive integer k, a partition $\{A_1, A_2, \ldots A_k\}$ of $V(G)$ is called a k-independent partition in G if each A_i is a non-empty independent set of G. Let $\alpha(G, k)$ denote the number of *k–independent partitions* in G.
For any bipartite graph $G = (A, B; E)$, define

$$\alpha'(G,3) = \alpha(G,3) - \left(2^{|A|-1} + 2^{|B|-1} - 2\right)$$

In [1], the authors found the following sharp bounds for $\alpha'(G,3)$.

Theorem 1.2*:* For $G \in \kappa^{-s}(p,q)$ with $p \geq q \geq 3$ and $0 \leq s \leq q-1$

$$s \leq \alpha'(G,3) \leq 2^s - 1,$$

where $\alpha'(G,3) = s \text{ iff } \Delta(G') = 1$ and $\alpha'(G,3) = 2^s - 1 \text{ iff } \Delta(G') = s$.

For $t = 0,1,2,\ldots$, let $B(p, q, s, t)$ denote the set of graphs $G \in K^{-s}(p,q)$ with $\alpha'(G,3) = s + t$. Thus, $K^{-s}(p,q)$ is partitioned into the following subsets:

$$B(p,q,s,0), \quad B(p,q,s,1), \quad \ldots, B(p,q,s,2^s - s - 1).$$

Assume that $B(p, q, s, t) = \phi$ for $t > 2^s - s - 1$.

Lemma 1.1 : (Dong et al. [2]) For $p \leq q \geq 3$ and $0 \leq s \leq q-1$,if $0 \leq 1 \leq 2^{q-1} - q - 1$, then

$$\mathrm{B}(p,q,s,t) \subseteq \kappa_2^{-s}(p,q).$$

Dong et al. [1] have shown that if G is a 2-connected graph in $B(p,q,s,0)\cup B(p,q,s,2^s-s-1)$, then G is χ–unique. In [2], Dong *et al.* proved that every 2-connected graph in B*(p, q, s, t)* is χ–unique for $1 \le t \le 4$. In [4] and [5], we extended this result for t = 5 and t = 6, respectively. In this paper, we prove the chromatic uniqueness of graphs in B (*p, q, s,* 7).

PRELIMINARY RESULTS AND NOTATION

For any graph G of order n, we have (see [3]):

$$P(G,\lambda)=\sum_{k=1}^{n}\alpha(G,k)\lambda(\lambda-1)\dots(\lambda-k+1)$$

Thus, we have

Lemma 2.1 : If $G \sim H$, then $\alpha(G, k) = \alpha(H, k)$ for k = 1, 2,....

By Theorem 1.1, the following two results were obtained in [2].

Theorem 2.1 The set $\mathrm{B}(p,q,s,t)\cap\kappa_2^{-s}(p,q)$ is $\chi-closed$ for all $t\geq 0$.

Corollary 2.1 If $0 \le t \le 2^{q-1}-q-1$, then $\mathrm{B}(p,q,s,t)$ is $\chi-closed$ ...

Let $\beta_i(G)$, or simply β_i denote the number of vertices in G with degree i, $n_i(G)$ denote the number of i-cycles in G and P_n denote the path with n vertices. Then Dong et al. [2] established the next two results.

Lemma 2.2: For $G=(A,B;E)\in\kappa^{-s}(p,q)$,

i. if $\Delta(G')\le 2$, then $\alpha'(G,3)=s+\beta_2(G')+n_4(G')$;

ii if $\Delta(G')=3$, then $\alpha'(G,3)\ge s+\beta_2(G')+n_4(G')$, where equality holds *iff* $|N_{G'}(u)\cap N_{G'}(\upsilon)|\le 2$ for all $u,\upsilon\in \mathrm{B}'$; or $u,\upsilon\in\mathrm{B}'$

iii. $\alpha'(G,3)\ge 2^{\Delta(G')}+s-1-\Delta(G')$.

For two disjoint graphs H_1 and H_2, let $H_1\cup H_2$ denote the graph with vertex set $V(H_1)\cup V(H_2)$ and edge set $E(H_1)\cup\in(H_2)$. Let $kH=\underbrace{H\cup\dots\cup H}_{k}$ for k ≥ 1 and let kH be null if $k = 0$.

Lemma 2.3: Let $G \in \kappa^{-s}(p,q)$. If $\alpha'(G,3) = s+t \le s+4$, then either

(i) *each component of G'* is a path and $\beta_2(G') = t$, or

(ii) $G' \cong K_{1,3} \cup (s-3)K_2$

Now for convenience we define the graphs Y_n, Z_1, Z_2 and Z_3 as in Figure 1.

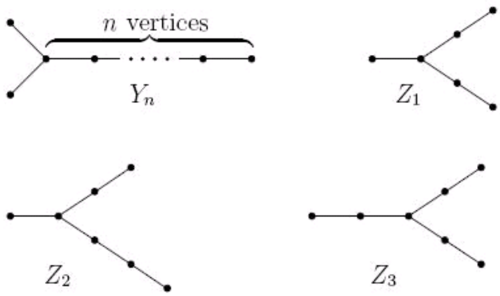

Figure 1: The graphs Y_n, Z_1, Z_2 and Z_3

The following result is an extension of Lemma 2.3.

Lemma 2.4: Let $G \in \kappa^{-s}(p,q)$. If $\alpha'(G,3) = s+7$, *then either*

i. *each component of G'* is a path and $\beta_2(G')=6$, or

ii. $G' \cong K_{1,3} \cup 3P_3 \cup (s-9)K_2$, or

iii. $G' \cong K_{1,3} \cup P_4 \cup P_3 \cup (s-8)K_2$, or

iv. $G' \cong K_{1,3} \cup P_5 \cup (s-8)K_2$, or

v. $G' \cong C_4 \cup 2P_3 \cup (s-8)K_2$, or

vi. $G' \cong C_4 \cup P_4 \cup (s-7)K_2$, or

vii. $G' \cong C_6 \cup P_3 \cup (s-8)K_2$, or

viii. $G' \cong Y_3 \cup P_4 \cup (s-7)K_2$, or

ix. $G' \cong Y_3 \cup 2P_3 \cup (s-8)K_2$, or

x. $G' \cong Y_4 \cup P_3 \cup (s-7)K_2$, or

xi. $G' \cong Y_5 \cup (s-6)K_2$, or

xii. $G' \cong Z_1 \cup P_3 \cup (s-7)K_2$, or

xiii. $G' \cong Z_2 \cup (s-6)K_2$, or

xiv. $G' \cong Z_3 \cup (s-6)K_2$

Proof. Since $\alpha'(G,3)=s+7, \Delta(G')=3$ by Lemma 2.2(iii). If $\Delta(G')\le 3$, by Lemma 2.2(ii), we have $\beta_2(G')=3, n_4(G')=0$ and $\beta_3(G')=1$.

Thus $G'\cong K_{1,3}\cup 3P_3\cup(s-9)K_2$ or $G'\cong K_{1,3}\cup P_4\cup P_3\cup(s-8)K_2$ or $G'\cong K_{1,3}\cup P_5\cup(s-7)K_2$ or $G'\cong Y_3\cup P_4\cup(s-7)K_2$ or $G'\cong Y_3\cup 2P_3\cup(s-8)K_2$ or $G'\cong Y_4\cup P_3\cup(s-7)K_2$ or $G'\cong Y_5\cup(s-6)K_2$ or $G'\cong Z_1\cup P_3\cup(s-7)K_2$ or $G'\cong Z_2\cup(s-6)K_2$ or $G'\cong Z_3\cup(s-6)K_2$. If $\Delta(G')=2$, we have $\beta_2(G')+n_4(G')=7$ by Lemma 2.2(i), and thus either *G'* contains no cycles or only have one cycle. Hence, when $\Delta(G')=2$, either each component of *G'* is a path, and $\beta_2(G')=7$, or $G'\cong K_{2,2}\cup 2P_3\cup(s-8)K_2$, $G'\cong K_{2,2}\cup P_4\cup(s-7)K_2$ and $G'\cong C_6\cup P_3\cup(s-8)K_2$ Lemma 2.2(i).

By Lemma 2.4, we have the following result.

Theorem 2.2 : Let $G\in k^{-s}(p,q)$ and $\alpha'(G,3)=s+7$, then

$$
\begin{aligned}
G'\in\{&P_9\cup(s-8)K_{2,}P_8\cup P_3\cup(s-9)K_2,\\
&P_7\cup 2P_3\cup(s-10)K_2, P_6\cup P_4\cup P_3\cup(s-10)K_2, P_6\cup 3P_3\cup(s-10)K_2,\\
&P_7\cup P_4\cup(s-9)K_{,}P_6\cup P_5\cup(s-9)K_2, 2P_5\cup P_3\cup(s-10)K_2,\\
&P_5\cup 4P_4\cup(s-12)K_2, P_5\cup P_4\cup 2P_3\cup(s-11)K_2, P_5\cup 2P_4\cup(s-10)K_2,\\
&3P_4\cup P_3\cup(s-11)K_2, 2P_4\cup 3P_3\cup(s-12)K_2, P_4\cup 5P_3\cup(s-13)K_2,\\
&7P_3\cup(s-14)K_2, K_{1,3}\cup 3P_3\cup(s-9)K_2, K_{1,3}\cup P_4\cup P_3\cup(s-8)K_2,\\
&K_{1,3}\cup P_5\cup(s-7)K_2, K_{2,2}\cup 2P_3\cup(s-8)K_2, K_{2,2}\cup P_4\cup(s-7)K_2,\\
&C_6\cup P_3\cup(s-8)K_2, Y_3\cup P_4\cup(s-7)K_2, Y_3\cup 2P_3\cup(s-8)K_2,\\
&Y_4\cup P_3\cup(s-7)K_2, Y_5\cup(s-6)K_2, Z_1\cup P_3\cup(s-7)K_2,\\
&Z_2\cup(s-6)K_2, Z_3\cup(s-6)K_2\}
\end{aligned}
$$

where $H\cup(s-i)K_2$ does not exist if $s<i$.

For a bipartite graph *G = (A, B; E)*, let

Ù(G) = {*Q* | *Q* is an independent set in *G* with $Q\cap A\neq\phi, Q\cap B\neq\phi\}$.

For a bipartite graph *G = (A, B; E)*, the number of 4−independent partitions $\{A_1,A_2,A_3,A_4\}$ in *G* with $A_i\subseteq A$ or $A_i\subseteq B$ for all *i* = 1, 2, 3,4 is

$$\left(2^{|A|-1}-1\right)\left(2^{|B|-1}-1\right)+\frac{1}{3!}\left(3^{|A|}-3.2^{|A|}+3\right)+\frac{1}{3!}\left(3^{|B|}-3.2^{|B|}+3\right)$$

$$=\left(2^{|A|-1}-2\right)\left(2^{|B|-1}-2\right)+\frac{1}{2}\left(3^{|A|-1}+3^{|B|-1}\right)-2$$

Define

$$\alpha'(G,4)=\alpha(G,4)-\left\{\left(2^{|A|-1}-2\right)\left(2^{|B|-1}-2\right)+\frac{1}{2}\left(3^{|A|-1}+3^{|B|-1}\right)-2\right\}.$$

Observed that for $G,H\in K^{-s}(p,q)$,

$$\alpha(G,4)=\alpha(H,4) \text{ iff } \alpha'(G,4)=\alpha'(H,4)$$

The following five lemmas (see [2]) will be used to prove our main results.

Lemma 2.5 *:* For $G=(A,B;E)\in K^{-s}(p,q)$ with $|A|=p$ and $|B|=q$,

$$\alpha'(G,4)=\sum_{Q\in\Omega(G)}\left(2^{p-1-|Q\cap A|}+2^{q-1|Q\cap B|}-2\right)+$$

$$\left|\left\{\{Q_1,Q_2\}\middle|Q_1,Q_2\in\Omega(G),Q_1\cap Q_2=\emptyset\right\}\right|.$$

Lemma 2.6 : *For a bipartite graph* $G=(A,B;E)$, if *uvw* is a path in *G'* with $d_{G'}(u)=1$ and $d_{G'}(v)=2$, then for any *k*≥2,

$$\alpha(G,K)=\alpha(G+uv,k)+\alpha(G-\{u,v,w\},k-1)$$

For a bipartite graph $G=(A,B;E)$ let $\beta_i(G,A)\left(resp.,\beta_i(G,B)\right)$ be the number of vertices in $A(resp.,B)$ with degree *i.*

Lemma 2.7: For $G\in \mathrm{B}(p,q,s,t)$, *if each component of G' is a path, then*

$$\sum_{Q\in\Omega(G)}(2^{p-1-|Q\cap A|}+2^{q-1-|Q\cap B|}-2)$$

$$=s\left(2^{p-2}+2^{q-2}-2\right)+t\left(2^{p-3}+2^{q-2}-2\right)+\left(2^{p-3}-2^{q-3}\right)\beta_2\left(G',A'\right).$$

Let $p_i(G)$ denote the number of paths P_i in *G*.

Lemma 2.8 : For $G\in\mathrm{B}(p,q,s,t)$, *if each component of G' is a path, then*

$$\left|\left\{\{Q_1,Q_2\}\middle|Q_1,Q_2\in\Omega(G),Q_1\cap Q_2=\phi\right\}\right|=\binom{s+t}{2}-3t-3p_4\left(G'\right)-p_5\left(G'\right).$$

For $G \in \mathrm{B}(p,q,s,t)$, define

$$\alpha''(G,4) = \alpha(G,4) - \Big[s\left(2^{p-2} + 2^{q-2} - 2\right) + t\left(2^{p-3} + 2^{q-2} - 2\right) + (s+t)(s+t-1)/2 - 3t \Big] \qquad (1)$$

Observe that for $G, H \in \mathrm{B}(p,q,s,t)$,

$$\alpha''(G,4) = \alpha''(H,4) \ \textit{iff} \ \ \alpha(G,4) = \alpha(H,4)$$

Lemma 2.9*:* For $G \in \mathrm{B}(p,q,s,t)$, if each component of *G'* is a path, then

$$\alpha''(G,4) = \left(2^{p-3} - 2^{q-3}\right)\beta_2\left(G', A'\right) - 3p_4\left(G'\right) - p_5\left(G'\right).$$

MAIN RESULT

Dong et al. [1] have shown that any 2-connected graph G is $\mathrm{B}(p,q,s,0) \cup \mathrm{B}(p,q,s,2^s - s - 1)$ is ÷-unique. In [2], Dong et al. proved that every 2-connected graph in $\mathrm{B}(p,q,s,t)$ is ÷-unique for $1 \le t \le 4$. In [4] and [5], we proved this result for $t = 5$ and $t = 6$ respectively. In this section, we shall prove that every 2-connected graph in $\mathrm{B}(p,q,s,t)$ is ÷-unique for $t = 7$.

The following theorem is our main result:

Theorem 3.1 : Let *p, q* and *s* be integers with $p \ge q \ge 3$ and $0 \le s \le q - 1$. For every $G \in \mathrm{B}(p,q,s,7)$, if *G* is 2-connected, then *G* is ÷-unique.

Proof : By Theorem 2.1, $\mathrm{B}(p,q,s,t) \cap \kappa_2^{-s}(p,q)$ is ÷-closed for all t ≥ 0. Hence, to show that every 2-connected graph in $\mathrm{B}(p,q,s,t)$ is ÷-unique, it suffices to show that for every two graphs *G* and *H* in $\mathrm{B}(p,q,s,t)$, if *G* ® *H* then either $\alpha(G,4) \ne \alpha(H,4)$ or $\alpha(G,5) \ne (H,5)$. Recall that $\alpha''(G,4) = \alpha''(H,4)$ iff $\alpha(G,4) = \alpha(H,4)$ and $\alpha'(G,4) = \alpha'(H,4)$ *iff* $\alpha(G,4) = \alpha(H,4)$.

The set $\mathrm{B}(p,q,s,7)$ consists of 108 graphs by Theorem 2.2, named as $G_{7,1}, G_{7,2}, G_{7,3,\ldots}, G_{7,108}$ (see Table 1 in [7]). This graphs are shown in this table with the values $\alpha''(G_{7,1},4), \alpha''(G_{7,2},4), \ldots \alpha''(G_{7,108},4)$. For each graph $G_{7,i,}$ if every component of $G'_{7,i}$ is a path, then $\alpha''(G_{7,i},4)$ can be obtained by Lemma 2.9; otherwise, we must find

$\alpha'(G_{7,i},4)$ by Lemma 2.5, and then we find $\alpha''(G_{7,i},4)$ by using Equation (1).

$$T_1 = \{G_{7,1,} G_{7,2}\}$$
$$T_2 = \{G_{7,3,} G_{7,4,\ldots,} G_{7,7}\}$$
$$T_3 = \{G_{7,8,} G_{7,9,\ldots,} G_{7,17}\} =$$
$$T_4 = \{G_{7,18,} G_{7,19,\ldots} G_{7,23}\}$$
$$T_5 = \{G_{7,24,} G_{7,25,\ldots,} G_{7,31}\}$$
$$T_6 = \{G_{7,32,} G_{7,33}\}$$
$$T_7 = \{G_{7,34,} G_{7,35,\ldots} G_{7,37}\}$$
$$T_8 = \{G_{7,38,} G_{7,39,\ldots,} G_{7,43}\}$$
$$T_9 = \{G_{7,44,} G_{7,45,\ldots} G_{7,53}\}$$
$$T_{10} = \{G_{7,54,} G_{7,55,\ldots} G_{7,63}\}$$
$$T_{11} = \{G_{7,64,} G_{7,65,\ldots} G_{7,67}\}$$
$$T_{12} = \{G_{7,68,} G_{7,69,\ldots} G_{7,73}\}$$
$$T_{13} = \{G_{7,74,} G_{7,75,\ldots} G_{7,81}\}$$
$$T_{14} = \{G_{7,82,} G_{7,83}\}$$
$$T_{15} = \{G_{7,84}, G_{7,85}\}$$
$$T_{16} = \{G_{7,86,} G_{7,87,\ldots,} G_{7,91}\}$$
$$T_{17} = \{G_{7,92,} G_{7,93,\ldots,} G_{7,97}\}$$
$$T_{18} = \{G_{7,98}\}$$
$$T_{19} = \{G_{7,99,} G_{7,100}\}$$
$$T_{20} = \{G_{7,101,} G_{7,102,\ldots,} G_{7,104}\}$$
$$T_{21} = \{G_{7,105,} G_{7,106,\ldots,} G_{7,108}\}$$

For non-empty sets $W_1, W_2, ..., W_k$ of graphs, let $\eta(W_1, W_{2,...}, W_k) = 0$ if $\alpha(G_{1,} 4) \neq \alpha(G_{2,} 4)$ for every two graphs $G_1 \in W_i$ and $G_2 \in W_{j,}$ where $i \neq j$ and let $\eta(W_1, W_{2,...}, W_k) = 1$ otherwise. The values of $\alpha''(G_{7,i}, 4)$ for i=5,6,7;12,13,...,17;46,47,...,53;82,83,...,108 are not given by Lemma 2.9, but they can be obtained by Lemma 2.5 and Equation (1). For example, we show a detailed computation of $\alpha''(G_{7,5,} 4)$ below :

i. $$\alpha''(G_{7,5}, 4) = \Big[s(2^{p-2} + 2^{q-2} - 2) + 4(2^{p-3} + 2^{q-2} - 2) + 2(2^{2q-3} + 2^{p-2} - 2) + (2^{p-3} + 2^{q-3} - 2) + \left\{ 15(s-8) + \binom{s-8}{2} + 65 \right\} \Big] - \left[s(2^{p-2} + 2^{q-2} - 2) + 7(2^{p-3} + 2^{q-2} - 2) + \binom{s-7}{2} - 21 \right] = 2^{p-2} - 3 \cdot 2^{q-3} - 19$$

Similarly, we obtain $\alpha''(G_{7,i}, 4)$ for other values of i as follows (see Table 1 in [7].

ii. $\alpha''(G_{7,6}, 4) = 3 \cdot 2^{p-3} - 2^{q-1} - 19.$

iii. $\alpha''(G_{7,7}, 4) = 2^{p-1} - 5 \cdot 2^{q-3} - 19.$

iv. $\alpha''(G_{7,12}, 4) = 2^{p-4} - 2^{q-3} - 18.$

v. $\alpha''(G_{7,13}, 4) = 3 \cdot 2^{p-4} - 2^{q-2} - 18.$

vi. $\alpha''(G_{7,14}, 4) = 5 \cdot 2^{p-4} - 3 \cdot 2^{q-3} - 18.$

vii. $\alpha''(G_{7,15}, 4) = 2^{p-1} - 9 \cdot 2^{q-4} - 18.$

viii. $\alpha''(G_{7,16}, 4) = 5 \cdot 2^{p-3} - 11 \cdot 2^{q-4} - 18.$

ix. $\alpha''(G_{7,17}, 4) = 6 \cdot 2^{p-3} - 13 \cdot 2^{q-4} - 18.$

x. $\alpha''(G_{7,46}, 4) = -2^{p-4} - 9.$

xi. $\alpha''(G_{7,47}, 4) = 2^{p-4} - 2^{q-3} - 9.$

xii. $\alpha''(G_{7,48}, 4) = 3 \cdot 2^{p-4} 2^{q-2} - 9.$

xiii. $\alpha''(G_{7,49}, 4) = 5 \cdot 2^{p-4} - 3 \cdot 2^{q-3} - 9.$

xiv. $\alpha''(G_{7,50}, 4) = 2^{p-1} - 9 \cdot 2^{q-4} - 9.$

xv. $\alpha''(G_{7,51}, 4) = 5.2^{p-3} - 11 \cdot 2^{q-4} - 9.$

xvi. $\alpha''\left(G_{7,52},4\right)=6\cdot 2^{p-3}-13\cdot 2^{q-4}-9.$

xvii. $\alpha''\left(G_{7,53},4\right)=7\cdot 2^{p-3}-15\cdot 2^{q-4}-9.$

xiix. $\alpha''\left(G_{7,82},4\right)=5.2^{p-4}-3\cdot 2^{q-3}-39.$

xix. $\alpha''\left(G_{7,83},4\right)=2^{p-1}-9\cdot 2^{q-4}-39.$

xx. $\alpha''\left(G_{7,84},4\right)=3\cdot 2^{p-4}-2^{q-2}-34.$

xxi. $\alpha''\left(G_{7,85},4\right)=5\cdot 2^{p-3}-11\cdot 2^{q-4}-34.$

xxii. $\alpha''\left(G_{7,86},4\right)=3\cdot 2^{p-4}-2^{q-2}-28.$

xxiii. $\alpha''\left(G_{7,87},4\right)=5\cdot 2^{p-4}-3\cdot 2^{q-3}-28.$

xxiv. $\alpha''\left(G_{7,88},4\right)=2^{p-1}-9\cdot 2^{q-4}-28.$

xxv. $\alpha''\left(G_{7,89},4\right)=5.2^{p-3}-11\cdot 2^{q-4}-28.$

xxvi. $\alpha''\left(G_{7,90},4\right)=3\cdot 2^{p-4}-2^{q-2}-28.$

xxvii. $\alpha''\left(G_{7,91},4\right)=5\cdot 2^{p-3}-11\cdot 2^{q-4}-28.$

xxviii. $\alpha''\left(G_{7,92},4\right)=3\cdot 2^{p-3}-3\cdot 2^{q-3}-24.$

xxix. $\alpha''\left(G_{7,93},4\right)=4\cdot 2^{p-3}-4\cdot 2^{q-3}-24.$

xxx. $\alpha''\left(G_{7,94},4\right)=2^{p-4}-2^{q-3}-24.$

xxxi. $\alpha''\left(G_{7,95},4\right)=3.2^{p-4}-2^{q-2}-24.$

xxxii. $\alpha''\left(G_{7,96},4\right)=5\cdot 2^{p-3}-11\cdot 2^{q-4}-24.$

xxxiii. $\alpha''\left(G_{7,97},4\right)=6\cdot 2^{p-3}-2^{q-4}-24.$

xxxiv. $\alpha''\left(G_{7,98},4\right)=3\cdot 2^{p-3}-2^{q-1}-22.$

xxxv. $\alpha''\left(G_{7,99},4\right)=3\cdot 2^{p-4}-2^{q-2}-21.$

xxxvi. $\alpha''\left(G_{7,100},4\right)=5\cdot 2^{p-3}-11\cdot 2^{q-4}-21.$

xxxvii. $\alpha''\left(G_{7,101},4\right)=2^{p-4}-2^{q-3}-16.$

xxxviii. $\alpha''\left(G_{7,102},4\right)=3\cdot 2^{p-4}-2^{q-2}-16.$

xxxix. $\alpha''\left(G_{7,103},4\right)=5\cdot 2^{p-3}-11.2^{q-4}-16.$

xxxx. $\alpha''\left(G_{7,104},4\right)=6\cdot 2^{p-3}-13\cdot 2^{q-4}-16.$

xli. $\alpha''\left(G_{7,105},4\right)=2^{p-4}-2^{q-3}-12.$

xlii. $\alpha''(G_{7,106},4)=3\cdot 2^{p-4}-2^{q-2}-12.$

xliii. $\alpha''(G_{7,107},4)=5\cdot 2^{p-3}-11\cdot 2^{q-4}-12.$

xliv. $\alpha''(G_{7,108},4)=6\cdot 2^{p-3}-13\cdot 2^{q-4}-12.$

Claim 1. $\eta(T_1,T_2,T_3,...,T_{21})=0$:

Proof of Claim 1. Note that if 2^k (k is an integer ≥ 1) is not a factor of x, then 2^h is also not a factor of x for any integer $h \geq k$. Similarly, if 2^k (k is an integer ≥ 1) is a factor of x, then 2^h is also a factor of x for any integer $1 \leq h \leq k$.

a. For s=6,T_{14},T_{15} and T_{16} are non-empty. Observe that 2^3 is a factor of $\alpha''(G,4)+39$ for $G \in \mathrm{T}_{14}$ but 2^3 is not a factor of $\alpha''(G,4)+39$ for $G \in \mathrm{T}_{15} \cup \mathrm{T}_{16}$. Hence, $\eta(\mathrm{T}_{14,}\mathrm{T}_{15} \cup \mathrm{T}_{16})=0$.

b. For s=6,2^3 is a factor of $\alpha''(G,4)+34$ for $G \in \mathrm{T}_{15}$ but 2^3 is not a factor of $\alpha''(G,4)+34$ for $G \in \mathrm{T}_{16}$. Hence, $\eta(\mathrm{T}_{15},\mathrm{T}_{16})=0$.

c. For $s \geq 7, \alpha''(G,4)$ is odd if $G \in \mathrm{T}_1 \cup \mathrm{T}_2 \cup \mathrm{T}_4 \cup \mathrm{T}_6 \cup \mathrm{T}_7 \cup \mathrm{T}_9 \cup \mathrm{T}_{10} \cup \mathrm{T}_{12} \cup \mathrm{T}_{14} \cup \mathrm{T}_{19}$ and even if $G \in \mathrm{T}_3 \cup \mathrm{T}_5 \cup \mathrm{T}_8 \cup \mathrm{T}_{11} \cup \mathrm{T}_{13} \cup \mathrm{T}_{15} \cup \mathrm{T}_{16} \cup \mathrm{T}_{17} \cup \mathrm{T}_{18} \cup \mathrm{T}_{20} \cup \mathrm{T}_{21}$. Hence
$\eta(\mathrm{T}_1 \cup \mathrm{T}_2 \cup \mathrm{T}_4 \cup \mathrm{T}_6 \cup \mathrm{T}_7 \cup \mathrm{T}_9 \cup \mathrm{T}_{10} \cup \mathrm{T}_{12} \cup \mathrm{T}_{14} \cup \mathrm{T}_{19} \cup \mathrm{T}_3 \cup \mathrm{T}_5 \cup \mathrm{T}_8 \cup \mathrm{T}_{11} \cup \mathrm{T}_{13} \cup \mathrm{T}_{15} \cup \mathrm{T}_{16} \cup \mathrm{T}_{17} \cup \mathrm{T}_{18} \cup \mathrm{T}_{20} \cup \mathrm{T}_{21})=0$

d. For $s \geq 7$, 2^3 is a factor of $\alpha''(G,4)+39$ for $G \in \mathrm{T}_1 \cup \mathrm{T}_4 \cup \mathrm{T}_{10} \cup \mathrm{T}_{14}$ but 2^3 is not factor of $\alpha''(G,4)+39$ for $G \in \mathrm{T}_2 \cup \mathrm{T}_6 \cup \mathrm{T}_7 \cup \mathrm{T}_9 \cup \mathrm{T}_{12} \cup \mathrm{T}_{19}$. Hence
$\eta(\mathrm{T}_1 \cup \mathrm{T}_4 \cup \mathrm{T}_{10} \cup \mathrm{T}_{14}, \cup \mathrm{T}_2 \cup \mathrm{T}_6 \cup \mathrm{T}_7 \cup \mathrm{T}_9 \cup \mathrm{T}_{12} \cup \mathrm{T}_{19})=0$.

e. For s≥7,2^6 is a factor of $\alpha''(G,4)+23$ for $G \in \mathrm{T}_{1,}$ 2^4 is not a factor of $\alpha''(G,4)+23$ for $G \in \mathrm{T}_4$, and 2^4 is a factor of $\alpha''(G,4)+23$ but 2^6 is not for $G \in \mathrm{T}_{10} \cup \mathrm{T}_{14}$. Hence $\eta(\mathrm{T}_{1,}\mathrm{T}_{4,}\mathrm{T}_{10} \cup \mathrm{T}_{14})=0$.

f. For s≥7,2^6 is a factor of $\alpha''(G,4)+7$ for $G \in \mathrm{T}_{10}$ but 2^6 is not a factor of $\alpha''(G,4)+7$ for $G \in \mathrm{T}_{14}$. Hence $\eta(\mathrm{T}_{10,}\mathrm{T}_{14})=0$.

g. For s≥7,2^4 is a factor of $\alpha''(G,4)+21$ for $G \in \mathrm{T}_{19,}$ 2^3 is not a factor of $\alpha''(G,4)+21$ for $G in \mathrm{T}_2 \cup \mathrm{T}_7 \cup \mathrm{T}_9 \cup \mathrm{T}_{12,}$ and 2^3 is a factor of $\alpha''(G,4)+21$ but 2^4 is not for $G \in \mathrm{T}_6$. Hence $\eta(\mathrm{T}_{19,}\mathrm{T}_6 \cup \mathrm{T}_2 \cup \mathrm{T}_7 \cup \mathrm{T}_9 \cup \mathrm{T}_{12})=0$.

h. For $s \geq 7, 2^5$ is a factor of $\alpha''(G,4)+19$ for $G \in \mathrm{T}_2, 2^3$ is not a factor of $\alpha''(G,4)+19$ for $G \in \mathrm{T}_9$, and 2^3 is a factor of $\alpha''(G,4)+19$ but 2^5 is not for $G \in \mathrm{T}_7 \cup \mathrm{T}_{12}$. Hence $\eta(\mathrm{T}_2,\mathrm{T}_9,\mathrm{T}_7 \cup \mathrm{T}_{12})=0$.

i. For $s \geq 7, 2^4$ is a factor of $\alpha''(G,4)+11$ for $G \in T_7$ but is not a factor of $\alpha''(G,4)+11$ for $G \in T_{12}$. Hence $\eta(T_7, T_{12}) = 0$.

j. For $s \geq 7, 2^3$ is a factor of $\alpha''(G,4)+34$ for $G \in T_3 \cup T_8 \cup T_{15}$ but 2^3 is not factor of $\alpha''(G,4)+34$ for $G \in T_5 \cup T_{11} \cup T_{13} \cup T_{16} \cup T_{17} \cup T_{18} \cup T_{20} \cup T_{21}$. Hence $\eta(T_3 \cup T_8 \cup T_{15} \cup T_5 \cup T_{11} \cup T_{13} \cup T_{16} \cup T_{17} \cup T_{18} \cup T_{20} \cup T_{21}) = 0.$

k. For $s \geq 7, 2^5$ is a factor of $\alpha''(G,4)+18$ for $(G \in T_3, 2^4)$ is not a factor of $\alpha''(G,4)+18$ for $G \in T_8$, and 2^4 is a factor of $\alpha''(G,4)+18$ but 2^5 is not for $G \in T_{15}$. Hence $\eta(T_3, T_8, T_{15}) = 0$.

l. For $s \geq 7, 2^3$ is a factor of $s \geq 7, 2^3$ for $\alpha''(G,4)+28$ for $G \in T_{16} \cup T_{21}$ but 2^3 is not a factor of $\alpha''(G,4)+28$ for $G \in T_5 \cup T_{11} \cup T_{13} \cup T_{17} \cup T_{18} \cup T_{20}$. Hence $\eta(T_{16} \cup T_{21}, T_5 \cup T_{11} \cup T_{13} \cup T_{17} \cup T_{18} \cup T_{20}) = 0$.

m. For $s \geq 7, 2^4$ is not factor of $\alpha''(G,4)+12$ for $G \in T_{21}$ but it is a factor of $\alpha''(G,4)+12$ for $G \in T_{16}$. Hence $\eta(T_{16}, T_{21}) = 0$.

n. For $s \geq 7, 2^4$ is a factor of $\alpha''(G,4)+24$ for $G \in T_{17}, 2^3$ is not a factor of $\alpha''(G,4)+24$ for $G \in T_5 \cup T_{11} \cup T_{18}$, and 2^3 is a factor of $\alpha''(G,4)+24$ but 2^4 is not for $G \in T_{13} \cup T_{20}$. Hence $\eta(T_{17}, T_5 \cup T_{11} \cup T_{18}, T_{13} \cup T_{20}) = 0.$

o. For $s \geq 7, 2^5$ is a factor of $\alpha''(G,4)+22$ for $G \in T_{18}, 2^4$ is not a factor of $\alpha''(G,4)+22$ for $G \in T_5$, and 2^4 is a factor of $\alpha''(G,4)+22$ but 2^5 is not for $G \in T_{11}$. Hence $\eta(T_5, T_{11}, T_{18}) = 0$.

p For $s \geq 7, 2^5$ is a factor of $\alpha''(G,4)$ for $G \in T_{13}$ but 2^5 is not a factor of $\alpha''(G,4)$ for $G \in T_{20}$. Hence $\eta(T_{13}, T_{20}) = 0$.

By (a) to (p), Claim 1 holds.
The remaining work is to compare every two graphs in each T_i except T_{18} since it contains only one graph. Since this comparison process is standard, long and rather repetitive, we shall not discuss all here. In the following we only show the detail comparisons of every two graphs in T_i for i = 1, 2 and 3. The reader may refer to [6] for complete comparisons.

1. T_i

1.1. When $p = q, G_{7,1} \cong G_{7,2}$.

1.2. When $p>q$ from Table 1 [7], we can easily see that $\alpha''(G_{7,1},4)-\alpha''(G_{7,2},4)<0.$

2. T_2

2.1. When $p=q, G_{7,3}\cong G_{7,4,} G_{7,5}\cong G_{7,7}$ and

$$\begin{aligned}
&\alpha''(G_{7,5,}4)-\alpha''(G_{7,6,}4)\\
&=\left(2^{p-2}-3\cdot 2^{q-3}-19\right)-\left(3\cdot 2^{p-3}-2^{q-1}-19\right)\\
&=-2^{p-3}+2^{q-3}=0,
\end{aligned}$$

$$\begin{aligned}
&\alpha''(G_{7,5,}4)-\alpha''(G_{7,3,}4)\\
&=\left(2^{p-2}-3\cdot 2^{q-3}-19\right)-\left(3\cdot 2^{p-3}-3\cdot 2^{q-3}-19\right)\\
&=-2^{p-3}.
\end{aligned}$$

Thus, we have

$$\alpha''(G_{7,5},4)=\alpha''(G_{7,6},4)<\alpha''(G_{7,3},4).$$

Since $\alpha''(G_{7,5},4)=\alpha''(G_{7,6},4)$, we need to calculate $\alpha(G_{7,5},5)-\alpha(G_{7,6},5)$. By using Lemma 2.6, we have

$$\begin{aligned}
&\alpha(G_{7,5},5)-\alpha(G_{7,6},5)\\
&=\left[\alpha(G_{7,5}+a_1b_1,5)+\alpha(G_{7,5}-\{a_1,b_1\},4)\alpha(G_{7,5}-\{a_1,b_1,c_1\},4)\right]-\\
&\quad\left[\alpha(G_{7,6}+a_1'b_1',5)+\alpha(G_{7,6}-\{a_1'b_1'\},4)+\alpha(G_{7,6}-\{a_1',b_1',c_1'\},4)\right]\\
&=\alpha(G_{7,5}-\{a_1,b_1,c_1\},4)-\alpha(G_{7,6}-\{a_1',b_1',c_1'\},4)\\
&\qquad\text{since } G_{7,5}+a_1b_1\cong G_{7,6}+a_1'b_1', \text{ and } G_{7,5}-\{a_1,b_1\}\cong G_{7,6}-\{a_1'b_1'\}\\
&=\alpha'(G_{7,5}-\{a_1,b_1,c_1\}4)-\alpha'(G_{7,6}-\{a_1',b_1',c_1'\}4).
\end{aligned}$$

Since $G_{7,5}-\{a_1,b_1,c_1\}\in \mathrm{B}(p-2,q-1,s-2,6)$ and

$G_{7,6}-\{a_{1,}b_{1,}c_1\}\in \mathrm{B}(p-1,q-2,s-2,6)$,

by Lemma 2.5, we have

$$\begin{aligned}
&\alpha(G_{7,5},5)-\alpha(G_{7,6},5)\\
&=\alpha'(G_{7,5}-\{a_1,b_1,c_1\},4)-\alpha'(G_{7,6}-\{a_1',b_1',c_1'\},4)
\end{aligned}$$

$$=\Big[(s-2)\big(2^{p-4}+2^{q-3}-2\big)+3\big(2^{p-5}+2^{q-3}-2\big)+2\big(2^{p-4}+2^{q-4}-2\big)+$$

$$\big(2^{p-5}+2^{q-4}-2\big)+\left\{12(s-8)+29+\binom{s-8}{2}\right\}\Big]-$$

$$\Big[(s-2)\big(2^{p-3}+2^{q-4}-2\big)+2\big(2^{p-3}+2^{q-5}-2\big)+$$

$$\big(2^{p-4}+2^{q-5}-2\big)+\left\{12(s-8)+29+\binom{s-8}{2}\right\}\Big]$$

$$=2^{p-5}\big[2(s-2)+8-2^2(s-2)-16\big]+2^{q-5}\big[2^2(s-2)+18-2(7)\big]$$

$= 2^{q-5}$ (since $p=q$).

Thus, we have

$$\alpha\left(G_{7,5},5\right)>\alpha\left(G_{7,6},5\right).$$

.

(2.2) When $p > q$

$$\alpha''\left(G_{7,5},4\right)>\alpha''\left(G_{7,6},4\right)$$

$$=\left(2^{p-2}-3\cdot 2^{q-3}-19\right)-\left(3\cdot 2^{p-3}-2^{q-1}-19\right)$$

$$=-2^{p-3}+2^{q-3}<0,$$

$$\alpha''\left(G_{7,6},4\right)-\alpha''\left(G_{7,3},4\right)$$

$$=\left(3\cdot 2^{p-3}-2^{q-1}-19\right)-\left(3\cdot 2^{p-3}-3\cdot 2^{q-3}-19\right)$$

$$=2^{q-3}<0,$$

$$\alpha''\left(G_{7,6},4\right)-\alpha''\left(G_{7,7},4\right)$$

$$=\left(3\cdot 2^{p-3}-2^{q-1}-19\right)-\left(2^{p-1}-5\cdot 2^{q-3}-19\right)$$

$$=2^{p-3}+2^{q-3}<0,$$

$$\alpha''\left(G_{7,3},4\right)-\alpha''\left(G_{7,4},4\right)$$

$$=-2^{p-3}+2^{q-3}<0,$$

$$\alpha''\left(G_{7,7},4\right)-\alpha''\left(G_{7,4},4\right)$$

$$=\left(2^{p-1}-5\cdot 2^{q-3}-19\right)-\left(4\cdot 2^{p-3}-4\cdot 2^{q-3}-19\right)$$

$$=2^{q-3}<0,$$

$$\alpha''(G_{7,3},4)-\alpha''(G_{7,7},4)$$
$$=(3\cdot 2^{p-3}-3\cdot 2^{q-3}-19)-(2^{p-1}-5\cdot 2^{q-3})$$
$$=-2^{p-3}+2^{q-2}$$
$$\begin{cases} =0, \ if \ \ p=q+1; \\ <0, \ if \ \ p\geq q+2. \end{cases}$$

Therefore, if p ≥ q + 2, we have

$$\alpha''(G_{7,5},4)<\alpha''(G_{7,6},4)<\alpha''(G_{7,3},4)<\alpha''(G_{7,7},4)<\alpha''(G_{7,4},4);$$

And if $p=q+1$, we have

$$\alpha''(G_{7,5},4)<\alpha''(G_{7,6},4)<\alpha''(G_{7,3},4)=\alpha''(G_{7,7},4)<\alpha''(G_{7,4},4).$$

Since $\alpha''(G_{7,3},4)-\alpha''(G_{7,7},4)=0$ for the case $p=q+1$ we need to find $\alpha(G_{7,3},5)-\alpha(G_{7,7},5)$. Since Lemma 2.6 cannot be used to determine $\alpha(G_{7,3},5)-\alpha(G_{7,7},5)$, we shall use direct counting. We first need the following definitions.

For a graph G and $x\in V(G)$, let $N_G(x)$ or simply $N(x)$ denote the set of vertices y such that $xy\in E(G)$. Since $s\leq q-1\leq p-1$, there exist $x\in A$ and $y\in B$ such that $N_G(x)=B$ and $N_G(y)=A$. Thus, for any 5-independent partition $\{A_1,A_2,A_3,A_4,A_5\}$, there are at least two A_i and A_j with $A_i\subseteq A$ and $A_j\subseteq B$. This means that G has only four types of 5-independent partition. We call G has partition of type k for k=0,1,2,3, if there are exactly k A_i's with $A_i\in\Omega(G)$. For our purpose, we only need to consider the case for k=1,2 and 3. Recalled that for $G,H\in K^{-s}(p,q),\alpha(G,5)=\alpha(H,5)$ iff $\alpha'(G,5)=\alpha'(H,5)$.

Let $\alpha_i'(G_{7,3},5)$ and $\alpha_i'(G_{7,7},5)$ denote the number of 5-independent partition of type *i,* where $1\leq i\leq 3$. Observe that for $G,H\in\kappa^{-s}(p,q),\alpha(G,5)=\alpha(H,5)$ iff $\alpha'(G,5)=\alpha'(H,5)$. By direct counting, we obtained the following :

i. $\alpha_i'(G_{7,3},5)-\alpha_i'(G_{7,7},5)$

$$= \left[2\left(2^{q-2}-1\right)\left(2^{q-2}-1\right)+2\left(\frac{1}{2}\left(3^{q-2}+1\right)-2^{q-2}\right)+\right.$$

$$\left. 2\left(\frac{1}{2}\left(3^{q-2}+1\right)-2^{q-2}\right) \quad \right]\left[\quad \left(2^{q-1}-1\right)\left(2^{q-3}-1\right)+\right.$$

$$\left(\frac{1}{2}\left(3^{q-1}+1\right)-2^{q-1}\right)+\left(\frac{1}{2}\left(3^{q-3}+1\right)-2^{q-3}\right)+$$

$$\left.\left(2^{q-2}-1\right)\left(2^{q-3}-1\right)+\left(\frac{1}{2}\left(3^{q-2}+1\right)-2^{q-2}\right)+\left(\frac{1}{2}\left(3^{q-3}+1\right)-2^{q-3}\right)\right]$$

$$= 2^{q-5} - 3^{q-3}.$$

ii. $\alpha_2^{'}\left(G_{7,3},5\right)-\alpha_2^{'}\left(G_{7,7},5\right)$

$$= \left[4\left(2^{q-4}+2^{q-3}-2\right)+\left(2s-5\right)\left(2^{q-2}-2\right)\right]-$$

$$\left[\left(2^{q-3}+2^{q-4}-2\right)+4\left(2^{q-2}+2^{q-5}-2\right)+2\left(2^{q-3}+2^{q-5}-2\right)+\right.$$

$$\left.\left(s-1\right)\left(2^{q-2}+2^{q-4}-2\right)+\left(s-4\right)\left(2^{q-3}+2^{q-4}-2\right)\right]$$

$$= -34\cdot 2^{q-5}+6.$$

iii. $\alpha_3^{'}\left(G_{7,3},5\right)-\alpha_3^{'}\left(G_{7,7},5\right)$

$$= \left[7\binom{s-9}{2}+51\left(s-9\right)+107\right]-\left[7\binom{s-8}{2}+42\left(s-8\right)+65\right]$$

$$= 2_{s} - 18.$$

Thus, we have

$$\alpha\left(G_{7,3},5\right)-\alpha\left(G_{7,7},5\right)$$

$$= \left(\alpha_1^{'}\left(G_{7,3},5\right)-\alpha_1^{'}\left(G_{7,7},5\right)\right)+\left(\alpha_2^{'}\left(G_{7,3},5\right)-\alpha_2^{'}\left(G_{7,7},5\right)\right)+$$

$$\left(\alpha_3^{'}\left(G_{7,3},5\right)-\alpha_3^{'}\left(G_{7,7},5\right)\right)$$

$$= \left(2^{2q-5}-3^{q-3}\right)+\left(-34\cdot 2^{q-5}+6\right)+\left(2s-18\right)$$

$$= 2^{2q-5}-34\cdot 2^{q-5}-3^{q-3}+2s-12\neq 0.$$

3. T_3

3.1. When $p=q, G_{7,8}\cong G_{7,9,}G_{7,10}\cong G_{7,11,}G_{7,12}\cong G_{7,17,}G_{7,13}\cong G_{7,16,}G_{7,14}\cong G_{7,15}$ and

$$\alpha''(G_{7,8},4)-\alpha''(G_{7,10},4)=0 \qquad (2)$$

$$\alpha''(G_{7,8},4)-\alpha''(G_{7,12},4)$$
$$=\left(3\cdot 2^{p-3}-3\cdot 2^{q-3}-18\right)-\left(2^{p-4}-2^{q-3}-18\right)$$
$$=2^{p-4}>0,$$

$$\alpha''(G_{7,12},4)-\alpha''(G_{7,13},4)$$
$$=\left(2^{p-4}-2^{q-3}-18\right)-\left(3\cdot 2^{p-4}-2^{q-2}-18\right)$$
$$=-2^{p-3}+2^{q-3} \qquad (3)$$
$$=0,$$

$$\alpha''(G_{7,12},4)-\alpha''(G_{7,14},4)$$
$$=\left(2^{p-4}-2^{q-3}-18\right)-\left(5\cdot 2^{p-4}-3\cdot 2^{q-3}-18\right)$$
$$=-2^{p-2}+2^{q-2}=0.$$

Thus we have

$$\alpha''(G_{7,12},4)=\alpha''(G_{7,13},4)=\alpha''(G_{7,14},4)<\alpha''(G_{7,8},4)=\alpha''(G_{7,10},4).$$

Hence, we need to compare $\alpha(G_{7,8},5)$ with $\alpha(G_{7,10},5)$ and for each pair of $\alpha(G_{7,i},5)$, where $12 \le i \le 14$ By using Lemma 2.6, we have

$$\alpha(G_{7,8},5)-\alpha(G_{7,10},5)$$
$$=\left[\alpha(G_{7,8}+u_1v_1,5)+\alpha(G_{7,8}-\{u_1,v_1\},4)+\alpha(G_{7,8}-\{u_1,v_1,w_1\},4)\right]-$$
$$\left[\alpha(G_{7,10}+u_2v_2,5)+\alpha(G_{7,10}-\{u_2,v_2\},4)+\alpha(G_{7,10}-\{u_2,v_2,w_2\},4)\right]$$
$$=\left[\alpha(G_{7,8}-\{u_1,v_1\},4)+\alpha(G_{7,8}-\{u_1,v_1,w_1\},4)\right]-$$
$$\left[\alpha(G_{7,10}-\{u_2,v_2\},4)+\alpha(G_{7,10}-\{u_2,v_2,w_2\},4)\right]$$

since $G_{7,8}+u_1v_1 \cong G_{7,10}+u_2b_2$

$$=\left[\alpha(G_{7,8}-\{u_1v_1\},4)-\alpha(G_{7,10}-\{u_2,v_2\},4)\right]+$$
$$\left[\alpha(G_{7,8}-\{u_1,v_1,w_1\},4)-\alpha(G_{7,10}-\{u_2,v_2,w_2\},4)\right]$$

$$=\left[\alpha^{"}\left(G_{7,8}-\{u_1,v_1\},4\right)-\alpha^{"}\left(G_{7,10}-\{u_2,v_2\},4\right)\right]+$$

$$\left[\alpha^{"}\left(G_{7,8}-\{u_1,v_1,w_1\},4\right)-\alpha^{"}\left(G_{7,10}-\{u_2,v_2,w_2\},4\right)\right].$$

Since

$$G_{7,8}-\{u_1,v_1\}\in \mathrm{B}\left(p-1,q-1,s-2,5\right),$$
$$G_{7,10}-\{u_2,v_2\}\in \mathrm{B}\left(p-1,q-1,s-2,5\right),$$

$$G_{7,8}-\{u_1,v_1,w_1\}\in \mathrm{B}\left(p-2,q-1,s-3,4\right),$$
$$G_{7,10}-\{u_2,v_2,w_2\}\in \mathrm{B}\left(p-2,q-1,s-3,4\right), \text{ and}$$

by Lemma 2.9, we have

$$\alpha\left(G_{7,8}-\{u_1,v_1\},4\right)-\alpha\left(G_{7,10}-\{u_2,v_2\},4\right)$$
$$=\alpha^{"}\left(G_{7,8}-\{u_1,v_1\},4\right)-\alpha^{"}\left(G_{7,10}-\{u_2,v_2\},4\right)$$
$$=\left[2\left(2^{(p-1)-3}-2^{(q-1)-3}\right)-10\right]-\left[2\left(2^{(p-1)-3}-2^{(q-1)-3}-11\right)\right]$$
$$=1,$$

and

$$\alpha\left(G_{7,8}-\{u_1,v_1,w_1\},4\right)-\alpha\left(G_{7,10}-\{u_2,v_2,w_2\},4\right)$$
$$=\alpha^{"}\left(G_{7,8}-\{u_1,v_1,w_1\},4\right)-\alpha^{"}\left(G_{7,10}-\{u_2,v_2,w_2\},4\right)$$
$$=\left[2\left(2^{(p-2)-3}-2^{(q-1)-3}-6\right)\right]-\left[2\left(2^{(p-2)-3}-2^{(q-1)-3}\right)-11\right]$$
$$=5.$$

Thus, we have

$$\alpha\left(G_{7,8},5\right)-\alpha\left(G_{7,10},5\right)=6>0. \tag{4}$$

Similarly, by Lemma 2.6, we have

$$\alpha\left(G_{7,12},5\right)-\alpha\left(G_{7,13},5\right)$$
$$=\left[\alpha\left(G_{7,12}+a_2b_2,5\right)+\alpha\left(G_{7,12}-\{a_2,b_2\},4\right)+\alpha\left(G_{7,12}-\{a_2,b_2,c_2\},4\right)\right]-$$
$$\left[\alpha\left(G_{7,13}+a_2^{'}b_2^{'},5\right)+\alpha\left(G_{7,13}-\{a_2^{'}b_2^{'}\},4\right)+\alpha\left(G_{7,13}-\{a_2^{'},b_2^{'},c_2^{'}\},4\right)\right]$$

$$= \alpha\left(G_{7,12} - \{a_2, b_2, c_2\}, 4\right) - \alpha\left(G_{7,13} - \{a_2^{'}, b_2^{'}, c_2^{'}\}, 4\right)$$

Since $G_{7,12} + a_2 b_2 \cong G_{7,13} + a_2^{'} b_2^{'}, G_{7,12} - \{a_2, b_2\} \cong G_{7,13} - \{a_2^{'}, b_2^{'}\}$

$$= \alpha^{'}\left(G_{7,12} - \{a_2, b_2, c_2\}, 4\right) - \alpha^{'}\left(G_{7,13} - \{a_2^{'}, b_2^{'}, c_2^{'}\}, 4\right).$$

Since $G_{7,12} - \{a_2, b_2, c_2\} \in \mathrm{B}\left(p-2, q-1, s-2, 6\right)$ and

$G_{7,13} - \{a_2^{'}, b_2^{'}, c_2^{'}\} \in \mathrm{B}\left(p-1, q-2, s-2, 6\right),$

by Lemma 2.5, we have

$$\alpha\left(G_{7,12} - \{a_2, b_2, c_2\}, 4\right) - \alpha\left(G_{7,13} - \{a_2^{'}, b_2^{'}, c_2^{'}\}, 4\right)$$

$$= \alpha^{'}\left(G_{7,12} - \{a_2, b_2, c_2\}, 4\right) - \alpha^{'}\left(G_{7,13} - \{a_2^{'}, b_2^{'}, c_2^{'}\}, 4\right)$$

$$= \Big[(s-2)\left(2^{p-4} + 2^{q-3} - 2\right) + 4\left(2^{p-5} + 2^{q-3} - 2\right) + \left(2^{p-4} + 2^{q-4} - 2\right) +$$

$$\left(2^{p-6} + 2^{q-3} - 2\right) + \left\{12(s-8) + 30 + \binom{s-8}{2}\right\}\Big] -$$

$$\Big[(s-2)\left(2^{p-3} + 2^{q-4} - 2\right) + 4\left(2^{p-4} + 2^{q-4} - 2\right) + \left(2^{p-3} + 2^{q-5} - 2\right) +$$

$$\left(2^{p-5} + 2^{q-4} - 2\right) + \left\{12(s-8) + 30 + \binom{s-8}{2}\right\}\Big]$$

$$= 2^{q-6}\left[-4(s-2) - 13\right] + 2^{q-5}\left[2(s-2) + 11\right], \text{ (since } p = q \text{)}$$

$$= 9 \cdot 2^{q-6} > 0.$$

Similarly, by Lemma 2.6, we have

$$\alpha\left(G_{7,13}, 5\right) - \alpha\left(G_{7,14}, 5\right)$$

$$= \left[\alpha\left(G_{7,13} + a_3 b_3, 5\right) + \alpha\left(G_{7,13} - \{a_3, b_3\}, 4\right) + \alpha\left(G_{7,13} - \{a_3, b_3, c_3\}, 4\right)\right] -$$

$$\left[\alpha\left(G_{7,14} + a_3^{'} b_3^{'}, 5\right) + \alpha\left(G_{7,14} - \{a_3^{'}, b_3^{'}\}, 4\right) + \alpha\left(G_{7,14} - \{a_3^{'}, b_3^{'}, c_3^{'}\}, 4\right)\right]$$

$$= \alpha\left(G_{7,13} - \{a_3, b_3, c_3\}, 4\right) - \alpha\left(G_{7,14} - \{a_3^{'}, b_3^{'}, c_3^{'}\}, 4\right)$$

since $G_{7,13} + a_{3,} b_3 \cong G_{7,14} + a_3^{'} b_3^{'}$, and $G_{7,13} - \{a_{3,} b_3\} \cong G_{7,14} - \{a_{3,}^{'} b_3^{'}\}$

$$= \alpha^{'} \left(G_{7,13} - \{a_3, b_3, c_3\}, 4 \right) - \alpha^{'} \left(G_{7,14} - \{a_3^{'}, b_3^{'}, c_3^{'}\}, 4 \right).$$

Since $G_{7,13} - \{a_{3,} b_{3,} c_3\} \in \mathrm{B}(p-2, q-1, s-2, 6)$ and

$G_{7,14} - \{a_3^{'}, b_3^{'}, c_3^{'}\} \in \mathrm{B}(p-1, q-2, s-2, 6)$,

by Lemma 2.5 we have

$$\alpha \left(G_{7,13} - \{a_{3,} b_{3,} c_3\}, 4 \right) - \alpha^{'} \left(G_{7,14} - \{a_3^{'}, b_3^{'}, c_3^{'}\}, 4 \right)$$

$$= \alpha^{'} \left(G_{7,13} - \{a_{3,} b_{3,} c_3\}, 4 \right) - \alpha^{'} \left(G_{7,14} - \{a_3^{'}, b_3^{'}, c_3^{'}\}, 4 \right)$$

$$= \Big[(s-2)\left(2^{p-4} + 2^{q-3} - 2\right) + 3\left(2^{p-5} + 2^{q-}3 - 2\right) + 2\left(2^{p-4} + 2^{q-4} - 2\right) \Big] +$$

$$\left(2^{p-6} + 2^{q-3} - 2\right) + \left\{12(s-8) + 30 + \right\} \Big] - \binom{s-8}{2}$$

$$\Big[(s-2)\left(2^{p-3} + 2^{q-4} - 2\right) + 3\left(2^{p-4} + 2^{q-4} - 2\right) + 2\left(2^{p-3} + 2^{q-5} - 2\right) +$$

$$\left(2^{p-5} + 2^{q-4} - 2\right) + \left\{ 12(s-8) + 30 + \binom{s-8}{2} \right\} \Big]$$

$$= 2^{q-6}\left[-4(s-2) - 15\right] + 2^{q-5}\left[2(s-2) + 10\right] \quad (\text{since } p = q)$$

$$= 5 \cdot 2^{q-6} > 0.$$

3.2 When $p>q$ from Equation (3), we have

$$\alpha^{''} \left(G_{7,12,} 4 \right) - \alpha^{''} \left(G_{7,13,} 4 \right) = -2^{p-3} + 2^{q-3} < 0,$$

$$\alpha^{''} \left(G_{7,13,} 4 \right) - \alpha^{''} \left(G_{7,14,} 4 \right)$$

$$= \left(3 \cdot 2^{p-4} - 2^{q-2} - 18\right) - \left(5 \cdot 2^{p-4} - 3 \cdot 2^{q-3} - 18\right)$$

$$= -2^{p-3} + 2^{q-3} < 0,$$

$$\alpha''\left(G_{7,14,}4\right)-\alpha''\left(G_{7,8,}4\right)$$
$$=\left(5\cdot 2^{p-4}-3\cdot 2^{q-3}-18\right)-\left(3\cdot 2^{p-3}-3\cdot 2^{q-3}-18\right)$$
$$=-2^{p-4}<0,$$

From Equation (2), we have

$$\alpha''\left(G_{7,8,}4\right)-\alpha''\left(G_{7,10,}4\right)=0,$$
$$\alpha''\left(G_{7,8,}4\right)-\alpha''\left(G_{7,15,}4\right)$$
$$=\left(3\cdot 2^{p-3}-3\cdot 2^{q-3}-18\right)-\left(2^{p-1}-9\cdot 2^{q-4}-18\right)$$
$$=-2^{p-3}+3\cdot 2^{q-4}<0,$$
$$\alpha''\left(G_{7,15,}4\right)-\alpha''\left(G_{7,9,}4\right)$$
$$=\left(2^{p-1}-9\cdot 2^{q-4}-18\right)-\left(4\cdot 2^{p-3}-4\cdot 2^{q-3}-18\right)$$
$$=-2^{p-2}-2^{q-4}<0,$$
$$\alpha''\left(G_{7,9,}4\right)-\alpha''\left(G_{7,11,}4\right)=0,$$
$$\alpha''\left(G_{7,9,}4\right)-\alpha''\left(G_{7,16,}4\right)$$
$$=\left(4\cdot 2^{p-3}-4\cdot 2^{q-3}-18\right)-\left(5\cdot 2^{p-3}-11\cdot 2^{q-4}-18\right)$$
$$=-2^{p-3}+3\cdot 2^{q-4}<0,$$
$$\alpha''\left(G_{7,16,}4\right)-\alpha''\left(G_{7,17,}4\right)$$
$$=\left(5\cdot 2^{p-3}-11\cdot 2^{q-4}-18\right)-\left(6\cdot 2^{p-3}-13\cdot 2^{q-4}-18\right)$$
$$=-2^{p-3}+2^{q-3}<0.$$

Therefore,

$$\alpha''\left(G_{7,12},4\right)<\alpha''\left(G_{7,13},4\right)<\alpha''\left(G_{7,14},4\right)<\alpha''\left(G_{7,8},4\right)=\alpha''\left(G_{7,10},4\right)<$$
$$\alpha''\left(G_{7,15},4\right)<\alpha''\left(G_{7,9},4\right)=\alpha''\left(G_{7,11},4\right)<\alpha''\left(G_{7,16},4\right)<\alpha''\left(G_{7,17},4\right).$$

Since $\alpha''\left(G_{7,8},4\right)=\alpha''\left(G_{7,10},4\right)$ and $\alpha''\left(G_{7,9},4\right)=\alpha''\left(G_{7,11},4\right)$, we need to calculate $\alpha\left(G_{7,8},5\right)-\alpha\left(G_{7,10},5\right)$ and $\alpha\left(G_{7,9},5\right)-\alpha\left(G_{7,11},5\right)$. Hence,by Equation (4), we have $\alpha\left(G_{7,8},5\right)-\alpha\left(G_{7,10},5\right)=6>0$. By using Lemma 2.6, we have

$$\alpha\left(G_{7,9},5\right)-\alpha\left(G_{7,11},5\right)$$
$$=\left[\alpha\left(G_{7,9}+u_3v_3,5\right)+\alpha\left(G_{7,9}-\left\{u_3v_3\right\},4\right)+\alpha\left(G_{7,9}-\left\{u_{3,}v_{3,}w_3\right\},4\right)\right]-$$
$$\left[\alpha\left(G_{7,11}+u_4v_4,5\right)+\alpha\left(G_{7,11}-\left\{u_{4,}v_4\right\},4\right)+\alpha\left(G_{7,11}-\left\{u_{4,}v_{4,}w_4\right\},4\right)\right]$$
$$=\left[\alpha\left(G_{7,9}-\left\{u_{3,}v_3\right\},4\right)+\alpha\left(G_{7,9}-\left\{u_{3,}v_{3,}w_3\right\},4\right)\right]-$$
$$\left[\alpha\left(G_{7,11}-\left\{u_{4,}v_4\right\},4\right)+\alpha\left(G_{7,11}-\left\{u_{4,}v_{4,}w_4\right\},4\right)\right]$$
since $G_{7,9}+u_3v_3\cong G_{7,11}+u_{4,}v_4$
$$=\left[\alpha\left(G_{7,9}-\left\{u_{3,}v_3\right\},4\right)-\alpha\left(G_{7,11}-\left\{u_{4,}v_4\right\},4\right)\right]+$$
$$\left[\alpha\left(G_{7,9}-\left\{u_3,v_3,w_3\right\},4\right)-\alpha\left(G_{7,11}-\left\{u_4,v_4,w_4\right\},4\right)\right]+$$
$$=\left[\alpha'\left(G_{7,9}-\left\{u_3,v_3\right\},4\right)-\alpha'\left(G_{7,11}-\left\{u_4,v_4\right\},4\right)\right]+$$
$$\left[\alpha'\left(G_{7,9}-\left\{u_3,v_3,w_3\right\},4\right)-\alpha'\left(G_{7,11}-\left\{u_4,v_4,w_4\right\},4\right)\right].$$

Since $G_{7,9}-\left\{u_3,v_3\right\}\in \mathrm{B}\left(p-1,q-1,s-2,5\right)$ and
$G_{7,11}-\left\{u_4,v_4\right\}\in \mathrm{B}\left(p-1,q-2,s-2,5\right)$,

by Lemmas 2.5, 2.7 and 2.8, we have

$$\alpha\left(G_{7,9}-\left\{u_3,v_3\right\},4\right)-\alpha\left(G_{7,11}-\left\{u_4,v_4\right\},4\right)$$
$$=\alpha'\left(G_{7,9}-\left\{u_3,v_3\right\},4\right)-\alpha'\left(G_{7,11}-\left\{u_4,v_4\right\},4\right)$$
$$=\left[\left(s-2\right)\left(2^{p-3}+2^{q-3}-2\right)+5\left(2^{p-4}+2^{q-3}-2\right)+3\left(2^{p-4}-2^{q-4}\right)+\right.$$

$$\left\{\binom{s-3}{2}-25\right\}\Bigg]-\Big[(s-2)\left(2^{p-3}+2^{q-3}-2\right)+5\left(2^{p-4}+2^{q-3}-2\right)+$$

$$3\left(2^{p-4}-2^{q-4}\right)+\left\{\binom{s+3}{2}-26\right\}\Bigg]$$

$$=1.$$

Similarly, since

$$G_{7,9}-\{u_3,v_3,w_3\}\in \mathrm{B}\left(p-1,q-2,s-3,4\right) and$$
$$G_{7,11}-\{u_4,v_4,w_4\}\in \mathrm{B}\left(p-1,q-2,s-3,4\right),$$

by Lemma 2.5, 2.7 and 2.8, we have

$$\alpha\left(G_{7,9}-\{u_3,v_{3,}w_3\},4\right)-\alpha\left(G_{7,11}-\{u_4,v_4,w_4\},4\right)$$
$$=\alpha'\left(G_{7,9}-\{u_3,v_3,w_3\},4\right)-\alpha'\left(G_{7,11}-\{u_4,v_4,w_4\},4\right)$$
$$=\Big[(s-3)\left(2^{p-3}+2^{q-4}-2\right)+4\left(2^{p-4}+2^{q-4}-2\right)+2\left(2^{p-4}-2^{q-5}\right)+$$

$$\left\{\binom{s+1}{2}-18\right\}\Bigg]-\Big[(s-3)\left(2^{p-3}+2^{q-4}-2\right)+4\left(2^{p-4}+2^{q-4}-2\right)$$

$$2\left(2^{p-4}-2^{q-5}\right)+\left\{\binom{s+1}{2}-23\right\}\Bigg]$$

$$=5.$$

Thus we have $\alpha\left(G_{7,9},5\right)-\alpha\left(G_{7,11},5\right)=6>0.$

For the remaining working every two graphs in T_4 to T_{21} , the reader may refer to [6]. This completes the proof of Theorem 3.1. □

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