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Generalized Weyl's Theorem for Log-Hyponormal

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ABSTRACT

In this paper, we prove that if T is $\log$-hyponormal then the generalized Weyl's theorem holds for T ; that is, $\sigma_{BW}(T) = \sigma(T) - E(T)$.

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INTRODUCTION

Let H be a complex Hilbert space and let $\mathrm{B(H)}$ be the algebra of all bounded linear operators acting on H. If $T \in \mathrm{B(H)}$, we shall write $\ker(T)$, $ran(T)$ for the null space and range of T ,respectively. For $T \in \mathrm{B(H)}$, we denote the spectrum, the point spectrum and the approximate point spectrum of T by $\sigma(T), \sigma_p(T)$ and $\sigma_a(T)$, respectively.

If $T \in \mathrm{B(H)}$ set $\alpha(T) := \dim \ker(T)$, the dimension of the null space, and $\beta(T) := co - \dim ran(T)$, the co-dimension of the range.

The class of all upper semi-Fredholm operators is defined as the set $SF_+(\mathrm{H})$ of all $T \in \mathrm{B(H)}$ such that $\alpha(T) < \beta(T)$ and $ran(T)$ is closed. The class of all lower semi-Fredholm operators is defined as the set $SF_-(\mathrm{H})$ of all $T \in \mathrm{B(H)}$ such that $\beta(T) < \infty$. The class of all semi-Fredholm operators is denoted by $SF_\pm(\mathrm{H})$, while by $F(\mathrm{H}) = SF_+(\mathrm{H}) \cap SF_-(\mathrm{H})$ we shall denote the class of all Fredholm operators.

The index of $T \in SF_\pm(H)$ is defined by $ind(T) = \alpha(T) - \beta(T)$. The other two quantities associated with a linear operator T are the ascent $a := a(T)$, defined as the smallest non-negative integer s (if it does exist) such that $\ker(T^s) = \ker(T^{s+1})$ and the descent $d := d(T)$, defined as the smallest non-negative integer t (if it does exist) such that $ran(T^t) = ran(T^{t+1})$. It is well known that if $a(T-\lambda I)$ and $d(T-\lambda I)$ are both finite, then $a(T-\lambda I) = d(T-\lambda I)$ and λ is a pole of the resolvent $\lambda \to (T-\lambda I)^{-1}$, in particular an isolated point of the spectrum $\sigma(T)$ (see proposition 1.49 and theorem 1.52 of Dowson [9]). The class of Weyl's operators is defined by $W(T) := \{T \in F(H) : ind(T) = 0\}$ while the class of Browder operators is defined by:

$$Bro(H) = \{T \in F(H) : a(T) < \infty, d(T) < \infty\}.$$

Obviously $Bro(H) \subseteq W(H)$. The Weyl's spectrum and the Browder's spectrum of $T \in \mathrm{B(H)}$ are defined by:

$$\sigma_W(T) = \{\lambda \in C : \lambda I - T \notin W(H)\}, \text{ and}$$

$$\sigma_B(T) = \{\lambda \in C : \lambda I - T \notin Bro(H)\}.$$

Berkani (1990) introduced the concept of $B-$Fredholm as follows: For each integer n, define T_n to be the restriction of T to $ran(T^n)$ viewed as a map from $ran(T^n)$ into $ran(T^n)$ (in particular $T_0 = T$).

If for some integer n the space $ran(T^n)$ is closed and T_n is a Fredholm operator, then T is called a $B-$Fredholm operator. In this case T_m is a Fredholm operator and $ind(T_n) = ind(T_m)$ for each $m \geq n$.

Let $BF(H)$ be the class of all $B-$Fredholm operators. It is known that $F(H) \subseteq BF(H)$. Moreover, an operator $T \in B(H)$ is $B-$Fredholm if and only if $T = Q \oplus F$, where Q is a nilpotent operator and F is Fredholm. [3,Theorem 2.7]

Definition 1.1. [5] Let $T \in B(H)$. The $B-$Fredholm spectrum $\sigma_{BF}(T)$ is defined by:

$$\sigma_{BF}(T) = \{\lambda \in C : \lambda I - T \notin BF(H)\}.$$

Definition 1.2. [3] Let $T \in B(H)$ be a $B-$Fredholm operator and let n be any integer such that T_n is a Fredholm operator. Then the index $ind(T)$ of is T defined as the index of the Fredholm operator T_n.

Definition 1.3. [5] An operator $T \in B(H)$ is called a $B-$Weyl operator if it is a $B-$Fredholm operator of index 0. The $B-$Weyl spectrum $\sigma_{BW}(T)$ of T is defined by:

$$\sigma_{BW}(T) = \{\lambda \in C : \lambda I - T \notin BW(H)\}.$$

In the case of a hyponormal operator acting on a Hilbert space H, Berkani [6] showed that:

$$\sigma_{BW}(T) = \sigma(T) - E(T),$$

where $\sigma_{BW}(T)$ is the $B-$Weyl spectrum of T and $E(T)$ is the set of all eigenvalues of T which are isolated in the spectrum of T.

Definition 1.4. [5] Let $T \in B(H)$. We will say that:

a) T satisfies Weyl's theorem if $\sigma_W(T) = \sigma(T) - E_0(T)$, where $E_0(T)$ is the set of all eigenvalues of finite multiplicity isolated in $\sigma(T)$.
b) T satisfies generalized Weyl's theorem if $\sigma_{BW}(T) = \sigma(T) - E(T)$.
c) T satisfies Browder's theorem if $\sigma_W(T) = \sigma(T) - \pi_0(T)$, where $\pi_0(T)$ is the set of all poles of finite rank.
d) T satisfies Browder's theorem if $\sigma_W(T) = \sigma(T) - \pi(T)$, where $\pi(T)$ is the set of all poles.

From [3, 5, 7, 11] we have the following implication:

generalized Weyl's theorem $\Rightarrow$ Weyl's theorem
$\Rightarrow$ Browder's theorem

generalized Browder's theorem $\Leftrightarrow$ Browder's theorem

MAIN RESULTS

Following [8], an operator T is called log$-$hyponormal if T is invertible and satisfies $\log(T^*T) \geq \log(TT^*)$. Let $T = U|T|$ be the decomposition of T. If T is log$-$hyponormal, then the operator U is unitary.

$Ch\ddot{o}$ Showed that if T is log$-$hyponormal operator, then so is T^{-1}. We write $r(T)$ and $W(T)$ for the spectral radius and numerical range, respectively. It is well-known that $r(T) \leq \|T\|$ and that is $W(T)$ convex with convex hull $conv\sigma(T) \subseteq \overline{W(T)}$. T is called convexoid if $conv\,\sigma(T) = \overline{W(T)}$, and normaloid if $r(T) = \|T\|$.

Lemma 2.1[8]
If $T = U|T|$ is a log$-$hyponormal operator, then T is normaloid; i.e., the spectral radius $r(T) = \|T\|$.

Lemma 2.2
Let $T = U|T|$ be a log$-$hyponormal operator and let $\lambda \in C$. Assume that $\sigma(T) = \{\lambda\}$. Then $T = \lambda I$.

Proof. Since T is log$-$hyponormal, then T is invertible and $\lambda \neq 0$. We see that T, T^{-1} are normaloid. On the hand $\sigma(T^{-1}) = \{\frac{1}{\lambda}\}$, so $\|T\|\|T^{-1}\| = |\lambda||\frac{1}{\lambda}| = 1$. It follows that T **is** convexoid, so $W(T) = \{\lambda\}$. Therefore $T = \lambda I$.

Recall that an operator $T \in B(H)$ is called isoloid if all isolated points $iso(\sigma(T))$ of $\sigma(T)$ are eigenvalues of T. As a consequence of lemma 4 and [12, theorem 14] we have immediately

Corollary 2.3. *Let $T \in B(H)$. If T is log$-$hyponormal, then T is isoloid.*

Theorem 2.4. *Let $T \in B(H)$ be a log$-$hyponormal operator, then T is of finite ascent.*

Proof. Let $x \in \ker(T^2)$, then $\|Tx\|^2 \le \|T^2x\| = 0,$ and so $x \in \ker(T)$. Since by [9] the eigenvalues of $\log-$hyponormal operators are normal eigenvalues of T, if $0 \ne \lambda \in \sigma_p(T)$ and $x \in \ker((T-\lambda I)^2)$, $(T-\lambda I)(T-\lambda I) = 0 = (T-\lambda I)^*(T-\lambda I)x$ and $\|(T-\lambda I)x\|^2 = \langle (T-\lambda I)^*(T-\lambda I)x, x\rangle = 0$.

Hence, if T is $\log-$hyponormal, then $a(T-\lambda I) = 1$.

Definition 2.5. [2, definition 1.1] Let $Hol(\sigma(T))$ be the space of all functions that are analytic in an open neighborhoods of $\sigma(T)$. Let $T \in B(H)$. The operator T is said to have the single-valued extension property at $\lambda_0 \in C$ (SVEP at λ_0), if for every open disc D_{λ_0} centered at λ_0 the only analytic function $f : D_{\lambda_0} \to H$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in D_{\lambda_0}$ is the function $f \equiv 0$.

Recall [1, 2, 11, 13, 14, 15] that an operator $T \in B(H)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in C$. Trivially, an operator $T \in B(H)$ has the SVEP at every point of the resolvent $\rho(T) = C - \sigma(T)$. Moreover, from the identity theorem for analytic function it easily follows that T has the SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has the SVEP at every isolated point of the spectrum. Hence, we have the following implication
$\sigma(T)$ does not cluster at $\lambda_0 \Rightarrow$ T has the SVEP at λ_0 .

In [15], Laursen proved that if T is of finite ascent, then T has SVEP.

Theorem 2.6. *If* $T \in B(H)$ *is* $\log-$*hyponormal operator. Then* T *and* T^* *satisfy Weyl's theorem.*

Proof. Since T is $\log-$hyponormal, then T has SVEP. Then T satisfies Browder's theorem if and only if T^* satisfies Browder's theorem if and only if:

$$\pi_0(T) = \sigma(T) - \sigma_W(T) \subseteq E_0(T) \text{ and } \pi_0(T^*) = \sigma(T^*) - \sigma_W(T^*) \subseteq E_0(T^*).$$

If $\lambda \in E_0(T^*)$ then T and T^* both has SVEP at λ and $0 < a((T-\lambda I)^*) = d(T-\lambda I) < \infty$. Thus the ascent and descent of $T-\lambda I$ and $(T-\lambda I)^*$ are finite and hence equal [10]. Then $T-\lambda I$ and $(T-\lambda I)^*$ are Fredholm operators of index zero.

Consequently, $E_0(T) \subseteq \sigma(T) - \sigma_W(T)$ and $E_0(T^*) \subseteq \sigma(T^*) - \sigma_W(T^*)$. This implies that both T and T^* satisfy Weyl's theorem.

Definition 2.7. [7, definition 2.2] Let $T \in B(H)$. We will say that T is of stable sign index if for each $\lambda, \mu \in \rho_{BF}(T)$, $ind(\lambda I - T)$ and $ind(\mu I - T)$ have the same sign.

Proposition 2.8. *Let $T \in B(H)$ be a* $\log-$*hyponormal operator. Then T is of stable index.*

Proof. Let T be a $\log-$hyponormal operator. Then $\ker(T) = \ker(T^*) = ran(T)^{\perp}$. Since $a(T) = 1$, then $\ker(T) = \ker(T^2)$. Moreover, if T is also a $B-$Fredholm operator, then there exists an integer n such that $ran(T^n)$ is closed and such that $T_n : ran(T^n) \to ran(T^n)$ is a Fredholm operator. We have:

$$
\begin{aligned}
ind(T) = ind(T_n) &= \dim(\ker(T) \cap ran(T^n) - \dim\left(ran(T^n) / ran(T^{n+1}) \right) \\
&= -\dim\left(ran(T^n) / ran(T^{n+1}) \right)
\end{aligned}
$$

so $ind(T) \le 0$.

Further, if $\lambda \in \rho_{BF}(T)$, then $\lambda I - T$ is a $B-$Fredholm operator, and $\lambda I - T$ is also a $\log-$hyponormal operator. From the preceding argument, we have $ind(\lambda I - T) \le 0$. Therefore T is of stable index.

Since a $\log-$hyponormal operator is of stable sign index, then from [7, Theorem 2.4] we have immediately the following corollary.

Corollary 2.9. *Let $T \in B(H)$ be a* $\log-$*hyponormal operator and let $f \in Hol(\sigma(T))$. Then $f(\sigma_{BW}(T)) = \sigma_{BW}(f(T))$.*

Berkani [6] proved that if $T \in B(H)$ is a hyponormal. Then T satisfies generalized Weyl's theorem $\sigma_{BW}(T) = \sigma(T) - E\ (T)$. In the following theorem, we extend this result to the case of a $\log-$hyponormal operator.

Theorem 2.10. *Let $T \in B(H)$ be a* $\log-$*hyponormal. Then T satisfies generalized Weyl's theorem* $\sigma_{BW}(T) = \sigma(T) - E\ (T)$.

Proof. Let $\lambda \in \sigma(T) - \sigma_{BW}(T)$. Then $T - \lambda I$ is a $B-$Fredholm operator of index zero. Hence it follows from [10] that there exist M, N closed subspaces of H such that $H = M \oplus N, T - \lambda I\,|_M$ is a Fredholm operator of index zero and $T - \lambda I\,|_N$ is a nilpotent operator.

Let $R = T\,|_M, S = T\,|_N$, and $I_1 = I\,|_M, I_2 = I\,|_N$. Since T is a $\log-$hyponormal then so is R. Hence it follows from [8] that:

$$\sigma(R) - \sigma_W(R) = E_0(R).$$

We have two cases:

Case 1: $\lambda \in \sigma(R)$.. Since $R - \lambda I_1$ is a Fredholm operator of index zero then $\lambda \in E_0(R)$ and so λ is isolated in $\sigma(R)$.

Since $T - \lambda I = (R - \lambda I_1) \oplus (S - \lambda I_2)$ is nilpotent then $\sigma(T - \lambda I) - \{0\} = \sigma(R - \lambda I_1) - \{0\}$. Therefore 0 is isolated in $\sigma(T - \lambda I), i.e.,$, λ is isolated in $\sigma(T)$. But since $\lambda \in \sigma_p(R)$ then $\lambda \in E(T)$.
Case 2: $\lambda \notin \sigma(R)$. In this case we also deduce from $T - \lambda I = (R - \lambda I_1) \oplus (S - \lambda I_2)$, that λ is isolated in $\sigma(T)$. Since $T - \lambda I$ is not invertible then $\lambda \in E(T)$.

Conversely let $\lambda \in E(T)$. Then λ is isolated in $\sigma(T)$ and we can represent T as a direct sum $T = T_1 \oplus T_2$ where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) - \{\lambda\}$.

Since T is $\log-$hyponormal operator then T_1 is also $\log-$hyponormal operator. Since T is invertible then $0 \notin \sigma(T)$. Therefore we can write:

$$0 = F(T_1) = c(T_1 - \lambda I_1)^n \prod_{j=1}^{k} (T_1 - \lambda_j I_1),$$

with $n \neq 0$, and $\lambda_j \neq \lambda, j = 1, \cdots, k.$

Since $T_1 - \lambda_j I_1$ is invertible for every $j = 1, \cdots, k,$, then $(T_1 - \lambda I_1)^n = 0$ and so $T_1 - \lambda I_1$ is nilpotent. Since $T_2 - \lambda I_2$ is invertible it follows from [5] that $T - \lambda I$ is a Fredholm operator of index 0. Therefore $\lambda \in \sigma(T) - \sigma_{BW}(T)$.

Definition 2.11. [8, definition 3.1] An operator $T \in B(H)$ is called polaroid if all isolated points of the spectrum of T are poles of the resolvent of T .

Corollary 2.12. *Let $T \in B(H)$ be a $\log-$hyponormal operator, then T is a Polaroid operator.*

Proof. This is an immediate consequence of lemma 2.2, theorem 2.10 and [7].

The following result is a consequence of corollary 2.12 and [7].

Corollary 2.13. *Let $T \in B(H)$ be a $\log-$hyponormal, then $E(f(T)) = \pi(f(T))$ for every $f \in Hol(\sigma(T))$.*

Theorem 2.14. *Let $T \in B(H)$ be a $\log-$hyponormal. Then $f(T)$ satisfies generalized Weyl's theorem $\sigma(f(T)) - \sigma_{BW}(f(T)) = E(f(T))$ for every $f \in Hol(\sigma(T))$.*

Proof. T satisfies generalized Weyl's theorem by theorem 2.10 and isoloid by corollary 2.3. Moreover, from corollary 2.9 we have $\sigma_{BW}(f(T)) = f(\sigma_{Bw}(T))$. From [6, theorem 2.10], it follows that $f(T)$ satisfies generalized Weyl's theorem.

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