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Comparison of Bayes Estimators of the Parameter and Reliability Function for Rayleigh Distribution under Different Loss Functions

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ABSTRACT

In this paper we derive Bayes' estimators for the parameter and reliability function of the Rayleigh distribution. These estimators are obtained on the basis of squared error loss function and LINEX loss function. Comparisons in terms of risks of those under linex loss and squared error loss functions with Bayes estimators relative to squared error loss function have been made. Finally, numerical study is given to illustrate the results.

Keywords: Bayes estimator, LINEX loss function, Reliability function, Risk function, Squared error loss function.

INTRODUCTION

The Rayleigh distribution is a suitable model for life testing experiments and clinical studies which age with time as its hazard rate $H(t) = t/\sigma^2$ is a linear function of time. Polovko, (1968) and Dyer and Whisenand, (1973) demonstrated the importance of this distribution in electro vacuum devices and communication engineering. Ariyawansa and Templeton, (1984) have also discussed some of its applications. Howlader and Hossian, (1995) obtained Bayes estimators for the scale parameter and the reliability function (R(t)) in the case of type-II censored sampling. The origin and other aspects of this distribution can be found in Siddiqui, (1962) and Hirano, (1986). Abd Elfattah *et al.,*(2006) studied the efficiency of the maximum likelihood estimates of the parameter under three cases, namely, type-I, type-II and progressive type-II censored sampling schemes.

The probability distribution function (PDF) and the reliability function of the Rayleigh distribution are respectively given by :

$$f\left(x\middle|\sigma^2\right)=\frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right);\qquad x\geq 0,\quad \sigma>0 \tag{1.1}$$

and $$R(t)=\overline{F(t)}=P(x>t)=\exp\left(-\frac{t^2}{2\sigma^2}\right);\qquad t\geq 0,\sigma>0 \tag{1.2}$$

In the estimation of the reliability function, the use of symmetric loss function may be inappropriate as was recognized by Canfield, (1970). Varian, (1975) and Zellner, (1986) proposed an asymmetric loss function known as LINEX loss function which has been found to be appropriate in situation where overestimation is more serious than underestimation or vice-versa.

Suppose $\Delta=\frac{\hat{\sigma}}{\sigma}-1$, where $\hat{\sigma}$ is an estimate of σ. Consider the following convex loss function,

$$L(\Delta)=e^{a\Delta}-a\Delta-1;\quad a\neq 0 \tag{1.3}$$

The sign and magnitude of '*a*' represent the direction and degree of asymmetry respectively. The positive value of '*a*' is used when overestimation is more costly than underestimation while negative value of '*a*' is used in the reverse situation. If '*a*' is close to zero, this loss function is approximately squared error loss and therefore almost symmetric. Several authors including Basu and Ebrahimi, (1991), Rojo, (1987), Soliman, (2000) and Zellner, (1986) have used this loss function in various estimation and prediction problems.

If we define $\Delta_1=\hat{\sigma}-\sigma$, then $L(\Delta_1)$ is equivalent to the loss function used by Varian, (1975) and Zellner, (1986). If we define $\Delta_2=\left(\frac{\hat{\sigma}}{\sigma}\right)^2-1$ then $L(\Delta_2)$ is equivalent to the loss function used by Soliman, (2000). Here, we consider the natural conjugate family of priors:

$$g(\sigma)\propto\frac{\exp\left(-\frac{\beta}{2\sigma^2}\right)}{\sigma^{\alpha+1}};\qquad \alpha,\beta>0 \tag{1.4}$$

If $\beta = 0, \alpha = 0$, we get a non-informative prior (Jeffrey's, (1961)). Also, if $\beta = 0, \alpha = 2$, we get the asymptotically invariant prior, proposed by Hartigan, (1964).

In section 2, we obtain Bayes estimator of σ. The estimates are based on the squared error loss function and LINEX loss function $L(\Delta_2)$ where $\Delta_2 = \left(\frac{\hat{\sigma}}{\sigma}\right)^2 - 1$. By using $g(\sigma)$ as the prior distribution, the risk of the estimates are obtained. Comparison in terms of risk with the estimates of σ under squared error loss and LINEX loss functions are made. A numerical example is given to compare our results.

In section 3, we obtain Bayes estimator of R(t) when the LINEX loss function is $L(\Delta_1)$, where $\Delta_1 = \hat{\sigma} - \sigma$. This is compared with those corresponding to the squared error loss function using Monte Carlo Simulation.

BAYES ESTIMATE OF σ

In this section we estimate the unknown parameter σ of the Rayleigh distribution based on a complete random sample of size *n*. The likelihood function (LF) is given by :

$$L(x|\sigma) = \prod_{i=1}^{n} x_i \sigma^{-2n} \exp\left(-\frac{s}{2\sigma^2}\right) \qquad ; \sigma > 0 \qquad (2.1)$$

where $x = (x_1, x_2, ..., x_n)$ is a random sample from the pdf (1.1) and

$$s = \sum_{i=1}^{n} x_i^2$$

Note that, when x has one-parameter Rayleigh distribution as in (1.1), then it is easy to see that x^2 has one parameter exponential distribution which implies $\frac{s}{\sigma^2}$ follows a Chi-square distribution with 2*n* degrees of freedom giving the probability density function of '*s*' as :

$$h(s) = \frac{1}{2^n\left(\sigma^2\right)^n \Gamma n} \exp\left(-\frac{s}{2\sigma^2}\right) s^{n-1} \quad ; \quad s > 0 \tag{2.2}$$

Bayes Estimator of σ Based On Squared Error Loss Function

Using Bayes theorem, the posterior PDF of σ is

$$\pi(\sigma|x) = \frac{L(x|\sigma)\,g(\sigma)}{\int_0^\infty L(x|\sigma)\,g(\sigma)\,d\sigma}$$

$$= \frac{\prod_{i=1}^{n} x_i \sigma^{-2n} \exp\left(-\frac{s}{2\sigma^2}\right) \frac{\exp\left(-\frac{\beta}{2\sigma^2}\right)}{\sigma^{\alpha+1}}}{\int_0^\infty \prod_{i=1}^{n} x_i \sigma^{-2n} \exp\left(-\frac{s}{2\sigma^2}\right) \frac{\exp\left(-\frac{\beta}{2\sigma^2}\right)}{\sigma^{\alpha+1}} d\sigma}$$

$$= \frac{2\left(\frac{s+\beta}{2}\right)^{\frac{2n+\alpha}{2}}}{\Gamma\left(\frac{2n+\alpha}{2}\right)} \frac{\exp\left(-\frac{s+\beta}{2\sigma^2}\right)}{\sigma^{2n+\alpha+1}} \quad ; \quad \sigma, \alpha, \beta > 0 \tag{2.3}$$

Considering the squared error loss ($(L(\hat{\sigma},\sigma) = (\hat{\sigma}-\sigma)^2)$), the Bayes estimator of σ denoted by $\hat{\sigma}_{SB}$ for the above prior , given the posterior mean of σ is

$$\widehat{\sigma}_{SB} = \int_0^\infty \sigma\, \pi(\sigma|x)\, d\sigma$$

$$= \int_0^\infty \sigma \frac{2\left(\frac{s+\beta}{2}\right)^{\frac{2n+\alpha}{2}}}{\Gamma\left(\frac{2n+\alpha}{2}\right)} \frac{\exp\left(-\frac{s+\beta}{2\sigma^2}\right)}{\sigma^{2n+\alpha+1}} d\sigma$$

$$= \frac{\Gamma\left(\frac{2n+\alpha-1}{2}\right)}{\Gamma\left(\frac{2n+\alpha}{2}\right)}\left(\frac{s+\beta}{2}\right)^{1/2} \tag{2.4}$$

Bayes Estimator of σ Based on LINEX Loss Function

Under the LINEX loss function (1.3), the posterior expectation of the loss function $L(\Delta_2)$ with respect to $\pi(\sigma|x)$ in (2.3) is

$$E\left[L(\Delta_2)\right] = \int_0^\infty \left\{ e^{a\left[\left(\frac{\hat{\sigma}}{\sigma}\right)^2 - 1\right]} - a\left[\left(\frac{\hat{\sigma}}{\sigma}\right)^2 - 1\right] - 1 \right\} \pi(\sigma|x)\, d\sigma \tag{2.5}$$

$$= e^{-a} E\left[\exp\left\{a\left(\frac{\hat{\sigma}}{\sigma}\right)^2\right\}\right] - aE\left[\left(\frac{\hat{\sigma}}{\sigma}\right)^2 - 1\right] - 1 \tag{2.6}$$

The value of $\hat{\sigma}$ that minimizes the posterior expectation of the loss function $L(\Delta_2)$, denoted by $\hat{\sigma}_{LB}$ is obtained by solving the equation:

$$\frac{\partial E\left[L(\Delta_2)\right]}{\partial \hat{\sigma}} = E\left[e^{-a}\frac{\hat{\sigma}}{\sigma^2}\exp\left(a\left(\frac{\hat{\sigma}}{\sigma}\right)^2\right)\right] - E\left(\frac{\hat{\sigma}}{\sigma^2}\right) = 0 \tag{2.7}$$

that is, $\hat{\sigma}_{LB}$ is the solution of the equation

$$E\left[\frac{\hat{\sigma}_{LB}}{\sigma^2}\exp\left(a\left(\frac{\hat{\sigma}_{LB}}{\sigma}\right)^2\right)\right] = e^{a}\, E\left(\frac{\hat{\sigma}_{LB}}{\sigma^2}\right) \tag{2.8}$$

provided that all expectation exists and are finite.

$$\Rightarrow \frac{\hat{\sigma}_{LB}}{\sigma^2}\int_0^\infty \exp\left(a\left(\frac{\hat{\sigma}_{LB}^2}{\sigma^2}\right)\right)\frac{2\left(\frac{s+\beta}{2}\right)^{\frac{2n+\alpha}{2}}}{\Gamma\left(\frac{2n+\alpha}{2}\right)}\frac{\exp\left(-\frac{s+\beta}{2\sigma^2}\right)}{\sigma^{2n+\alpha+1}}d\sigma$$

$$= e^a\int_0^\infty \frac{\hat{\sigma}_{LB}}{\sigma^2}\times\frac{2\left(\frac{s+\beta}{2}\right)^{\frac{2n+\alpha}{2}}}{\Gamma\left(\frac{2n+\alpha}{2}\right)}\frac{\exp\left(-\frac{s+\beta}{2\sigma^2}\right)}{\sigma^{2n+\alpha+1}}d\sigma$$

On simplification, we get the optimal estimate of σ relative to L(Δ_2) is

$$\hat{\sigma}_{LB} = \left(\frac{s+\beta}{2a}\right)\left[1-\exp\left(-\frac{2a}{2n+\alpha+2}\right)\right]^{1/2} \tag{2.9}$$

The Risk Efficiency of Estimators $\hat{\sigma}_{LB}$ with Respect to $\hat{\sigma}_{SB}$ Under LINEX Loss L(Δ_2)

The risk functions of estimators $\hat{\sigma}_{LB}$ and $\hat{\sigma}_{SB}$ relative to L(Δ_2) are of interest. These risk functions are denoted by R_L($\hat{\sigma}_{LB}$) and R_L($\hat{\sigma}_{SB}$), where subscript L denotes risk relative to L(Δ_2) and are given by using h(*s*) in (2.2) as follows:

$$R_L(\hat{\sigma}_{LB}) = E\left[L(\Delta_2)\right] = \int_0^\infty\left\{e^{a\left[\left(\frac{\hat{\sigma}_{LB}}{\sigma}\right)^2-1\right]} - a\left[\left(\frac{\hat{\sigma}_{LB}}{\sigma}\right)^2-1\right]-1\right\}h(s)ds$$

$$= e^{-a}\int_0^\infty e^{a\left\{\frac{\left[\frac{(s+\beta)}{2a}\left\{1-\exp\left(-\frac{2a}{2n+\alpha+2}\right)\right\}\right]}{\sigma^2}\right\}}\frac{1}{\left(2\sigma^2\right)^n\Gamma n}s^{n-1}\exp\left(-\frac{s}{2\sigma^2}\right)ds$$

$$-\int_0^\infty a\left\{\frac{\left[\frac{(s+\beta)}{2a}\left\{1-\exp\left(-\frac{2a}{2n+\alpha+2}\right)\right\}\right]}{\sigma^2}-1\right\}\frac{1}{\left(2\sigma^2\right)^n\Gamma n}\exp\left(-\frac{s}{2\sigma^2}\right)s^{n-1}ds-1$$

$$=\exp\left[\frac{2an}{2n+\alpha+2}+\frac{\beta}{2\sigma^2}\left\{1-\exp\left(-\frac{2a}{2n+\alpha+2}\right)\right\}-a\right]$$

$$-\left(n+\frac{\beta}{2\sigma^2}\right)\left[1-\exp\left(-\frac{2a}{2n+\alpha+2}\right)\right]+a-1 \tag{2.10}$$

In the same manner, we get

$$R_L\left(\hat{\sigma}_{SB}\right)=E\left[L(\Delta_2)\right]=\int_0^\infty\left\{e^{a\left[\left(\frac{\hat{\sigma}_{SB}}{\sigma}\right)^2-1\right]}-a\left[\left(\frac{\hat{\sigma}_{SB}}{\sigma}\right)^2-1\right]-1\right\}h(s)\,ds$$

$$=e^{-a}\int_0^\infty\exp\left[a\left(\frac{\frac{s}{2}A^2}{\sigma^2}\right)\right]\frac{1}{\left(2\sigma^2\right)^n\Gamma n}s^{n-1}\exp\left(-\frac{s}{2\sigma^2}\right)ds$$

$$-\int_0^\infty\left[a\left(\frac{\frac{s}{2}A^2}{\sigma^2}\right)\right]\frac{1}{\left(2\sigma^2\right)^n\Gamma n}s^{n-1}\exp\left(-\frac{s}{2\sigma^2}\right)ds-1$$

$$=e^{-a}\left[\exp\left(\frac{a\beta}{2\sigma^2}A^2\right)\left(1-aA^2\right)^{-n}\right]-aA^2\left(n+\frac{\beta}{2\sigma^2}\right)+a-1 \tag{2.11}$$

where,

$$A = \frac{\Gamma\left(\frac{2n+\alpha-1}{2}\right)}{\Gamma\left(\frac{2n+\alpha}{2}\right)}$$

The risk efficiency of $\hat{\sigma}_{LB}$ with respect to $\hat{\sigma}_{SB}$ under LINEX Loss $L(\Delta_2)$ may be defined as follows:

$$RE_L\left(\hat{\sigma}_{LB}, \hat{\sigma}_{SB}\right) = \frac{R_L\left(\hat{\sigma}_{SB}\right)}{R_L\left(\hat{\sigma}_{LB}\right)} \tag{2.12}$$

The Risk Efficiency of Estimators $\hat{\sigma}_{LB}$ with Respect to $\hat{\sigma}_{SB}$ Under Squared Error Loss

The risk functions of the estimators $\hat{\sigma}_{LB}$ and $\hat{\sigma}_{SB}$ under squared error loss are denoted by $R_S\left(\hat{\sigma}_{LB}\right)$ and $R_S\left(\hat{\sigma}_{SB}\right)$ and are given by :

$$R_S\left(\hat{\sigma}_{LB}\right) = \int_0^\infty \left(\hat{\sigma}_{LB} - \sigma\right)^2 h(s)\,ds$$

$$= \int_0^\infty \left(\hat{\sigma}_{LB}^2 - 2\hat{\sigma}_{LB}\sigma + \sigma^2\right) \frac{1}{\left(2\sigma^2 \Gamma n\right)} \exp\left(-\frac{s}{2\sigma^2}\right) s^{n-1}\,ds$$

$$= \frac{1}{2a}\left[1 - \exp\left(-\frac{2a}{2n+\alpha+2}\right)\right] \int_0^\infty (s+\beta) \frac{1}{\left(2\sigma^2\right)^n \Gamma n} \exp\left(-\frac{s}{2\sigma^2}\right) s^{n-1}\,ds$$

$$-2\sigma \frac{1}{2a}\left[1 - \exp\left(-\frac{2a}{2n+\alpha+2}\right)\right] \int_0^\infty (s+\beta)^{1/2} \frac{1}{\left(2\sigma^2\right)^n \Gamma n} \exp\left(-\frac{s}{2\sigma^2}\right) s^{n-1}\,ds + \sigma^2$$

On simplification, we get,

$$R_S\left(\hat{\sigma}_{LB}\right) \simeq \phi_1\beta + 2n\sigma^2\phi_1 - 2\sigma\left(\phi_1\right)^{1/2} \frac{e^{\frac{\beta}{2\sigma^2}}}{\Gamma n}\left[\frac{\xi_1}{2\sigma^2} - \frac{(n-1)\beta}{\left(2\sigma^2\right)^{3/2}}\xi_2 + \frac{(n-1)(n-2)}{2!\left(2\sigma^2\right)^{5/2}}\beta^2\xi_3 - \ldots\right] \tag{2.13}$$

where,

$$\varphi_1 = \frac{1}{2a}\left[1-\exp\left(\frac{2a}{2n+\alpha+2}\right)\right]$$

$$\xi_1 = \Gamma\left\{\left(n+\frac{1}{2}\right), \frac{\beta}{2\sigma^2}\right\}$$

$$\xi_2 = \Gamma\left\{\left(n-\frac{1}{2}\right), \frac{\beta}{2\sigma^2}\right\}$$

$$\xi_3 = \Gamma\left\{\left(n-\frac{3}{2}\right), \frac{\beta}{2\sigma^2}\right\}$$

Again,

$$R_S\left(\widehat{\sigma}_{SB}\right) = \int_0^\infty \left(\widehat{\sigma}_{SB} - \sigma\right)^2 h(s)ds$$

$$= \int_0^\infty \left(\widehat{\sigma}_{SB}^2 - 2\widehat{\sigma}_{SB}\sigma + \sigma^2\right)\frac{1}{\left(2\sigma^2\right)^n \Gamma n} s^{n-1} \exp\left(-\frac{s}{2\sigma^2}\right)ds$$

$$= \int_0^\infty \left[\left\{\left(\frac{s+\beta}{2}\right)^{1/2} A\right\}^2 - 2\left\{\left(\frac{s+\beta}{2}\right)^{1/2} A\right\}\sigma + \sigma^2\right]\frac{1}{\left(2\sigma^2\right)^n \Gamma n}\exp\left(-\frac{1}{2\sigma^2}\right)s^{n-1}ds$$

$$= \frac{A^2}{2}\int_0^\infty (s+\beta)\frac{1}{\left(2\sigma^2\right)^n \Gamma n}\exp\left(-\frac{s}{2\sigma^2}\right)s^{n-1}ds$$

$$-2\sigma A\frac{1}{\sqrt{2}}\int_0^\infty (s+\beta)^{1/2}\frac{1}{\left(2\sigma^2\right)^n \Gamma n}\exp\left(-\frac{s}{2\sigma^2}\right)s^{n-1}ds + \sigma^2$$

On simplification, we get,

$$R_S\left(\hat{\sigma}_{SB}\right)=\sigma^2+\frac{A^2}{2}\left(2n\sigma^2+\beta\right)-\sqrt{2}A\sigma\frac{e^{\frac{\beta}{2\sigma^2}}}{\Gamma n}$$

$$\times\left[\frac{\xi_1}{2\sigma^2}-\frac{(n-1)\beta}{\left(2\sigma^2\right)^{3/2}}\xi_2+\frac{(n-1)(n-2)}{2!\left(2\sigma^2\right)^{5/2}}\beta^2\xi_3-\ldots\right] \tag{2.14}$$

The efficiency of $\hat{\sigma}_{LB}$ with respect to $\hat{\sigma}_{SB}$ under squared error loss is defined as:

$$RE_S\left(\hat{\sigma}_{LB},\hat{\sigma}_{SB}\right)=\frac{R_S\left(\hat{\sigma}_{SB}\right)}{R_S\left(\hat{\sigma}_{LB}\right)} \tag{2.15}$$

NUMERICAL EXAMPLE

To compare the proposed estimator $\hat{\sigma}_{LB}$ with the estimator $\hat{\sigma}_{SB}$, the risk functions are computed so as to see whether $\hat{\sigma}_{LB}$ out performs $\hat{\sigma}_{SB}$ under LINEX loss $L(\Delta_2)$ and how $\hat{\sigma}_{LB}$ performs as compared to $\hat{\sigma}_{SB}$ when true loss is squared error. This is essential to check whether an estimator is inadmissible under some loss function. For this purpose the risk efficiency is computed.

One sample does not tell us much, so we generated N = 3000 samples of sizes n =10, 20, 30 from (1.1) with $\sigma=1$.

From table 1 to 14, we note that the risk efficiency $RE_L(\hat{\sigma}_{LB},\hat{\sigma}_{SB})$ is greater than one for all values of 'a' (except for Hartigan Prior with a = -1). We also see that the risk efficiency $RE_S(\hat{\sigma}_{LB},\hat{\sigma}_{SB})$ is greater than one for all values of 'a', which indicates that the proposed estimator $\hat{\sigma}_{LB}$ is preferable to $\hat{\sigma}_{SB}$ for both LINEX and squared error loss functions. For the Hartigan prior, we see that (for a = ± 1), under squared error loss, the proposed estimator $\hat{\sigma}_{LB}$ has smaller risk than $\hat{\sigma}_{SB}$ (RE_S >1).

In general, the values of the risk efficiencies RE_L are very sensitive to variation in α and β .

TABLE 1: The Estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for the values of a =1 and α = β =0

n	$\hat{\sigma}_{SB}$ α=0, β=0	$\hat{\sigma}_{LB}$ α=0, β=0 a = 1	$RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$	$RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$
10	1.0231	0.9174	1.6735	1.1113
20	1.0254	0.9671	1.2906	1.0564
30	1.0278	0.9903	1.1813	1.0375

TABLE 2: The Estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for the values of a =1 and α =0, β =1

n	$\hat{\sigma}_{SB}$ α=1, β=1	$\hat{\sigma}_{LB}$ α=1, β=1 a = 1	$RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$	$RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$
10	1.0491	0.9407	2.0364	1.1162
20	1.0326	0.9770	1.4336	1.0571
30	1.0235	0.9862	1.2692	1.0378

TABLE 3: The Estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for the values of a =1 and α = 0, β =2

n	$\hat{\sigma}_{SB}$ α =1, β=1	$\hat{\sigma}_{LB}$ α =1, β=1 a = 1	$RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$	$RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$
10	1.0683	0.9580	2.4314	1.1191
20	1.0429	0.9867	1.5829	1.0578
30	1.0354	0.9977	1.3636	1.0382

TABLE 4: The Estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for the values of a =1 and α = 1, β =0

n	$\hat{\sigma}_{SB}$ α =1, β=1	$\hat{\sigma}_{LB}$ α =1, β=1 a = 1	$RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$	$RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$
10	0.9887	0.8911	1.3348	1.1051
20	0.9960	0.9436	1.1519	1.0543
30	0.9991	0.9633	1.0994	1.0366

TABLE 5: The Estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for the values of a =1 and α = β =1

n	$\hat{\sigma}_{SB}$ α =1, β =1	$\hat{\sigma}_{LB}$ α =1, β =1 a = 1	$RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$	$RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$
10	1.0308	0.9290	1.6182	1.1078
20	1.0310	0.9768	1.2768	1.0550
30	1.0310	0.9941	1.1806	1.0369

TABLE 6: The Estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for the values of a =1 and α =1 and β =2

n	$\hat{\sigma}_{SB}$ α =1, β =2	$\hat{\sigma}_{LB}$ α =1, β =2 a = 1	$RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$	$RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$
10	1.0350	0.9329	1.9709	1.1105
20	1.0361	0.9816	1.4213	1.0557
30	1.0392	1.0020	1.2649	1.0372

TABLE 7: The Estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for the values of a =1 and α =2 and β =0

n	$\hat{\sigma}_{SB}$ α =2, β =0	$\hat{\sigma}_{LB}$ α =2, β =0 a = 1	$RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$	$RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$
10	0.9803	0.8876	1.0917	1.0977
20	0.9998	0.9484	1.0369	1.0524
30	1.0142	0.9784	1.0244	1.0357

TABLE 8: The Estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for the values of a = -1 and α = β =0

n	$\hat{\sigma}_{SB}$ α = 0, β = 0	$\hat{\sigma}_{LB}$ α = 0, β = 0 a = -1	$RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$	$RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$
10	1.0277	0.9644	1.8081	1.0661
20	1.0283	0.9964	1.0826	1.0321
30	1.0290	1.0076	1.0552	1.0212

TABLE 9: The Estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for the values of a = -1 and α =0, β =1

n	$\hat{\sigma}_{SB}$ α =1, β =1	$\hat{\sigma}_{LB}$ α =1, β =1 a = 1	$RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$	$RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$
10	1.0302	0.9666	1.3241	1.0678
20	1.0112	0.9799	1.1502	1.0325
30	1.0064	0.9855	1.1006	1.0214

TABLE 10: The Estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for the values of a = -1 and α =0, β =2

n	$\hat{\sigma}_{SB}$ α =1, β =1	$\hat{\sigma}_{LB}$ α =1, β =1 a = 1	$RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$	$RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$
10	1.0617	0.9963	1.4812	1.0693
20	1.0462	1.0137	1.2217	1.0329
30	1.0375	1.0160	1.1465	1.0216

TABLE 11: The Estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for the values of a = -1 and α =1, β =0

n	$\hat{\sigma}_{SB}$ α =1, β =1	$\hat{\sigma}_{LB}$ α =1, β =1 a = -1	$RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$	$RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$
10	0.9871	0.9292	1.0355	1.0613
20	0.9924	0.9624	1.0163	1.0309
30	0.9986	0.9782	1.0122	1.0207

TABLE 12: The Estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for the values of a = -1 and α = β =1

n	$\hat{\sigma}_{SB}$ α =1, β =1	$\hat{\sigma}_{LB}$ α =1, β =1 a = -1	$RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$	$RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$
10	1.0145	0.9550	1.1620	1.0628
20	1.0168	0.9860	1.0826	1.0305
30	1.0255	1.0046	1.0506	1.0209

TABLE 13: The Estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for the values of a= -1 and α = 1 and β =2

n	$\hat{\sigma}_{SB}$ α=1,β=2	$\hat{\sigma}_{LB}$ α=1, β=2 a = -1	$RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$	$RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$
10	1.0191	0.9593	1.3149	1.0643
20	1.0283	0.9972	1.1486	1.0317
30	1.0315	1.0105	1.0974	1.0210

TABLE 14: The Estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for the values of a = -1 and α =2 and β =0

n	$\hat{\sigma}_{SB}$ α =2, β =0	$\hat{\sigma}_{LB}$ α =2, β =0 a = -1	$RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$	$RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$
10	0.9554	0.9018	0.9345	1.0569
20	0.9886	0.9594	0.9643	1.0298
30	1.0014	0.9813	0.9702	1.0202

BAYES ESTIMATOR OF $\overline{F(t)}$

Let $\gamma = \overline{F(t)}$ be the probability that a system will survive a specified mission time t. By substituting $\sigma^2 = \frac{t^2}{-2\log\gamma}$ in (2.3), we obtain the posterior PDF of γ as :

$$\pi_1(\gamma|x) = \left(\frac{\left((\frac{s+\beta}{2})^{\frac{2n+\alpha}{2}} \right)}{\Gamma\left(\frac{2n+\alpha}{2}\right)} \frac{\exp\left(\frac{s+\beta}{\left(-\frac{t^2}{\log\gamma}\right)} \right)}{\left(\left(-\frac{t^2}{2\log\gamma}\right)^{\frac{2n+\alpha+2}{2}} \right)} \left(\frac{t^2}{2}\right) (\log\gamma)^{-2} \frac{1}{\gamma} \right)$$

$$= \frac{1}{\Gamma\left(\frac{2n+\alpha}{2}\right)} \left(\frac{s+\beta}{t^2}\right)^{\frac{2n+\alpha}{2}} \gamma^{\frac{2n+\alpha}{2}-1} (-\log\gamma)^{\frac{2n+\alpha}{2}-1}, \quad 0<\gamma<1 \tag{4.1}$$

Using the convex loss function $L(\Delta_1)$, $\Delta_1 = \hat{\gamma} - \gamma$, it is seen that this loss function is quite asymmetric when a =1 (see eqn. 1.3) with overestimation being more serious than underestimation. Also, when $a < 0$ and $\Delta_1 < 0$, $L(\Delta_1)$ rises almost exponentially and almost linearly when $a < 0$ and $\Delta_1 > 0$. For small values of $|a|$ the loss function $L(\Delta_1)$ is approximately squared error loss and therefore almost symmetric.

The posterior expectation of the LINEX loss function $L(\Delta_1)$ is:

$$E_{\gamma_{LB}}\left[L(\Delta_1)\right]=\int_0^1\left[e^{a(\hat{\gamma}-\gamma)}-a\left(\hat{\gamma}-\gamma\right)-1\right]\pi_1\left(\gamma|x\right)d\gamma$$

$$=e^{a\hat{\gamma}}E_{\gamma}\left(e^{-a\gamma}\right)+E_{\gamma}\left(a\gamma\right)-a\hat{\gamma}-1$$

For a minimum to exist at $\Delta_1 = 0$

$$\frac{\partial E\left[L(\Delta_1)\right]}{\partial\hat{\gamma}}=ae^{a\hat{\gamma}}E_{\gamma}\left(e^{-a\gamma}\right)-a=0$$

$$\Rightarrow E_{\gamma}\left(e^{-a\gamma}\right)=e^{-a\hat{\gamma}}$$

leading to Bayes estimator of γ relative to $L(\Delta_1)$, denoted by $\hat{\gamma}_{LB}$ and is given by (after some algebraic manipulation)

$$\hat{\gamma}_{LB}=-\frac{1}{a}\log\sum_{j=0}^{\alpha}\frac{(-a)^j}{j!}\left(1+\frac{jt^2}{s+\beta}\right)^{-\frac{2n+\alpha}{2}} \tag{4.2}$$

Considering the squared error loss $\left(L\left(\hat{\gamma},\gamma\right)=\left(\hat{\gamma}-\gamma\right)^2\right)$, the Bayes estimator of γ, denoted by $\hat{\gamma}_{SB}$ for the above prior , given the posterior mean of γ is then

$$\hat{\gamma}_{SB}=\int_0^{\infty}\gamma\pi_1\left(\gamma|x\right)d\gamma$$

$$=\int_0^{\infty}\gamma\frac{1}{\Gamma\left(\frac{2n+\alpha}{2}\right)}\left(\frac{s+\beta}{t^2}\right)^{\frac{2n+\alpha}{2}}\gamma^{\frac{2n+\alpha}{2}-1}\left(-\log\gamma\right)^{\frac{2n+\alpha}{2}-1}d\gamma$$

After some algebraic manipulation, the Bayes' estimator of γ is obtained as:

$$\hat{\gamma}_{SB} = \left(1 + \frac{t^2}{s+\beta}\right)^{-\frac{2n+\alpha}{2}} \qquad (4.3)$$

It can be seen that the risk functions relative to loss function $L(\Delta_1)$ do not exist. Let us consider the set of data generated in example 3, let t = 1, we compute $\hat{\gamma}_{LB}$ and $\hat{\gamma}_{SB}$, the results with the corresponding values of '*a*' and '*n*' are given in Table 15. Entries within parentheses represent the corresponding RMSE.

It can be seen from Table 15 that, the Bayes estimate $\hat{\gamma}_{LB}$ of reliability function relative to $L(\Delta_1)$, for negative value of '*a*' has smaller RMSE than that of the Bayes estimates $\hat{\gamma}_{SB}$ under squared error loss function. From the results in Table 15, we can conclude that in situations involving reliability estimation, asymmetric loss function with a = -1.0 is more appropriate than squared error loss function i.e., when under estimation becomes more serious than over estimation. Further, it is to be noted that, Hartigan prior gives better results than Jeffreys prior.

TABLE 15: Bayes estimation of reliability function based on LINEX Loss ($\hat{R}_{LB}(t)$) and Squared Error Loss ($\hat{R}_{SB}(t)$) for variation in '*a*', 'α' and 'β' (for t = 1, the true value $R(t)_{t=1}$ = .6065)

n	α	β	*a*	$R(t)_{LB}$	$R(t)_{SB}$
10	0	0	1	.5870(0.1097)	.5915(0.1079)
20	0	0	1	.6022(0.0633)	.6044(0.0627)
30	0	0	1	.6121(0.0518)	.6135(0.0517)
10	2	0	1	.6206(0.1092)	.5650(0.1064)
20	2	0	1	.6211(0.0837)	.5934(0.0824)
30	2	0	1	.6228(0.0619)	.6045(0.0597)
10	1	1	1	.6211(0.0940)	.5959(0.0925)
20	1	1	1	.6239(0.0720)	.6113(0.0715)
30	1	1	1	.6242(0.0557)	.6154(0.0553)
10	1	2	1	.6251(0.0884)	.6000(0.0871)
20	1	2	1	.6261(0.0719)	.6136(0.0714)
30	1	2	1	.6295(0.0561)	.6211(0.0560)
10	0	0	-1	.5966(0.0984)	.5922(0.1004)
20	0	0	-1	.6092(0.0793)	.6070(0.0803)
30	0	0	-1	.6115(0.0563)	.6131(0.0570)
10	2	0	-1	.6173(0.1036)	.5513(0.1063)
20	2	0	-1	.6179(0.0737)	.5854(0.0748)
30	2	0	-1	.6200(0.0498)	.5985(0.0506)
10	1	1	-1	.6207(0.0868)	.5865(0.0884)
20	1	1	-1	.6212(0.0712)	.6041(0.0720)
30	1	1	-1	.6243(0.0551)	.6130(0.0552)
10	1	2	-1	.6265(0.0873)	.5923(0.0884)
20	1	2	-1	.6267(0.0607)	.6097(0.0611)
30	1	2	-1	.6270(0.0475)	.6158(0.0478)

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