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On Double Sumudu Transform and Double Laplace Transform

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ABSTRACT

In this study, first of all, we consider wave equations with constant coefficients. By using convolution we then produce a new equation with variable coefficients. Finally, we apply two techniques: double Laplace transform and double Sumudu transform to solve the new wave equation with non-constant coefficients and establish a relationship between double Sumudu transform and double Laplace transform.

Keywords: Double Laplace transform, single Laplace transform, double Sumudu transform and convolution theorem.

INTRODUCTION

The wave equation is known as one of fundamental equations in mathematical physics and occurs in many branches of physics, in applied mathematics as well as in engineering. It is also known that there are two types of these equations; The homogenous equation with constant coefficients has many classical solutions such as separation of variables (Lamb,1995), the methods of characteristics (Myint, 1980 and Constanda, 2002), single Laplace transform and Fourier transform (Duffy, 2004), and non-homogenous equations with constant coefficient which is solved by double Laplace transform (Babakhani and Dahiya, 2001), and operation calculus (Brychkov *et al.*, 1992).

In this study we use double Laplace transform and double Sumudu transform to solve non homogenous wave equation with non-constant coefficient where the non homogenous terms are double convolution functions. In this study we follow the method that was proposed by A. Kiliçman and H. Eltayeb (Kilicman and Eltayeb, 2008) and (Estrin and Higgins, 1951), where they extended one dimensional convolution theorem to two dimensional case.

First of all we recall the following definitions which were given by Tchuenche and Nyimvua (Tchuenche and Nyimvua, 2007).

Definition 1:

The double Sumudu transform is defined by

$$F(u,v) = S_2\left[f(t,x);(u,v)\right] = \frac{1}{uv}\int_0^{\infty}\int_0^{\infty} e^{-(\frac{t}{v}+\frac{x}{u})} f(t,x)\,dtdx$$

and the double Sumudu transform of second partial derivative with respect to x is of form

$$S_2\left[\frac{\partial^2 f(x,t)}{\partial x^2};(u,v)\right] = \frac{1}{uv}\int_0^{\infty}\int_0^{\infty} e^{-\left(\frac{t}{v}+\frac{x}{u}\right)} \frac{\partial^2 f(x,t)}{\partial x^2}\,dtdx$$
$$= \frac{1}{v}\int_0^{\infty} e^{-\frac{t}{v}}\left(\frac{1}{u}\int_0^{\infty} e^{-\frac{x}{u}} \frac{\partial^2 f(x,t)}{\partial x^2}\,dx\right)dt.$$

The integral inside the bracket can be computed as

$$\frac{1}{u}\int_0^{\infty} e^{-\frac{x}{u}} \frac{\partial^2 f(x,t)}{\partial x^2}dx = \frac{1}{u^2}F(u,t) - \frac{1}{u^2}f(t,0) - \frac{1}{u}\frac{\partial f(0,t)}{\partial x}. \tag{1}$$

By taking Sumudu transform with respect to t for equation (1), we get double Sumudu transform in the form of

$$S_2\left[\frac{\partial^2 f(x,t)}{\partial x^2};(u,v)\right] = \frac{1}{u^2}F(u,v) - \frac{1}{u^2}F(0,v) - \frac{1}{u}\frac{\partial F(0,v)}{\partial x}. \tag{2}$$

Similarly, the double Sumudu transform of $\dfrac{\partial^2 f(t,x)}{\partial t^2}$ is given by

$$S\left[\frac{\partial^2 f(x,t)}{\partial t^2};(u,v)\right] = \frac{1}{v^2}F(u,v) - \frac{1}{v^2}F(u,0) - \frac{1}{v}\frac{\partial f(u,0)}{\partial t} \tag{3}$$

and Estrin and Higgins (Estrin and Higgins, 1951) defined double Laplace transform by

$$L_x L_t\left[f(x,t);(p,s)\right] = F(p,s) = \int_0^\infty e^{-px}\int_0^\infty e^{-st} f(x,t)dtdx$$

where $x,t>0$ and p,s complex value

and defined the first order partial derivative as

$$L_x L_t\left[\frac{\partial f(x,t)}{\partial x};(p,s)\right] = pF(p,s) - F(0,s).$$

The double Laplace transform for second partial derivative with respect to x given by

$$L_x L_t\left[\frac{\partial^2 f(x,t)}{\partial^2 x};(p,s)\right] = p^2F(p,s) - pF(0,s) - \frac{\partial F(0,s)}{\partial x}$$

and double Laplace transform for second partial derivative with respect to t similarly as above given by

$$L_x L_t\left[\frac{\partial^2 f(x,t)}{\partial^2 t};(p,s)\right] = s^2F(p,s) - sF(p,0) - \frac{\partial F(p,0)}{\partial t}.$$

In a similar manner the double Laplace transform of a mixed partial derivative can be deduced from the single Laplace transform as

$$L_x L_t\left[\frac{\partial^2 f(x,t)}{\partial x\partial t};(p,s)\right] = psF(p,s) - pF(p,0) - sF(0,s) - F(0,0)\ .$$

The double Sumudu transform and double Laplace transform having strong relation (see Tchuenche and Nyimvua, 2007.) may be expressed either as

(I) $uvF(u,v) = £_2\left(f(x,y);\left(\frac{1}{u},\frac{1}{v}\right)\right)$

or

(II) $psF(p,s) = £_2\left(f(x,y);\left(\frac{1}{p},\frac{1}{s}\right)\right),$

where $\pounds_2$ represents the operation of double Laplace transform. In particular, this relation is best illustrated by the fact that the double Sumudu and double Laplace transform interchange the image of $\sin(x+t)$ and $\cos(x+t)$ that are

$$S_2\left[\sin(x+t)\right]=\pounds_2\left[\cos(x+t)\right]=\frac{u+v}{(1+u)^2(1+v)^2},$$

and

$$S_2\left[\cos(x+t)\right]=\pounds_2\left[\sin(x+t)\right]=\frac{1}{(1+u)^2(1+v)^2}.$$

We note that the relation between double Sumudu of convolution and double Laplace transform of convolution is

$$S_2\left[(f**g)(t,x);(u,v)\right]=\frac{1}{uv}\pounds_2(f**g)(t,x)),$$

where

$$F_1(x,y)**F_2(x,y)=\int_0^y\int_0^x F_1(x-\theta_1,y-\theta_2)F_2(\theta_1,\theta_2)d\theta_1 d\theta_2.$$

WAVE EQUATION IN ONE DIMENSIONAL SPACE AND DOUBLE LAPLACE TRANSFORM AND DOUBLE SUMUDU TRANSFORM

Consider non-homogenous one dimensional wave equation with non-constant coefficient in the form

$$p(x,t)**\left[u_{tt}-u_{xx}\right]=\sum_{i=1}^{n} f(x,t)**g_i(x,t) \tag{4}$$

where the conditions

$$\begin{aligned} &u(0,t)=\overline{g}_1(t),\ u(x,0)=h_1(x)\\ &u_x(0,t)=\frac{\partial}{\partial t}\overline{g}_1(t),\quad u_t(x,0)=\frac{\partial}{\partial t}h_1(x), \end{aligned} \tag{5}$$

where $p(x,t)$ and $g_i(x,t)$ are polynomials.

Now we let $F(x,t)$ be a solution of

$$u_{tt}(x,t)-u_{xx}(x,t)=\sum_{i=1}^{n} f(x,t)**g_i(x,t) \qquad (x,t)\in \mathrm{R}_+^2 \tag{6}$$

and further we consider $K(x,t)$ as a solution of

$$p(x,t)**\left(u_{tt}(x,t)-u_{xx}(x,t)\right)=\sum_{i=1}^{n} f(x,t)**g_i(x,t) \quad (t,x)\in \mathrm{R}_+^2 \tag{7}$$

then $F(x,t)$ satisfies equation (6):

$$F_{tt}(x,t)-F_{xx}(x,t)=\sum_{i=1}^{n} f(x,t)**g_i(x,t) \tag{8}$$

and similarly, $K(x,t)$ satisfies equation (8):

$$K_{tt}(x,t)-K_{xx}(x,t)=\frac{1}{i!\,j!} f(x,t)\ . \tag{9}$$

Now, we can easily check what the convolution $F(x,t)**K(x,t)$ is not a solution of (6). Indeed from (6) it follows that

$$\left(F(x,t)**K(x,t)\right)_{tt}-\left(F(x,t)**K(x,t)\right)_{xx}\overset{?}{=}\sum_{i=1}^{n} f(x,t)**g_i(x,t) \tag{10}$$

on using the partial derivatives of the convolution; thus the left hand side of equation (10) gives

$$F_{tt}(x,t)**K(x,t)-F_{xx}(x,t)**K(x,t)=F(x,t)**K_{tt}(x,t)-F(x,t)**K_{xx}(x,t)$$

and then equation (10) can be written in the form

$$F(x,t)**\left[K_{tt}(x,t)-K_{xx}(x,t)\right]\overset{?}{=}\sum_{i=1}^{n} f(x,t)**g_i(x,t) \tag{11}$$

and

$$\left[F_{tt}(x,t)-F_{xx}(x,t)\right]**K(x,t)\overset{?}{=}\sum_{i=1}^{n} f(x,t)**g_i(x,t) \tag{12}$$

by substituting equation (7) into (9) and equation (6) into (10) we have

$$F(x,t) ** \frac{1}{i!\,j!} f(x,t) \neq \sum_{i=1}^{n} f(x,t) ** g_i(x,t) \tag{13}$$

and

$$f(x,t) ** K(x,t) \neq \sum_{i=1}^{n} f(x,t) ** g_i(x,t) \tag{14}$$

and thus we can easily see from the equations (13) and (14) that the convolution $F(x,t) ** K(x,t)$ is not a solution of equation (6). In general, however, it is a solution for another type of equation as given in the following theorem.

Theorem 1: If $F(x,t)$ is a solution of

$$u_{tt} - u_{xx} = \sum_{i=1}^{n} f(x,t) ** g_i(x,t) \tag{15}$$

under the initial condition

$$u(0,t) = \overline{g}_1(t), \; u(x,0) = h_1(x)$$

$$u_x(0,t) = \frac{\partial}{\partial t}\overline{g}_1(t), \quad u_t(x,0) = \frac{\partial}{\partial t} h_1(x),$$

and $K(x,t)$ is a solution of

$$p(x,t) ** \left(u_{tt} - u_{xx}\right) = \sum_{i=1}^{n} f(x,t) ** g_i(x,t) \qquad (t,x) \in \mathrm{R}_+^2 \tag{16}$$

under the same conditions, then $F(x,t) ** K(x,t)$ is a solution of the following equation

$$u_{tt}(x,t) - u_{xx}(x,t) - h(x,t) = \sum_{i=1}^{n} f(x,t) ** g_i(x,t) \quad (t,x) \in \mathrm{R}_+^2 \tag{17}$$

where $f(x,t)$ is an exponential function and $p(x,t)$ is a polynomial.

Proof: Since $F(x,t)$ is a solution of equation (15) then

$$F_{tt}(x,t) - F_{xx}(x,t) = \sum_{i=1}^{n} f(x,t) ** g_i(x,t) \tag{18}$$

holds and $K(x,t)$ is a solution of equation (16) then

$$K_{tt}(x,t)-K_{xx}(x,t)=\frac{1}{i!j!}f(x,t) \tag{19}$$

is also true and then by substitution what we have

$$\left(F(x,t)**K(x,t)\right)_{tt}-\left(F(x,t)**K(x,t)\right)_{xx}-h(x,t)=\sum_{i=1}^{n}f(x,t)**g_i(x,t) \tag{20}$$

on using the partial derivative of convolution, we obtain

$$F_{tt}(x,t)**K(x,t)-F_{xx}(x,t)**K(x,t)=F(x,t)**K_{tt}(x,t)-F(x,t)**K_{xx}(x,t)$$

and then equation (20) is followed by

$$\left[F_{tt}(x,t)-F_{xx}(x,t)\right]**K(x,t)-h(x,t)=\sum_{i=1}^{n}f(x,t)**g_i(x,t) \tag{21}$$

by substituting equation (18) into (21) we have

$$f(x,t)**K(x,t)-h(x,t)=\sum_{i=1}^{n}f(x,t)**g_i(x,t) \tag{22}$$

this shows that the convolution $F(x,t)**K(x,t)$ is a solution of equation (17).

In the next two examples we apply double Laplace transform and double Sumudu transform for wave equation. We compare the solution of one dimensional wave equation having constant and non constant coefficients by using two techniques.

Example: Consider the one dimensional wave equation in the form

$$\begin{aligned} &u_{tt}-u_{xx}=e^{x+t}**xt^2+e^{x+t}**x^2t^3 \qquad (x,t)\in \mathrm{R}_+^2 \\ &u(x,0)=xe^x,\ \ u_t(x,0)=xe^x+e^x \\ &u(0,t)=te^t,\ \ u_x(0,t)=te^t+e^t \end{aligned} \tag{23}$$

by taking double Laplace Transform and single Laplace Transform for equation (23), we obtain

$$U(p,s)=\frac{s}{(p-1)^2\left(s^2-p^2\right)}+\frac{p}{(p-1)^2\left(s^2-p^2\right)}-\frac{p}{(s-1)^2\left(s^2-p^2\right)}$$
$$-\frac{s}{(s-1)^2\left(s^2-p^2\right)}+\frac{2}{p^2s^3(p-1)(s-1)\left(s^2-p^2\right)} \tag{24}$$
$$+\frac{12}{p^3s^4(p-1)(s-1)\left(s^2-p^2\right)}.$$

Now by taking double inverse Laplace transform with respect to s, p for equation (24), we obtain the solution of equation (23) in the form

$$u(x,t)=26+\frac{13}{12}t^4+14xt+\frac{5}{2}e^{-t+x}+6x^2-6e^tx^2-14e^tx+7e^{t+x}t+2e^xt^3$$
$$+7e^xt^2+26e^xt+x^2t^3+7xt^2+\frac{7}{3}t^3x-\frac{61}{2}e^{t+x}+13t^2+\frac{13}{3}t^3$$
$$+\frac{1}{420}t^7+\frac{1}{20}t^5x^2+\frac{1}{10}t^5x+28e^x+\frac{7}{12}t^4x+26t+14x$$
$$+\frac{1}{5}t^5+3x^2t^2+6x^2t+\frac{1}{60}t^6-26e^t+\frac{1}{4}t^4x^2.$$

Now, we multiply the left hand side equation of (23) by non-constant coefficient $x^3t^4 **$ where the symbol $**$ means a double convolution with respect to x and t respectively, and apply the same technique that we used above then we get the solution in the form of

$$v(x,t)=\frac{1}{32}e^{t+x}-\frac{5}{288}e^{-t+x}+\frac{7}{144}e^{t+x}t\ . \tag{25}$$

If we take second derivatives with respect to t and x for equation (25), and taking the difference between them and multiply the result by $x^3t^4 **$, we obtain the non-homogenous term plus a function $h(x,t)$. That is

$$(x^3t^4)**\left(v_{tt}-v_{xx}\right)=\left(u_{tt}-u_{xx}\right)+h(x,t).$$

In the next Theorem we discuss the Sumudu transform of double convolution as follows.

Theorem 2: Let $f(x,t)$ and $g(x,t)$ be double Sumudu transformable. Then double Sumudu transform of the double convolution of the $f(x,t)$ and $g(x,t)$ exists

$$(f**g)(x,t)=\int_0^t\int_0^x f(\zeta,\eta)g(x-\zeta,t-\eta)d\zeta d\eta$$

and is given by

$$S_2\left[(f**g)(x,t);(u,v)\right]=uvF(u,v)G(u,v).$$

Proof: By using the definition of double Sumudu transform and double convolution, we have

$$S_2\left[(f**g)(x,t);(u,v)\right]=\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(\frac{t}{v}+\frac{x}{u})}(f**g)(t,x)dtdx$$
$$=\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-(\frac{t}{v}+\frac{x}{u})}\left(\int_0^t\int_0^x f(\zeta,\eta)g(x-\zeta,t-\eta)d\zeta d\eta\right)dtdx$$

let $\alpha=x-\zeta$ and $\beta=t-\eta$ and using the valid extension of upper bound of integrals to $t\to\infty$ and $x\to\infty$, it yields

$$S_2\left[(f**g)(x,t);(u,v)\right]=\frac{1}{uv}\int_0^\infty\int_0^\infty e^{-\left(\frac{\zeta}{v}+\frac{\eta}{u}\right)}f(x-\alpha,t-\beta,)d\zeta d\eta\int_{-\zeta}^\infty\int_{-\eta}^\infty e^{-\left(\frac{\alpha}{v}+\frac{\beta}{u}\right)}g(\alpha,\beta)d\alpha d\beta$$

where both function $f(x,t)$ and $g(x,t)$ have zero value for $t<0,$ and $x<0,$ and it follows with respect to lower limit of integrations that

$$S_2\left[(f**g)(t,x);v,u\right]=\frac{1}{uv}\int_0^\infty\int_0^\infty f(\zeta,\eta)e^{-\left(\frac{\zeta}{v}-\frac{\eta}{u}\right)}d\zeta d\eta\int_0^\infty\int_0^\infty e^{-\left(\frac{\alpha}{v}-\frac{\beta}{u}\right)}g(\alpha,\beta)d\alpha d\beta$$

then, it is easy to see that

$$S_2\left[(f**g)(t,x);(u,v)\right]=uvF(v,u)G(v,u).$$

In particular, consider the functions $f(x,t)=\sin(x+t)$ and $g(x,t)=\cos(x+t).$ We can easily prove that

$$S_2\left[(f**g)(t,x);(u,v)\right]=uvF(v,u)G(v,u),$$

the left hand side of the above equation is given by

$$\begin{aligned} &S_2\left[(\sin(x+t)**\cos(x+t))(t,x);(u,v)\right] \\ &=\frac{-vu(-v+vu^2-u+v^2u)}{(1+v^2)^2(1+u^2)^2)} \end{aligned} \tag{26}$$

and the right hand side is given by

$$\begin{aligned} uvF(v,u)G(v,u) &= uv\left[\frac{(v+u)}{(1+v^2)(1+u^2)}\right]\left[-\frac{(-1+uv)}{(1+v^2)(1+u^2)}\right] \\ &=\frac{-uv\left(-v+vu^2-u+v^2u\right)}{(1+v^2)^2(1+u^2)^2}. \end{aligned} \tag{27}$$

From equations (26) and (27) we have

$$S_2\left[(f**g)(t,x);(u,v)\right]=uvF(v,u)G(v,u).$$

Now, we are going back to apply the double Sumudu transform for the wave equation that is given in example1 as follows

$$\begin{aligned} &F_{tt}-F_{xx}=e^{x+t}**xt^2+e^{x+t}**x^2t^3 \qquad (x,t)\in \mathrm{R}_+^2 \\ &F(x,0)=xe^x,\ F_t(x,0)=xe^x+e^x \\ &F(0,t)=te^t,\ F_x(0,t)=te^t+e^t. \end{aligned} \tag{26}$$

By taking the double Sumudu transform wave equation and single Sumudu transform of equation for the conditions, we obtain

$$\begin{aligned} F(u,v) &= \frac{v^3u^2}{u^2(v-1)^2(u^2-v^2)}+\frac{v^2u^2}{u(v-1)^2(u^2-v^2)} \\ &-\frac{v^2u^3}{v^2(u-1)^2(u^2-v^2)}-\frac{v^2u^2}{v(u-1)^2(u^2-v^2)} \\ &+\frac{2v^5u^4}{(1-u)(1-v)(u^2-v^2)}+\frac{12v^6u^5}{(1-u)(1-v)(u^2-v^2)}. \end{aligned} \tag{27}$$

In order to find the inverse double Sumudu transform of equation (27), we use the next theorem.

Theorem 3: Let $G(u)$ be the Sumudu transform of $f(t)$ such that

(i) $\frac{G(\frac{1}{s})}{s}$ is a meromorphic function, with singularities having $\mathrm{Re}(s) < \gamma$, and

(ii) there exists a circular region Γ with radius R and positive constants, M and K, with

$$\left|\frac{G(\frac{1}{s})}{s}\right| < MR^{-K},$$

then the function $f(t)$ is given by

$$S^{-1}(G(s)) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} G\left(\frac{1}{s}\right)\frac{ds}{s} = \sum \text{residual}\left[e^{st}\frac{G(\frac{1}{s})}{s}\right].$$

For the proof see (Watugala, 1993).

Now, by taking double inverse Sumudu transform for both sides of equation (27) we obtain the solution of equation (26) as follows

$$\begin{aligned} F(x,t) = {} & 26 + \frac{13}{3}t^3 + 7e^{t+x}t - 14e^t x + 7e^x t^2 + 26e^x t + \frac{5}{2}e^{-t+x} + 28e^x + 14xt \\ & -26e^t + 26t + 14x + 7xt^2 - 6e^t x^2 + \frac{1}{60}t^6 + 3x^2t^2 + \frac{13}{12}t^4 + \frac{7}{12}t^4 x \\ & +6x^2 + 13t^2 + \frac{1}{5}t^5 + \frac{1}{420}t^7 + \frac{7}{3}t^3 x + \frac{1}{20}t^5 x^2 + 6x^2 t - \frac{61}{2}e^{t+x} \\ & + \frac{1}{10}t^5 x + \frac{1}{4}t^4 x^2 + 2e^x t^3 + x^2 t^3. \end{aligned}$$

Now, we multiplying the left hand side equation of (26) by non-constant coefficient $x^3t^4 **$ where the symbol $**$ means a double convolution with respect to x and t respectively, then equation (26) becomes

$$x^3t^4 ** \left(F_{tt} - F_{xx}\right) = e^{x+t} ** xt^2 + e^{x+t} ** x^2t^3 \qquad (x,t) \in \mathrm{R}_+^2$$

$$F(x,0) = xe^x,\ F_t(x,0) = xe^x + e^x \tag{28}$$

$$F(0,t) = te^t,\ F_x(0,t) = te^t + e^t.$$

Similarly, we apply the double Sumudu transform technique and single Sumudu transform for equation (28) to obtain

$$\begin{aligned} F(u,v) = & \frac{v^3u^2}{u^2(v-1)^2(u^2-v^2)} + \frac{v^2u^2}{u(v-1)^2(u^2-v^2)} \\ & - \frac{v^2u^3}{v^2(u-1)^2(u^2-v^2)} - \frac{v^2u^2}{v(u-1)^2(u^2-v^2)} \\ & + \frac{vu}{72(1-u)(1-v)(u^2-v^2)} + \frac{v^2u^2}{12(1-u)(1-v)(u^2-v^2)}. \end{aligned} \tag{29}$$

Now, by taking double inverse Sumudu transform for both sides of equation (29) we obtain the solution of equation (28) as follows

$$F_1(x,t) = -\frac{5}{288}e^{t+x} + \frac{5}{288}e^{-t+x} + \frac{7}{144}e^{t+x}t.$$

CONCLUSION

Thus we note that the wave equation in one dimensional with the non-constant coefficients (polynomials), under the initial conditions, give similar results when we use the double Laplace and double Sumudu transforms.

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