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On Subalgebras of Genetic Algebras Arising on Mathematical Models of Population Genetics

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ABSTRACT

In the present work we consider genetic algebras which are generated by a quadratic stochastic operator, tournaments and their algebras in the case of small dimensions.

Keywords*:* Associator, simplex, quadratic stochastic operator, evolutionary operator, genetic algebra, Volterra operator.

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INTRODUCTION

While modern understanding of genetic inheritance initiated with the theories of Charles Darwin, it was the Augustinian monk George Mendel who began to uncover the mathematical nature of the subject. In fact, the symbolism Mendel used to describe his first result (e.g., see his 1866 paper *Experiments in Plant-Hybridization* (Mendel, (1959)) is quite algebraically suggestive. Seventy four years later, Etherington introduced the formal language of abstract algebra to the study of genetics in his series of seminal papers (Etherington, (1939)), Etherington, (1941)) and Etherington, (1941)). In 1997 by Mary Lynn Reed explored the non-associative algebraic structure that naturally occurs as genetic information gets passed down through the generations and was discussed the concepts of genetics that suggest the underlying algebraic structure of inheritance (Reed, (1997)).

So called genetic (evolutionary) algebras naturally appear in the problems of the population genetics. Mathematically, the algebras that arise in genetics are very interesting structures. They are generally commutative but non-associative. Therefore, one has to deal with problems of the classification such algebras. The notions of a quadratic stochastic operator, the vertices of simplex, the fixed points of Volterrian quadratic stochastic operators, the tournaments and some properties were sufficiently studied by Ganikhodjaev (Ganikhodjaev, (1993)). Moreover, Volterrian quadratic stochastic operators on infinite dimensional simplex were studied by Mukhamedov.

In this paper genetic algebras, generated by the quadratic stochastic operator were studied and some properties of tournaments and their algebras in small dimensions were discussed.

BASIC CONCEPTS AND NOTATIONS

In this section we will give definitions of some necessary algebras, Volterrian quadratic stochastic operator and biological meaning of these notions.

Let A be algebra over the real field R. We consider a finite dimensional commutative, but generally, non-associative algebra over the field R. The following expression is called an *associator*:

$$(x,y,z)=(x\circ y)\circ z-x\circ(y\circ z).$$

Depending on the additional identities put on the associator we obtain various classes of *nonassociative algebras.* The most important algebras among them are (Abraham, (1980), Bernstein, (1924) and Kesten, (1970)):

1) *Jordan algebras:* $[(x\circ x)\circ y]\circ x=(x\circ x)\circ(y\circ x)$,
2) *Elastic algebras:* $(x\circ y)\circ x=x\circ(y\circ x)$,
3) *Alternative algebras:*

$$(x,x,y)=(y,x,x)=0\Leftrightarrow \quad a)\ (x\circ x)\circ y=x\circ(x\circ y),$$
$$b)\ (y\circ x)\circ x=y\circ(x\circ x).$$

Denote by

$$S^{n-1}=\left\{x=(x_1,x_2,...,x_n)\in R^n:\sum_{k=1}^{n}x_k=1,x_k\geq 0\right\}$$

$(n-1)$ -dimensional simplex.

Let $\{p_{ij,k}\}_{i,j=\overline{1,n}}$ be the set of a nonnegative numbers satisfying the following conditions:

$$p_{ij,k}=p_{ji,k} \text{ и } \sum_{k=1}^{n}p_{ij,k}=1.$$

In biology $\{p_{ij,k}\}$ are called coefficients of heredity, and the transition from the distribution of specie's probability in this generation to the distribution of specie's probability in the next generations is determined by

$$x_k^{'} = \sum_{i,j=1}^{n} p_{ij,k} x_i x_j , \tag{1}$$

where $x = (x_1, x_2, ..., x_n) \in S^{n-1}$.

The last equality determines a mapping $V : S^{n-1} \to S^{n-1}$, and this mapping is called *a quadratic stochastic operator (q.s.o.).*

Notion of q.s.o. is used in the works of Bernstein, (1924) on problems of mathematical genetics.

In mathematical genetics V is called *an evolutionary operator* of population. The population is determined as a closed community of organisms concerning the process of reproduction. In the population successive generations $F_1, F_2, ...$ are distinguished. Suppose that between kinds of different generations never happens an interbreed. Every individual, which contains in population, belongs to certain (single) from n varieties ("indications", kinds): $1, 2, ..., n$. Composition of population is the set of elements $x = (x_1, ..., x_n) \in S^{n-1}$ probability of varieties (Bernstein, (1924) and Lyubich, (1992)).

We extend V from simplex S^{n-1} to all space R^n by (1), i.e. $V : R^n \to R^n$.

A multiplication on R^n is determined by

$$x \circ y = \frac{1}{4}\big(V(x+y) - V(x-y)\big). \tag{2}$$

Obtained algebra $(R^n, \circ)$ is called a genetic algebra.

For any genetic algebra $H^{n-1} = \left\{ x : \sum_{i=1}^{n} x_i = 1 \right\}$, S^{n-1} and

$L^{n-1} = \left\{ x : \sum_{i=1}^{n} x_i = 0 \right\}$ are invariants regarding to introduced operation of multiplication (2). Moreover L^{n-1} is an ideal of this algebra.

Indeed, we shall prove that, let H^{n-1} is an invariant. For all $x,y\in H^{n-1}$ we have

$$(x+y)'_k=\sum_{i,j=1}^{n}p_{ij,k}\left(x_i+y_i\right)\left(x_j+y_j\right),\ (x-y)'_k=\sum_{i,j=1}^{n}p_{ij,k}\left(x_i-y_i\right)\left(x_j-y_j\right).$$

Consequently,

$$(x\circ y)_k=\frac{1}{4}\left(\sum_{i,j=1}^{n}p_{ij,k}\left(x_i+y_i\right)\left(x_j+y_j\right)-\sum_{i,j=1}^{n}p_{ij,k}\left(x_i-y_i\right)\left(x_j-y_j\right)\right)$$

$$=\frac{1}{4}\sum_{i,j=1}^{n}p_{ij,k}\left(2x_iy_j+2x_jy_i\right)=\frac{1}{2}\sum_{i,j=1}^{n}p_{ij,k}\left(x_iy_j+x_jy_i\right).$$

Now calculate the sum of coordinate, i.e.:

$$\sum_{k=1}^{n}(x\circ y)_k=\sum_{k=1}^{n}\left(\frac{1}{2}\sum_{i,j=1}^{n}p_{ij,k}\left(x_iy_j+x_jy_i\right)\right)=\frac{1}{2}\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n}p_{ij,k}\left(x_iy_j+x_jy_i\right)\right)$$

$$=\frac{1}{2}\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n}p_{ij,k}\right)\left(x_iy_j+x_jy_i\right)=\frac{1}{2}\sum_{i,j=1}^{n}\left(x_iy_j+x_jy_i\right)=\frac{1}{2}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(x_iy_j+x_jy_i\right)\right)$$

$$=\frac{1}{2}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}x_iy_j+\sum_{j=1}^{n}x_jy_i\right)=\frac{1}{2}\sum_{i=1}^{n}\left(x_i+y_i\right)=1.$$

Thus $x\circ y\in H^{n-1}$ for any $x,y\in H^{n-1}$.

Definition 2.1 (Ganikhodjaev, (1993)). *The quadratic stochastic operator* $V:S^{n-1}\to S^{n-1}$ *is called a Volterrian operator, if*

$$p_{ij,k}=0,\ at\ k\notin\{i,j\}. \tag{3}$$

If $V:S^{n-1}\to S^{n-1}$ a Volterrian operator, then we may rewrite V as:

$$x'_k=x_k\left(1+\sum_{i=1}^{n}a_{ki}x_i\right),\quad k=\overline{1,n} \tag{4}$$

where $a_{ki} = 2p_{ik,k} - 1$ at $i \neq k$ and $a_{ki} = -a_{ik}$, $|a_{ki}| \leq 1$, i.e. $A_n = (a_{ki})_{k,i=1}^{n}$ is a skew-symmetrical matrix.

The biological treatment of condition (3) is clear: The offspring repeats the genotype of one of its parents.

Consider operator $V : R^n \to R^n$ defined by

$$x_k^{'} = x_k \left(\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} a_{ki} x_i \right), \quad k = \overline{1,n} \tag{5}$$

Definition 2.2 *A linear continuous functional* φ *on the genetic algebra* $(R^n, \circ)$ *is called a multiplicative, if for all* x *and* y $\varphi(x \circ y) = \varphi(x) \cdot \varphi(y)$ (Etherington, (1941)).

MAIN RESULTS

In this section we study a one-dimensional subalgebras of genetic algebras and properties of tournaments and their algebras.

Theorem 3.1 *On the genetic algebra* $(R^n, \circ)$ *a functional* $\varphi(x) = \sum_{i=1}^{n} x_i$ *is a multiplicative linear functional.*

Proof. Consider a functional $\varphi(x) = \sum_{i=1}^{n} x_i$. By definition 2.2 follows that φ is a multiplicative linear functional if $\varphi(x \circ y) = \varphi(x) \cdot \varphi(y)$.

$$x \circ y = \frac{1}{4} \left(\sum_{i,j=1}^{n} p_{ij,k} \left(x_i + y_i\right)\left(x_j + y_j\right) - \sum_{i,j=1}^{n} p_{ij,k} \left(x_i - y_i\right)\left(x_j - y_j\right) \right)$$

$$= \frac{1}{4} \sum_{i,j=1}^{n} p_{ij,k} \left(2x_i y_j + 2x_j y_i\right) \text{ and } \varphi(Vx) = \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} p_{ij,k} x_i x_j \right).$$

Since the sum is finite then

$$\varphi(x\circ y)=\frac{1}{4}\sum_{k=1}^{n}\left(\sum_{i,j=1}^{n}p_{ij,k}\left(2x_iy_j+2x_jy_i\right)\right)=\frac{1}{4}\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n}p_{ij,k}\right)\left(2x_iy_j+2x_jy_i\right)$$

$$=\frac{1}{4}\sum_{i,j=1}^{n}\left(2x_iy_j+2x_jy_i\right)=\frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i=1}^{n}x_iy_j+\sum_{i=1}^{n}x_jy_i\right)=\frac{1}{2}\sum_{j=1}^{n}\left(y_j\varphi(x)+x_j\varphi(y)\right)$$

$$=\frac{1}{2}\left(\sum_{j=1}^{n}y_j\varphi(x)+\sum_{j=1}^{n}x_j\varphi(y)\right)=\frac{1}{2}\left(\varphi(x)\varphi(y)+\varphi(y)\varphi(x)\right)=\varphi(x)\varphi(y).\ \blacksquare$$

Theorem 3.2 Any *one-dimensional subalgebra on* $(R^n,\circ)$ *contains a unique nonzero fixed point of the operator* V*. Conversely, if* $0\neq x_0=Vx_0$ *is a nonzero fixed point then* $L=\{\lambda x_0\}_{\lambda\in R}$ *is one-dimensional subalgebra on* $(R^n,\circ)$.

Proof. Let $L=\{\lambda x_0\}_{\lambda\in R}$ be one-dimensional subalgebra on $(R^n,\circ)$ and $0\neq x_0\in L$. Then $Vx_0=x_0^2=x_0\circ x_0=\frac{1}{4}\left(V(x_0+x_0)-V(x_0-x_0)\right)\in L$. From here $Vx_0=\lambda_0x_0$. Let $x\in L$, i.e., $x=\mu x_0$ and $V\mu x_0=\mu x_0$. On the other hand $V\mu x_0=\mu^2Vx_0=\mu^2\lambda_0x_0$. Therefore $\mu^2\lambda_0x_0=\mu x_0\Rightarrow\mu^2\lambda_0=\mu,\mu\neq0$. Hence, for $\mu=\frac{1}{\lambda_0}$ we have the unique fixed point.

Conversely, let $0\neq x_0=Vx_0$ be a fixed point on the concerning operator V. Then for all $x,y\in L$ we have $Vx=V(\lambda x_0)=\lambda^2Vx_0=\lambda^2x_0$ and $Vy=V(\mu x_0)=\mu^2Vx_0=\mu^2x_0$.

Consequently, $x\circ y\in L$, i.e.

$$x\circ y=\frac{1}{4}\left(V(x+y)-V(x-y)\right)=\frac{1}{4}\left(V(\lambda+\mu)x_0-V(\lambda-\mu)x_0\right)=\frac{1}{4}\left((\lambda+\mu)^2x_0-(\lambda-\mu)^2x_0\right)$$

$$=\frac{1}{4}\left((\lambda+\mu)^2-(\lambda-\mu)^2\right)x_0=\lambda\mu x_0=\eta x_0\in L.$$

Hence, $L=\{\lambda x_0\}_{\lambda\in R}$ is one-dimensional subalgebra on $(R^n,\circ)$.

A mapping $f : X \to Y$ is called a *homomorphism* from algebra X into Y, if it satisfies the following conditions:

$$f(x+y) = f(x) + f(y), \qquad (i)$$
$$f(\alpha x) = \alpha f(x), \qquad (ii)$$
$$f(xy) = f(x) f(y) \qquad (iii)$$

for all $x, y \in X$ and $\alpha \in R$.

A homomorphism f of algebra X into algebra Y is called *an isomorphism* of X onto Y if f is one-to-one and onto Y, satisfying the conditions (i) - (iii).

Let x_0 is a fixed point of V.

Corollary 3.3 *Let* x_0 *is a fixed point of* V. *Any one-dimensional subalgebra* $L = \{\lambda x_0\}_{\lambda \in R}$ *in algebra* $(R^n, \circ)$ *is isomorphic onto* R *with the simple multiplication.*

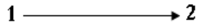
We discuss a classification of Volterrian genetic algebras. Suppose, $a_{ki} \neq 0$ at $k \neq i$. Alongside the dynamical system (4) we consider a full graph G_n consisting of n vertices: $1, 2, ..., n$.

Define *a* tournament T_n, as a graph consisting of n vertices labeled by $1, 2, ..., n$, corresponding to a skew-symmetrical matrix A_m by the following rule: there is an arrow from i to k if $a_{ki} < 0$, a reverse arrow otherwise. Note that if signs of two skew-symmetric matrices are the same, then the corresponding tournaments are the same as well.

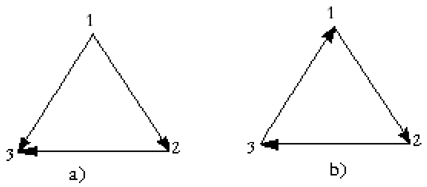
Recall that a tournament is said to be *strong* if it is possible to go from any vertex to any other vertex with directions taken into account. *A strong component* of a tournament is a maximal strong subtournament of the tournament. The tournament with the strong components of T_n as vertices and with the edge directions induced from T_n is called *the factor tournament* of the tournament T_n and denoted by $\tilde{T}_n$. *Transitivity* of the tournament means that there is no strong subtournament consisting of three vertices of the given tournament. A tournament containing fewer than three vertices is regarded as *transitive* by definition. As is known (Harrary, (1969)), the factor tournament $\tilde{T}_n$ of any tournament T_n is transitive. Further, after a suitable renumbering of the vertices of T_n we can assume that subtournament T_r contains the vertices of T_n as its vertices, i.e.,$\{1\},\{2\},...,\{r\}$.Obviously, $r \geq n$,and $r = n$ if and only if T_n is a strong tournament.

Let us describe the tournaments for small n. As an example in Figure 1 tournaments with two, three and four vertices are shown.

1) T_2:



2) T_3:



3) T_4:

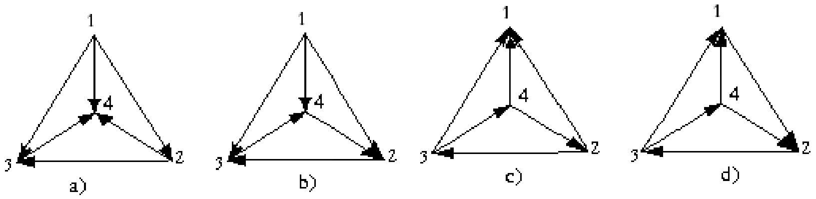


Figure 1

A tournament T_3 in case a) is called a transitive triple, in case b) called a cyclic triple. From Figure 1, one can easily see, that T_3 in case b) and T_4 in case d) is a strong tournament, and the others are non-strong tournaments.

Consider an operator $V : R^n \to R^n$ defined by (5). In case $n = 3$ this dynamical system takes the following form:

$$x_k' = x_k \left(\sum_{i=1}^{3} x_i + \sum_{i=1}^{3} a_{ki} x_i \right), \quad k = \overline{1,3} \qquad (6)$$

where $a_{ki} = 2p_{ik,k} - 1$ at $i \neq k$ and $a_{ki} = -a_{ik}$, $|a_{ki}| \leq 1$.

In particular case of T_3 we have the following dynamical systems:

$$V_1:\begin{cases} x_1^{'}=x_1\left(\sum_{i=1}^{3}x_i-x_2-x_3\right) \\ x_2^{'}=x_2\left(\sum_{i=1}^{3}x_i+x_1-x_3\right) \\ x_3^{'}=x_3\left(\sum_{i=1}^{3}x_i+x_1+x_2\right) \end{cases} \text{ and } V_2:\begin{cases} x_1^{'}=x_1\left(\sum_{i=1}^{3}x_i-x_2+x_3\right) \\ x_2^{'}=x_2\left(\sum_{i=1}^{3}x_i+x_1-x_3\right) \\ x_3^{'}=x_3\left(\sum_{i=1}^{3}x_i-x_1+x_2\right). \end{cases} \qquad (7)$$

Hence,

$$V_1:\begin{cases} x_1^{'}=x_1^2 \\ x_2^{'}=x_2(2x_1+x_2) \\ x_3^{'}=x_3(2x_1+2x_2+x_3) \end{cases} \quad \text{and} \quad V_2:\begin{cases} x_1^{'}=x_1(x_1+2x_3) \\ x_2^{'}=x_2(2x_1+x_2) \\ x_3^{'}=x_3(2x_2+x_3). \end{cases}$$

Now we consider the algebras, generated by the Volterrian operator, where operation of multiplication is defined by (2). The corresponding algebras are $A_1=(R^3,\circ_1)$ and $A_2=(R^3,\circ_2)$,

where

$$x\circ_1 y=(\,x_1y_1,\ x_1y_2+x_2y_1+x_2y_2,\ x_1y_3+x_3y_1+x_2y_3+x_3y_2+x_3y_3\,),$$

$$x\circ_2 y=(\,x_1y_1+x_1y_3+x_3y_1,\ x_2y_2+x_1y_2+x_2y_1,\ x_3y_3+x_2y_3+x_3y_2\,).$$

It is easy to check the following properties of the algebras:

1) *The algebras are commutative.*
2) *The algebra* A_1 *is associative.*
3) *The algebra* A_2 *is non-associative, non-Jordan and non-alternative.*

Here we shall prove 2): Let $x,y,z\in A_1$. Then,

$$t\circ_1 z=\left(t_1z_1,\ t_1z_2+t_2z_1+t_2z_2,\ t_1z_3+t_3z_1+t_2z_3+t_3z_2+t_3z_3\right),$$

$$x\circ_1 l=\left(x_1l_1,\ x_1l_2+x_2l_1+x_2l_2,\ x_1l_3+x_3l_1+x_2l_3+x_3l_2+x_3l_3\right),$$

where $t = x \circ_1 y$ and $l = y \circ_1 z$. Hence, $\left(x \circ_1 y\right) \circ_1 z = x \circ_1 \left(y \circ_1 z\right)$, i.e. the algebra A_1 is an associative .

Now in general case, we consider the algebras $B_1 = (R^3, \circ_1)$ and $B_2 = (R^3, \circ_2)$ with $(V_1 x)_k = x_k\left(\sum_{i=1}^{3} x_i + \sum_{i=1}^{3} a'_{ki} x_i\right),\ k = \overline{1,3}$

$$(V_2 x)_k = x_k\left(\sum_{i=1}^{3} x_i + \sum_{i=1}^{3} a''_{ki} x_i\right),\ k = \overline{1,3} \tag{8}$$

where $a'_{ki} = 2p'_{ik,k} - 1$ at $i \neq k$ and $a'_{ki} = -a'_{ik}$, $\left|a'_{ki}\right| \leq 1$, $a''_{ki} = 2p''_{ik,k} - 1$ at $i \neq k$ and $a''_{ki} = -a''_{ik}$, $\left|a''_{ki}\right| \leq 1$. Since the operator V_1 of (8) corresponding to the tournament T_3 (a) have 3 fixed points and the operator V_2 of (8) corresponding to T_3 (b) have 4 fixed points, the corresponding algebras are non-isomorphic.

In general case the following theorem holds:

Theorem 3.4 *If the numbers of cyclic triples of two tournaments are different, then corresponding algebras are non-isomorphic.*

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