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Λ_r -Sets and Separation Axioms

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ABSTRACT

Separation axioms are among the most common and important and interesting concepts in topology as well as in bitopologies. In this paper, we introduce Λ_r -sets and some weak separation axioms using Λ_r -open sets and Λ_r -closure operator.

Keywords: Λ_r -sets, Λ_r -open sets and Λ_r - T_k , $k = 0,1,2$ spaces.

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INTRODUCTION AND PRELIMINARIES

The separation axioms are important and interesting concepts among the topological spaces. Most of the definitions appeared simple, however the topological structure and properties might be complex and not always that easy to comprehend. For example, in digital topology, several spaces that fails to satisfy to be T_1 which are important in the study of the geometric and topological properties of digital images. Caldas and Dontchev (2000) characterized the concepts of Λ_s -sets and V_s -sets in topological spaces. By using the regularly open and regularly closed sets these structures can also be extended to the bitopological spaces. For more details on regularly pairwise open and closed sets, see for example, Fawakhreh and Kilicman (2002), Kilicman and Salleh (2007a), Kilicman and Salleh (2007b) and Kilicman and Salleh (2008). The purpose of this paper is to continue the research along these directions but this time by utilizing regularly-open sets. For details, see Fawakhreh and Kilicman (2004) , Fawakhreh and Kilicman (2006) and Kilicman and Salleh (2009). Caldas and Jafari (2004) introduced the notions of Λ_δ-T_0, Λ_δ-T_1, and Λ_δ-T_2 topological spaces. In this paper, we introduce some Λ_r-separation axioms in topological spaces. To define and investigate the axioms, we use the Λ_r-open sets. We call these axioms as Λ_r-T_0, Λ_r-T_1, and Λ_r-T_2.

Throughout the paper (X,τ) (or simply X) will always denote a topological space. Let (X,τ) be a topological space and S be a subset of X. Then S is called regularly-open if $S = \mathrm{Int}(\mathrm{cl}\ S)$. The complement $S^c(=X-S)$ of a regularly-open set S is called the regularly-closed set. The family of all regularly-open sets (resp. regularly-closed sets) will be denoted by $RO(X,\tau)$ (resp. $RC(X,\tau)$). A subset S of X is called Λ-set if it is the intersection of open sets containing S. The complement of Λ-set is called the V-set.

Λ_r-SETS AND V_r-SETS

Definition 2.1 Let S be a subset of a topological space (X,τ). We define the sets $\Lambda_r(S)$ and $V_r(S)$ as follows:

$$\Lambda_r(S) = \cap\{G/G \in RO(X,\tau) \text{ and } S \subseteq G\}$$
$$V_r(S) = \cup\{F/F \in RC(X,\tau) \text{ and } S \supseteq F\}$$

Lemma 2.2 For subsets S,Q and S_i, $i \in I$, of a topological space (X,τ), the following properties hold:

(1) $S \subseteq \Lambda_r(S)$

(2) $Q \subseteq S \Rightarrow \Lambda_r(Q) \subseteq \Lambda_r(S)$

(3) $\Lambda_r(\Lambda_r(S)) = \Lambda_r(S)$

(4) If $S \in RO(X,\tau)$, then $S = \Lambda_r(S)$

(5) $\Lambda_r(\bigcup_{i\in I} S_i) \supseteq \bigcup_{i\in I} \Lambda_r(S_i)$

(6) $\Lambda_r(\bigcap_{i\in I} S_i) \subseteq \bigcap_{i\in I} \Lambda_r(S_i)$

(7) $\Lambda_r(S^c) = (V_r(S))^c$

Proof.

(1) Let $x \notin \Lambda_r(S)$. Then there exists a regularly-open set G such that $S \subseteq G$ and $x \notin G$. Hence $x \notin S$ and so $S \subseteq \Lambda_r(S)$.

(2) Let $x \notin \Lambda_r(S)$. Then there exists a regularly-open set G such that $S \subseteq G$ and $x \notin G$. By our assumption $Q \subseteq S$, $Q \subseteq G$ and hence $x \notin \Lambda_r(Q)$. This shows (2).

(3) From (1) and (2), $\Lambda_r(S) \subseteq \Lambda_r(\Lambda_r(S))$. If $x \notin \Lambda_r(S)$, then there exists a regularly-open set G such that $S \subseteq G$ and $x \notin G$. From the definition of $\Lambda_r(S)$, $\Lambda_r(S) \subseteq G$ and hence $x \notin \Lambda_r(\Lambda_r(S))$.
Therefore $\Lambda_r(\Lambda_r(S)) \subseteq \Lambda_r(S)$. This proves (3).

(4) It directly follows from the definition of $\Lambda_r(S)$ and lemma 2.2(1).

(5) From (2), $\Lambda_r(S_i) \subseteq \Lambda_r(S)$ for each $i \in I$ where $S = \bigcup_{i\in I} S_i$ and hence $\bigcup_{i\in I} \Lambda_r(S_i) \subseteq \Lambda_r(S) = \Lambda_r(\bigcup_{i\in I} S_i)$.

(6) From (2), $\Lambda_r(S) \subseteq \Lambda_r(S_i)$ for each $i \in I$ where $S = \bigcap_{i\in I} S_i$ and hence $\Lambda_r(S) = \Lambda_r(\bigcap_{i\in I} S_i) \subseteq \bigcap_{i\in I}(S_i)$.

(7) Let $x \in \Lambda_r(S^c)$. Then for every regularly-open set G containing $S^c, x \in G$. Hence $x \notin G^c$, for every regularly-closed set $G^c \subseteq S$. Therefore $x \notin V_r(S)$ and hence $x \in (V_r(S))^c$.
Similarly, $(V_r(S))^c \subseteq \Lambda_r(S^c)$. Hence (7) is proved.

By using the above lemma, we can easily verify the next result.

Lemma 2.3 For subsets S, Q and S_i, $i \in I$, of a topological space (X, τ), the following properties hold:

(1) $V_r(S) \subseteq S$

(2) $Q \subseteq S \Rightarrow V_r(Q) \subseteq V_r(S)$

(3) $V_r(V_r(S)) = V_r(S)$

(4) If $S \in RC(X, \tau)$, then $S = V_r(S)$

(5) $V_r(\bigcap_{i\in I}(S_i)) \subseteq \bigcap_{i\in I} V_r(S_i)$

(6) $V_r(\bigcup_{i\in I} S_i) \supseteq \bigcup_{i\in I} V_r(S_i)$

In general, we have

$\Lambda_r(S \cap Q) \neq \Lambda_r(S) \cap \Lambda_r(Q)$ and $\Lambda_r(S \cap Q) \neq \Lambda_r(S) \cup \Lambda_r(Q)$ as the following examples show.

Example 2.4

Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b, c\}\}$.

Then $RO(X, \tau) = \{X, \phi, \{a\}, \{b, c\}\}$. Take $S = \{b\}$ and $Q = \{c\}$.

Then $\Lambda_r(S) = \{b, c\}$, $\Lambda_r(Q) = \{b, c\}$, $\Lambda_r(S) \cap \Lambda_r(Q) = \{b, c\}$ but $\Lambda_r(S \cap Q) = \phi$.

Example 2.5

Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$.

Then $RO(X, \tau) = \{X, \phi, \{a\}, \{b\}\}$. Take $S = \{a\}$ and $Q = \{b\}$.

Then $\Lambda_r(S) = \{a\}$, $\Lambda_r(Q) = \{b\}$, $\Lambda_r(S) \cup \Lambda_r(Q) = \{a, b\}$ but $\Lambda_r(S \cup Q) = X$.

Definition 2.6 A subset S of a space (X, τ) is called a

(1) regular-Λ-set, briefly Λ_r-set if $S = \Lambda_r(S)$

(2) regular-V-set, briefly V_r-set if $S = V_r(S)$

The set of all Λ_r-sets (resp. V_r-sets) is denoted by $\Lambda_r(X, \tau)$ (resp. $V_r(X, \tau)$).

Remark 2.7 Clearly regular-Λ-sets are Λ-sets and regular-V-sets are V-sets. Observe that a subset S is a regular-Λ-set if S^c is a regular-V-set. Also observe that every regular-Λ-set is a regularly-open set.

Proposition 2.8 For a space (X,τ), the following statements hold:

(1) ϕ and X are Λ_r -sets and V_r -sets
(2) Every union of V_r -sets is a V_r -set
(3) Every intersection of Λ_r -sets is a Λ_r -set.

Proof.

(1) It is obvious.
(2) Let $\{S_i / i \in I\}$ be a family of V_r -sets in (X,τ). Then $S_i = V_r(S_i)$ for each $i \in I$. Let $S = \underset{i=I}{\cup} S_i$. Then $V_r(S) = V_r(\underset{i=I}{\cup} S_i) \supseteq \underset{i=I}{\cup} V_r(S_i) = \underset{i=I}{\cup} S_i = S$. Also $V_r(S) \subseteq S$ and hence S is a V_r -set.
(3) By using lemma 2.2(6) and 2.2(1), we get (3).

The following example shows that union of Λ_r -sets need not be a Λ_r -set.

Example 2.9

Let $X = \{a,b,c\}$ and $\tau = \{X,\phi,\{a\},\{b\},\{a,b\}\}$.
Then $RO(X,\tau) = \{X,\phi,\{a\},\{b\}\}$ and $\Lambda_r(X,\tau) = \{X,\phi,\{a\},\{b\}\}$. Here $\{a\}$ and $\{b\}$ are Λ_r -sets but $\{a\} \cup \{b\} = \{a,b\}$ is not a Λ_r -set.

Similar to the previous case the following example shows that intersection of V_r -sets need not be a V_r -set.

Example 2.10

Let X and τ be defined as in example 2.9.
Then $V_r(X,\tau) = \{X,\phi,\{b,c\},\{a,c\}\}$. Here $\{b,c\}$ and $\{a,c\}$ are V_r -sets but $\{b,c\} \cap \{a,c\} = \{c\}$ is not a V_r -set.

In order to achieve our purpose, we recall the following definition (Jain (1980)).

Definition 2.11 Let (X,τ) be a topological space. Then the regular-closure of A, denoted by $rcl(A)$ is defined by

$$rcl(A) = \cap\{F/F \in RC(X,\tau) \text{ and } F \supset A\}.$$

Lemma 2.12 Let (X,τ) be a topological space and $x \in X$. Then $y \in \Lambda_r(\{x\})$ if $x \in rcl(\{y\})$.

Proof.

Suppose $y \in \Lambda_r(\{x\})$. Then for every regularly-open set $G \supseteq \{x\}$, $y \in G$. If $x \notin rcl(\{y\})$, then $\exists H \in RC(X,\tau)$ such that $\{y\} \subset H$ and $x \notin H$. That implies $x \in X - H$, $X - H \in RO(X,\tau)$ and $y \notin X - H$. Take $X - H = G$.

Then $G \in RO(X,\tau), \{x\} \subseteq G$ and $y \notin G$. By this contradiction, we get $x \in rcl(\{y\})$. Conversely, suppose $x \in rcl(\{y\})$. Then for every regularly-closed set $G \supset \{y\}, x \in G$. If $y \notin \Lambda_r(\{x\})$, then $\exists H \in RO(X,\tau)$ such that $\{x\} \subseteq H$ and $y \notin H$. Take $X - H = G$. Then $G \in RC(X,\tau), y \in G$ and $x \notin G$. So there exists a regularly-closed set $G \supset \{y\}$ such that $x \notin G$. By this contradiction, we get $y \notin \Lambda_r(\{x\})$.

Theorem 2.13 The following statements are equivalent for any points x and y in a topological space (X,τ)

(1) $\Lambda_r(\{x\}) \neq \Lambda_r(\{y\})$

(2) $rcl(\{x\}) \neq rcl(\{y\})$

Proof.

$(1) \to (2)$: Suppose $\Lambda_r(\{x\}) \neq \Lambda_r(\{y\})$. Then $\exists z \in X$ such that $z \in \Lambda_r(\{x\})$ and $z \in \Lambda_r(\{y\})$. Therefore $x \in rcl(\{z\})$ and $y \in rcl(\{z\})$. Hence $\{x\} \cap rcl(\{z\}) \neq \phi$ and $\{y\} \cap rcl(\{z\}) \neq \phi$. Since $x \in rcl(\{z\})$, $rcl(\{x\}) \subset rcl(\{z\})$ and hence $\{y\} \cap rcl(\{x\}) \neq \phi$. Thus $rcl(\{x\}) \neq rcl(\{y\})$.

$(2)\rightarrow(1)$: Suppose $rcl(\{x\})\neq rcl(\{y\})$. Then $\exists z\in X$ such that $z\in rcl(\{x\})$ and $z\notin rcl(\{y\})$. Therefore $x\in\Lambda_r(\{z\})$ and $y\notin\Lambda_r(\{z\})$. So there exists a regularly-open set $G\supset\{z\}$ such that $x\in G$ and $y\notin G$. Hence $y\notin\Lambda_r(\{x\})$ and hence $\Lambda_r(\{x\})\neq\Lambda_r(\{y\})$.

Lemma 2.14 Let (X,τ) be a topological space and $A\in RO(X,\tau)$. Then $\Lambda_r(A)=\{x\in X/rcl(\{x\})\cap A\neq\phi\}$.

Proof.

Let $x\in\Lambda_r(A)$. Since $A\in RO(X,\tau)$, $A=\Lambda_r(A)$. Also $x\in rcl(\{x\})$ and hence $rcl(\{x\})\cap A\neq\phi$. Conversely, let $x\in X$ such that $rcl(\{x\})\cap A\neq\phi$. If $x\notin\Lambda_r(A)$, then there exists $V\in RO(X,\tau)$ such that $A\subseteq V$ and $x\notin V$. Let $y\in rcl(\{x\})\cap A$. Since $y\in rcl(\{x\})$, $x\in\Lambda_r(\{y\})$. Therefore for every regularly-open set $G\supseteq\{y\}$ in (X,τ), $x\in G$. Since $y\in A$ and $A\subseteq V$, $y\in V$ where $V\in RO(X,\tau)$. Hence $x\in V$. By this contradiction, we get $x\in\Lambda_r(A)$.

Recall that a topological space (X,τ) is called a $r\text{-}R_0$ space (Jain (1980)) if for every regularly-open set G, $x\in G\Rightarrow rcl(\{x\})\subset G$.

Theorem 2.15 For a topological space (X,τ), the following properties are equivalent

(1) (X,τ) is a $r\text{-}R_0$ space
(2) For any $x\in X$, $rcl(\{x\})\subset\Lambda_r(\{x\})$

Proof.

$(1)\rightarrow(2)$: Let $y\notin\Lambda_r(\{x\})$. Then there exists $V\in RO(x,\tau)$ such that $V\supseteq\{x\}$, $y\notin V$. Since $x\in V\in RO(X,\tau)$, by (1) $rcl(\{x\})\subset V$. Hence $y\notin rcl(\{x\})$. Therefore $rcl(\{x\})\subset\Lambda_r(\{x\})$.

$(2) \to (1)$: Let $V \in RO(X,\tau)$ and $x \in V$. Suppose $y \in \Lambda_r(\{x\})$. Then for every regularly-open set $G \supseteq \{x\}, y \in G$. Hence $y \in V$ and hence $\Lambda_r(\{x\}) \subset V$. By (2), $rcl(\{x\}) \subset V$. Hence (X,τ) is a $r\text{-}R_0$ space.

Result 2.16 If F is regularly-open in (X,τ) and $x \in F$, then $\Lambda_r(\{x\}) \subset F$.

Proof. It directly follows from the definition of $\Lambda_r(\{x\})$.

Λ_r-CLOSED SETS AND ITS PROPERTIES

Definition 3.1

(1) Let A be a subset of a space (X,τ). Then A is called a Λ_r-closed set if $A = S \cap C$ where S is a Λ_r-set and C is a closed set.
(2) The complement of a Λ_r-closed set is called a Λ_r-open set.
(3) The collection of all Λ_r-open sets in (X,τ) is denoted by $\Lambda_r O(X,\tau)$. The collection of all Λ_r-closed sets in (X,τ) is denoted by $\Lambda_r C(X,\tau)$.
(4) A point $x \in X$ is called a Λ_r-cluster point of A if for every Λ_r-open set U containing x, $A \cap U \neq \phi$.
(5) The set of all Λ_r-cluster points of A is called the Λ_r-closure of A and is denoted by $\Lambda_r\text{-cl}(A)$.

Let (X,τ) be a topological space and A, B and A_k where $k \in I$, subsets of X. Then we have the following properties.

Property 3.2 $A \subset \Lambda_r\text{-cl}(A)$.

Proof. Let $x \notin \Lambda_r\text{-cl}(A)$. Then x is not a Λ_r-cluster point of A. So there exists a Λ_r-open set U containing x such that $A \cap U = \phi$ and hence $x \notin A$.

Property 3.3 $\Lambda_r\text{-cl}(A) = \cap\{F / A \subset F \text{ and } F \text{ is } \Lambda_r\text{-closed}\}$.

Proof. Let $x \notin \Lambda_r\text{-cl}(A)$. Then there exists a Λ_r-open set U containing x such that $A \cap U = \phi$. Take $F = U^c$. Then F is Λ_r-closed, $A \subset F$ and

$x \notin F$ and hence $x \notin \cap\{F/A \subset F$ and F is Λ_r - closed$\}$. Similarly, Λ_r - cl(A) $\subset \cap\{F/A \subset F$ and F is Λ_r - closed$\}$.

Property 3.4 If $A \subset B$, then Λ_r - cl(A) $\subset \Lambda_r$ - cl(B).

Proof. Let $x \notin \Lambda_r$ - cl(B). Then there exists a Λ_r - open set U containing x such that $B \cap U = \phi$. Since $A \subset U$, $A \cap U = \phi$ and hence x is not a Λ_r -cluster point of A. Therefore $x \notin \Lambda_r$ - cl(A).

Property 3.5 A is Λ_r - closed if $A = \Lambda_r$ - cl(A).

Proof.

Suppose A is Λ_r - closed. Let $x \notin A$. Then $x \in A^c$ and A^c is Λ_r -open. Take $A^c = U$. Then U is a Λ_r -open set containing x and $A \cap U = \phi$ and hence $x \notin \Lambda_r$ - cl(A). By using Property 3.2, we get $A = \Lambda_r$ - cl(A). Conversely, suppose $A = \Lambda_r$ - cl(A). Since $A = \cap\{F/A \subset F$ and F is Λ_r - closed$\}$ by Property 3.3, A is Λ_r -closed.

Property 3.6 Λ_r - cl(A) is Λ_r -closed.

Proof.

By using the Properties 3.2 and 3.4, we have Λ_r - cl(A) $\subset \Lambda_r$ - cl(Λ_r - cl(A)). Let $x \in \Lambda_r$ - cl(Λ_r - cl(A)). That implies x is a Λ_r -cluster point of Λ_r - cl(A). That implies for every Λ_r -open set U containing x, (Λ_r - cl(A)) $\cap U \neq \phi$. Let $y \in \Lambda_r$ - cl(A) $\cap U$. Then y is a Λ_r - cluster point of A. Therefore for every Λ_r -open set G containing y, $A \cap G \neq \phi$. Since U is Λ_r -open and $y \in U$, $A \cap U \neq \phi$ and hence $x \in \Lambda_r$ - cl(A). Hence Λ_r - cl(A) $= \Lambda_r$ - cl(Λ_r - cl(A)). By Property 3.5, Λ_r - cl(A) is Λ_r -closed.

Remark 3.7

(1) X and ϕ are both Λ_r -open and Λ_r -closed.

(2) By using the Properties 3.3 and 3.6, Λ_r - cl(A) is the smallest Λ_r - closed set containing A.

Property 3.8 If A_k is Λ_r-closed for each $k \in I$, then $\bigcap_{k\in I} A_k$ is Λ_r-closed.

Proof.

Let $A = \bigcap_{k\in I} A_k$ and $x \in \Lambda_r\text{-cl}(A)$. Then x is a Λ_r-cluster point of A. Hence for every Λ_r-open set U containing x, $A \cap U \neq \phi$. That implies $(\bigcap_{k\in I} A_k) \cap U \neq \phi$. That implies $A_k \cap U \neq \phi$ for each $k \in I$. If $x \notin A$, then for some $i \in I$, $x \notin A_i$. Since A_i is Λ_r-closed, $A_i = \Lambda_r\text{-cl}(A_i)$ and hence $x \notin \Lambda_r\text{-cl}(A_i)$. Therefore x is not a Λ_r-cluster point of A_i. So there exists a Λ_r-open set V containing x such that $A_i \cap V = \phi$. By this contradiction, $x \in A$. Therefore $\Lambda_r\text{-cl}(A) \subset A$ and hence $A = \Lambda_r\text{-cl}(A)$. By using the Property 3.5, A is Λ_r-closed. That is, $\bigcap_{k\in I} A_k$ is Λ_r-closed.

Remark 3.9 The union of Λ_r-closed sets need not be Λ_r-closed. For example, let $X = \{a,b,c,d\}$ and $\tau = \{X,\phi\{a\},\{b\},\{a,b\}\}$. Then $\{a\}$ and $\{b\}$ are Λ_r-closed but $\{a\} \cup \{b\} = \{a,b\}$ is not a Λ_r-closed set.

Property 3.10 If A_k is Λ_r-open for each $k \in I$, then $\bigcup_{k\in I} A_k$ is Λ_r-open.

Definition 3.11 Let (X,τ) be a topological space, $A \subset X$. Then Λ_r-kernel of A is defined by $\Lambda_r\text{-ker}(A) = \cap\{G/G \in \Lambda_r O(X,\tau)$ and $A \subset G\}$.

Let (X,τ) be a topological space and A,B be subsets of X. Let $x,y \in X$. Then we have the following lemmas.

Lemma 3.12 $A \subset \Lambda_r\text{-ker}(A)$

Proof. Let $x \notin \Lambda_r\text{-ker}(A)$. Then there exists $V \in \Lambda_r O(X,\tau)$ such that $A \subset V$ and $x \notin V$ and hence $x \notin A$.

Lemma 3.13 If $A \subset B$, then $\Lambda_r\text{-ker}(A) \subset \Lambda_r\text{-ker}(B)$.

Proof. Let $x \notin \Lambda_r\text{-ker}(B)$. Then there exists $G \in \Lambda_r O(X,\tau)$ such that $B \subset G$ and $x \notin G$. Since $A \subset B$, $A \subset G$ and hence $x \notin \Lambda_r\text{-ker}(A)$.

Lemma 3.14 $\Lambda_r\text{-ker}(A) = \Lambda_r\text{-ker}(\Lambda_r\text{-ker}(A))$.

Proof. Let $x \in \Lambda_r\text{-ker}(\Lambda_r\text{-ker}(A))$. Then for every Λ_r-open set $G \supset \Lambda_r\text{-ker}(A)$, $x \in G$. Since $A \subset \Lambda_r\text{-ker}(A)$, for every Λ_r-open set $G \supset A$, $x \in G$.

Hence $x \in \Lambda_r\text{-ker}(A)$. Therefore $\Lambda_r\text{-ker}(\Lambda_r\text{-ker}(A)) \subset \Lambda_r\text{-ker}(A)$. Also $\Lambda_r\text{-ker}(A) \subset \Lambda_r\text{-ker}(\Lambda_r\text{-ker}(A))$. Hence $\Lambda_r\text{-ker}(A) = \Lambda_r\text{-ker}(\Lambda_r\text{-ker}(A))$.

Lemma 3.15 $y \in \Lambda_r\text{-ker}(\{x\})$ if $x \in \Lambda_r\text{-cl}(\{y\})$.

Proof. $y \notin \Lambda_r\text{-ker}(\{x\}) \Leftrightarrow \exists$a Λ_r-open set $V \supset \{x\}$ such that $y \notin V \Leftrightarrow \exists$a Λ_r-open set $V \supset \{x\}$ such that $\{y\} \cap V = \phi \Leftrightarrow x$ is not a Λ_r-cluster point of $\{y\} \Leftrightarrow x \notin \Lambda_r\text{-cl}(\{y\})$.

Lemma 3.16 $\Lambda_r\text{-ker}(A) = \{x / \Lambda_r\text{-cl}(\{x\}) \cap A \neq \phi\}$.

Proof. Let $x \in \Lambda_r\text{-ker}(A)$. Then for every Λ_r-open set $G \supset A, x \in G$. Suppose $\Lambda_r\text{-cl}(\{x\}) \cap A \neq \phi$. Then $A \subset X - (\Lambda_r\text{-cl}(\{x\}))$. Take $V = X - (\Lambda_r\text{-cl}(\{x\}))$. Then V is a Λ_r-open set containing A and $x \notin V$. By this contradiction, we get $\Lambda_r\text{-cl}(\{x\}) \cap A \neq \phi$. Conversely, let $x \in X$ such that $\Lambda_r\text{-cl}(\{x\}) \cap A \neq \phi$. Let $y \in \Lambda_r\text{-cl}(\{x\}) \cap A$. Then y is a Λ_r-cluster point of $\{x\}$. Therefore for every Λ_r-open set U containing y, $U \cap \{x\} \neq \phi$ and hence $x \in U$. If $x \notin \Lambda_r\text{-ker}(A)$, then $\exists$ a Λ_r-open set $V \supset A$ such that $x \notin V$. Since $y \in A$, V is a Λ_r-open set containing y and $x \notin V$. By this contradiction, we get $x \in \Lambda_r\text{-ker}(A)$.

Λ_r - T_k SPACES

Definition 4.1 (X, τ) is $\Lambda_r\text{-}T_0$ if for each pair of distinct points x, y of X, $\exists$ a Λ_r-open set containing one of the points but not the other.

Theorem 4.2 (X,τ) is Λ_r-T_0 if for each pair of distinct points x, y of X, $\Lambda_r\text{-cl}(\{x\}) \neq \Lambda_r\text{-cl}(\{y\})$.

Proof.

Necessity: Let (X,τ) be a Λ_r-T_0 space. Let $x, y \in X$ such that $x \neq y$. Then $\exists$ a Λ_r-open set V containing one of the points but not the other, say $x \in V$ and $y \notin V$. Then V^c is a Λ_r-closed set containing y but not x. But $\Lambda_r\text{-cl}(\{y\})$ is the smallest Λ_r-closed set containing y. Therefore $\Lambda_r\text{-cl}(\{y\}) \subset V^c$ and hence $x \notin \Lambda_r\text{-cl}(\{y\})$. Thus $\Lambda_r\text{-cl}(\{x\}) \neq \Lambda_r\text{-cl}(\{y\})$.

Sufficiency: Suppose $x, y \in X$, $x \neq y$ and $\Lambda_r\text{-cl}(\{x\}) \neq \Lambda_r\text{-cl}(\{y\})$. Let $z \in X$ such that $z \in \Lambda_r\text{-cl}(\{x\})$ but $z \notin \Lambda_r\text{-cl}(\{y\})$. If $x \in \Lambda_r\text{-cl}(\{y\})$, then $\Lambda_r\text{-cl}(\{x\}) \subset \Lambda_r\text{-cl}(\{y\})$ and hence $z \in \Lambda_r\text{-cl}(\{y\})$. This is a contradiction. Therefore $x \notin \Lambda_r\text{-cl}(\{y\})$. That implies $x \in (\Lambda_r\text{-cl}(\{y\}))^c$. Therefore $(\Lambda_r\text{-cl}(\{y\}))^c$ is a Λ_r-open set containing x but not y. Hence (X,τ) is Λ_r-T_0.

Definition 4.3 (X,τ) is Λ_r-T_1 if for any pair of distinct points x, y of X, there is a Λ_r-open set U in X such that $x \in U$ and $y \notin U$ and there is a Λ_r-open set V in X such that $y \in U$ and $x \notin V$.

Remark 4.4 Every Λ_r-T_1 space is Λ_r-T_0 space. But the converse need not be true. For example, let $X = \{a,b,c\}$ and $\tau = \{X, \phi, \{a\}, \{a,b\}\}$. Then (X,τ) is Λ_r-T_0 space but not Λ_r-T_1 space.

Theorem 4.5 For a space (X,τ), the following are equivalent

(1) (X,τ) is Λ_r-T_1
(2) For every $x \in X$, $\{x\} = \Lambda_r\text{-cl}(\{x\})$
(3) For each $x \in X$, the intersection of all Λ_r-open sets containing x is $\{x\}$.

Proof.

$(1)\rightarrow(2)$: Suppose $y\neq x$ in X. Then $\exists$ a Λ_r-open set V such that $x\in V$ and $y\notin V$. If $x\in\Lambda_r\text{-cl}(\{y\})$, then x is a Λ_r-cluster point of $\{y\}$. That implies for every Λ_r-open set U containing $x, \{y\}\cap U\neq\phi$. Here V is a Λ_r-open set containing x. Therefore $\{y\}\cap V\neq\phi$ implies $y\in V$. This is a contradiction. Thus $x\notin\Lambda_r\text{-cl}(\{y\})$. Hence for a point $x, y\notin\Lambda_r\text{-cl}(\{x\})$. Thus $\{x\}=\Lambda_r\text{-cl}(\{x\})$.

$(2)\rightarrow(3)$: $x\in\Lambda_r\text{-cl}(\{y\})\Leftrightarrow x$ is a Λ_r-cluster point of $\{x\}\Leftrightarrow$ for every Λ_r-open set U containing x, $\{x\}\cap U\neq\phi\Leftrightarrow x\in\cap\{G/G\in\Lambda_r O(X,\tau)$ and $\{x\}\subset G\}$. Therefore $\Lambda_r\text{-cl}(\{x\})=\cap\{G/G\in\Lambda_r O(X,\tau)$ and $\{x\}\subset G\}$. By (2), $\{x\}=\cap\{G/G\in\Lambda_r O(X,\tau)$ and $\{x\}\subset G\}$.

$(3)\rightarrow(1)$: Let $x\neq y$ in X. By (3), and $\{\{x\}\subset G\}$. Hence $\exists$ one Λ_r-open set V containing x but not y. Similarly, $\exists$ one Λ_r-open set U containing y but not x. Hence (X,τ) is $\Lambda_r\text{-}T_1$.

Theorem 4.6 A space (X,τ) is $\Lambda_r\text{-}T_1$ if the singletons are Λ_r-closed sets.

Proof. Suppose (X,τ) is $\Lambda_r\text{-}T_1$. Let $x\in X$ and $y\in\{x\}^c$. Then $x\neq y$ and so $\exists$ a Λ_r-open set U_y such that $y\in U_y$ but $x\notin U_y$. Therefore $y\in U_y\subset\{x\}^c$. That is, $\{x\}^c=\cup\{U_y/y\in\{x\}^c$ is Λ_r-open. Hence $\{x\}$ is Λ_r-closed. Conversely, let $x, y\in X$ with $x\neq y$. Then $y\in\{x\}^c$ and $\{x\}^c$ is a Λ_r-open set containing y but not x. Similarly, $\{y\}^c$ is a Λ_r-open set containing x but not y. Hence (X,τ) is a $\Lambda_r\text{-}T_1$ space.

Definition 4.7 (X,τ) is $\Lambda_r\text{-}T_2$ if for each pair of distinct points x and y in X, $\exists$ a Λ_r-open set U and a Λ_r-open set V in X such that $x\in U$, $y\in V$ and $U\cap V=\phi$.

Remark 4.8 Every Λ_r - T_2 space is Λ_r - T_1.

Theorem 4.9 For a topological space (X,τ), the following are equivalent:

(1) (X,τ) is Λ_r - T_2
(2) If $x \in X$, then for each $y \neq x$, there is a Λ_r-open set U containing x such that $y \notin \Lambda_r$ - cl(U)
(3) For each $x \in X, \{x\} = \cap\{\Lambda_r\text{-cl}(U)/U$ is a Λ_r-open set containing $x\}$

Proof.

$(1) \to (2)$: Let $x \in X$. Then for each $y \neq x$, $\exists$ Λ_r-open sets A and B such that $x \in A$, $y \in B$ and $A \cap B = \phi$. Then $x \in A \subset X - B$. Take $X - B = F$. Then F is Λ_r-closed, $A \subset F$ and $y \notin F$. That implies $y \notin \cap\{F/F$ is Λ_r-closed and $A \subset F\} = \Lambda_r$ - cl(A).

$(2) \to (1)$: Let $x, y \in X$ and $x \neq y$. By (2), $\exists$ a Λ_r-open set U containing x such that $y \notin \Lambda_r$ - cl(U). Therefore $y \in X - (\Lambda_r$ - cl(U)), $X - (\Lambda_r$ - cl(U)) is Λ_r-open and $x \notin X - (\Lambda_r$ - cl(U)). Also $U \cap X - (\Lambda_r$ - cl(U)) $= \phi$. Hence (X,τ) is Λ_r - T_2.

$(2) \leftrightarrow (1)$: It is obvious.

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