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Numerical Solution of Second Kind Linear Fredholm Integral Equations Using QSGS Iterative Method with High-Order Newton-Cotes Quadrature Schemes

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ABSTRACT

The main purpose of this paper is to examine the effectiveness of the Quarter-Sweep Gauss-Seidel (QSGS) method in solving the dense linear systems generated from the discretization of the linear Fredholm integral equations of the second kind. In addition, the applications of the various orders of closed Newton-Cotes quadrature discretization schemes will be investigated in order to form linear systems. Furthermore, the basic formulation and implementation of the proposed method are also presented. The numerical results of test examples are also included in order to verify the performance of the proposed method.

Keywords: Linear Fredholm equations, Newton-Cotes quadrature, Gauss-Seidel, Quarter-sweep iteration

INTRODUCTION

Integral equations have been one of the principal tools in various areas of science such as applied mathematics, physics, biology and engineering. On the other hand, integral equations are encountered in numerous applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomena in physics and biology, renewal theory, quantum mechanics, radiation, optimization, optimal control systems, communication theory, mathematical economics, population genetics, queuing theory, medicine, mathematical problems of radiative equilibrium, particle transport problems of astrophysics and reactor theory, acoustics, fluid mechanics, steady state heat conduction, fracture mechanics, and radiative heat transfer problems (Wang (2006)). From few types of the integral equations, the most frequently investigated integral equations are Fredholm linear equations and its nonlinear counterpart. However, in this paper, linear Fredholm integral equations of the second kind are considered.

Generally, second kind linear integral equations of Fredholm type in the generic form can be defined as follows

$$\lambda y(x) - \int_{\Gamma} K(x,t)\, y(t)\, dt = f(x), \; \Gamma = [a,b] \;\; \lambda \neq 0 \tag{1}$$

where the parameter λ, kernel K and free term f are given, and y is the unknown function to be determined. Kernel K is called Fredholm kernel if the kernel in Equation (1) is continuous on the square $S = \{a \leq x \leq b, a \leq t \leq b\}$ or at least square integrable on this square and it is also assumed to be absolutely integrable and satisfy other properties that are sufficient to imply the Fredholm alternative theorem. Meanwhile, Equation (1) also can be rewritten in the equivalent operator form

$$(\lambda - \kappa)\, y = f \tag{2}$$

where the integral operator define as follows

$$\kappa y(t) = \int_{\Gamma} K(x,t)\, y(t)\, dt\,. \tag{3}$$

Theorem (Fredholm Alternative) (Atkinson (1997))

Let χ be a Banach space and let $\kappa : \chi \to \chi$ be compact. Then the equation $(\lambda - \kappa)\, y = f$, $\lambda \neq 0$ has a unique solution $x \in \chi$ if and only if the homogeneous equation $(\lambda - \kappa)\, z = 0$ has only the trivial solution $z = 0$. In such a case, the operator $\lambda - \kappa : \chi \underset{onto}{\overset{1-1}{\to}} \chi$ has a bounded inverse $(\lambda - \kappa)^{-1}$.

Definition (Compact operators) (Atkinson (1997))

Let χ and Y be normed vector space and let $\kappa : \chi \to \mathrm{Y}$ be linear. Then κ is compact if the set $\{\kappa x \mid \|x\|\, x \leq 1\}$ has compact closure in Y. This is equivalent to saying that for every bounded sequence $\{x_n\} \subset \chi$, the sequences $\{\kappa x_n\}$ has a subsequence that is convergent to some points in Y. Compact operators are also called completely continuous operators.

In many application areas, numerical approaches were used widely to solve Fredholm integral equations of the second kind than the analytical method. To solve Equation (2) numerically, we either seek to determine an approximate solution by using the quadrature method (Laurie (2001); Lin (2003); Muthuvalu and Sulaiman, (2008b; 2009))

$$\left(\lambda I-\kappa_{n}\right) y_{n}=f \tag{4}$$

where I is the identity matrix and κ_n is the approximation of the κ which is obtained by discretization of κ by an n point quadrature method, or use the projection method

$$\left(\lambda-P_{n} \kappa\right) y_{n}=P_{n} f \tag{5}$$

where $y_n \in V_n$ and $P_n : C \to V_n$ is a projection operator in a chosen finite dimensional space V_n; see Kaneko (1989), Chen *et al.* (2002), Maleknejad and Kajani (2003), Asady *et al.* (2005), Kajani and Vencheh (2005), Xiao *et al.* (2006), Chen *et al.* (2007), Long and Nelakanti (2007), and Oladejo *et al.* (2008). Such discretizations of integral equations lead to dense linear systems and can be prohibitively expensive to solve as n, the order of the linear system increases. Thus, iterative methods are the natural options for efficient solutions.

Consequently, the concept of the half-sweep iterative method has been proposed by Abdullah (1991) via the Explicit Decoupled Group (EDG) method to solve two-dimensional Poisson equations. Half-sweep iteration is also known as the complexity reduction approach (Hasan *et al.* (2007)). Since the implementation of half-sweep iterations will only consider half of all interior node points in a solution domain. Following to that, further studies on the applications of the half-sweep iterative methods have been reviewed by Yousif and Evans (1995), Abdullah and Ali (1996), Othman *et al.* (2000), Muthuvalu and Sulaiman (2008a; 2008b; 2009; 2011), Sulaiman *et al.* (2004a; 2007; 2008a) and Abdullah *et al.* (2006).

In 2000, Othman and Abdullah extended the concept of half-sweep iteration by introducing quarter-sweep iterative method via the Modified Explicit Group (MEG) iterative method to solve two-dimensional Poisson equations. Further studies to verify the effectiveness of the quarter-sweep iterative methods have been carried out by Othman and Abdullah (2001), Hasan *et al.* (2005), Sulaiman *et al.* (2004b), Hasan *et al.* (2008), Sulaiman *et al.* (2008b), and Sulaiman *et al.* (2010). However, in this paper, we examined the applications of the half- and quarter-sweep iteration concepts with Gauss-Seidel (GS) iterative method by using approximation equation based on Newton-Cotes quadrature schemes for solving problem (1). The standard GS iterative method is also called as the Full-Sweep Gauss-Seidel (FSGS) method. Meanwhile, combinations of the GS method with half- and quarter-sweep iterations are called as Half-Sweep Gauss-Seidel (HSGS) and Quarter-Sweep Gauss-Seidel (QSGS) methods respectively.

The remainder of this paper is organized in following way. In next section, the formulation of the full-, half- and quarter-sweep quadrature approximation equations based on repeated Newton-Cotes schemes will be elaborated. The latter section of this paper will discuss the formulations of the FSGS, HSGS and QSGS iterative methods and some numerical results will be shown to assert the effectiveness of the proposed method. Besides that, analysis on computational complexity is also given and the concluding remarks are given in final section.

QUARTER-SWEEP QUADRATURE APPROXIMATION EQUATION

As explained in previous section, discretization method based on quadrature schemes was used to construct approximation equations for problem (1) by replacing the integral to finite sums. Generally, quadrature method can be defined as follows

$$\int_a^b y(t)\,dt = \sum_{j=0}^{n} A_j y(t_j) + \varepsilon_n(y) \tag{6}$$

where $t_j \ (j = 0,1,2,\cdots,n)$ is the abscissas of the partition points of the integration interval $[a,b]$, $A_j \ (j = 0,1,2,\cdots,n)$ is numerical coefficients that do not depend on the function $y(t)$ and $\varepsilon_n(y)$ is the truncation error of Equation (6). Meanwhile, Figure 1 shows the finite grid networks in order to form the full-, half- and quarter-sweep quadrature approximation equations.

Based on Figure 1, the full-, half- and quarter-sweep iterative methods will compute approximate values onto node points of type ● only until the convergence criterion is reached. According to Abdullah (1991) and, Othman and Abdullah (2000), the approximation solutions for the remaining points are calculated by using direct methods.

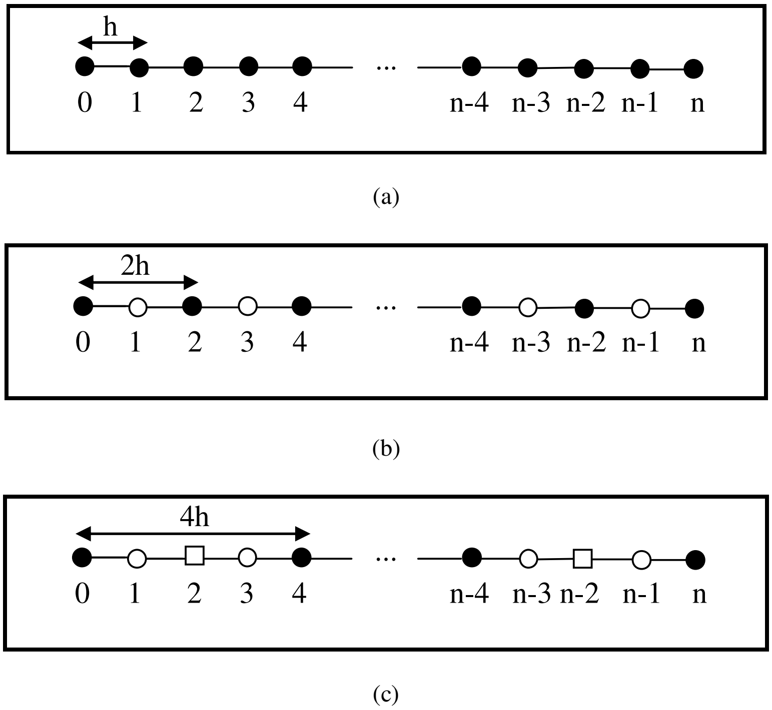

Figure 1: (a), (b) and (c) show distribution of uniform node points for the full-, half- and quarter-sweep cases respectively

However, in 2009, Muthuvalu and Sulaiman carried out a study to investigate the applications of the half-sweep iteration in solving dense linear system generated from the discretization of the second kind Fredholm integral equations using high-order Newton-Cotes schemes. From the results obtained, it has shown that applications of the half-sweep iteration with high-order Newton-Cotes discretization schemes reduce the accuracy of the numerical solutions and it is due to the computational technique for calculating the remaining points by using direct method.

Thus, in this paper, we will use second-order Lagrange interpolation method to compute the remaining points for both half- and quarter-sweep iterations in order to overcome the problem mentioned in Muthuvalu and Sulaiman (2009). Formulations to compute the remaining points using second order Lagrange interpolation for half- and quarter-sweep iterations are defined in Equations (7) and (8) respectively as follows

$$y_i = \begin{cases} \frac{3}{8} y_{i-1} + \frac{3}{4} y_{i+1} - \frac{1}{8} y_{i+3} \,, & i = 1,3,5,\cdots,n-3 \\ \frac{3}{4} y_{i-1} + \frac{3}{8} y_{i+1} - \frac{1}{8} y_{i-3} \,, & i = n-1 \end{cases} \tag{7}$$

$$y_i = \begin{cases} \frac{3}{8} y_{i-2} + \frac{3}{4} y_{i+2} - \frac{1}{8} y_{i+6} \,, & i = 2,6,10,\cdots,n-6 \\ \frac{3}{4} y_{i-2} + \frac{3}{8} y_{i+2} - \frac{1}{8} y_{i-6} \,, & i = n-2 \\ \frac{3}{8} y_{i-1} + \frac{3}{4} y_{i+1} - \frac{1}{8} y_{i+3} \,, & i = 1,3,5,\cdots,n-3 \\ \frac{3}{4} y_{i-1} + \frac{3}{8} y_{i+1} - \frac{1}{8} y_{i-3} \,, & i = n-1 \end{cases} . \tag{8}$$

By applying Equation (6) into Equation (1) and neglecting the error, $\varepsilon_n(y)$, a system of linear equations can be formed for approximation values of $y(t)$. The following linear system generated using quadrature method can be easily shown in matrix form as follows

$$M \underset{\sim}{y} = \underset{\sim}{f} \tag{9}$$

where

$$M = \begin{bmatrix} \lambda - A_0 K_{0,0} & -A_p K_{0,p} & -A_{2p} K_{0,2p} & \cdots & -A_n K_{0,n} \\ -A_0 K_{p,0} & \lambda - A_p K_{p,p} & -A_{2p} K_{p,2p} & \cdots & -A_n K_{p,n} \\ -A_0 K_{2p,0} & -A_p K_{2p,p} & \lambda - A_{2p} K_{2p,2p} & \cdots & -A_n K_{2p,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -A_0 K_{n,0} & -A_p K_{n,p} & -A_{2p} K_{n,2p} & \cdots & \lambda - A_n K_{n,n} \end{bmatrix}_{\left(\left(\frac{n}{p}\right)+1\right) x \left(\left(\frac{n}{p}\right)+1\right)} ,$$

$$\underset{\sim}{y} = \begin{bmatrix} y_0 & y_p & y_{2p} & \cdots & y_{n-2p} & y_{n-p} & y_n \end{bmatrix}^T,$$

$$\underset{\sim}{f} = \begin{bmatrix} f_0 & f_p & f_{2p} & \cdots & f_{n-2p} & f_{n-p} & f_n \end{bmatrix}^T.$$

In order to facilitate the formulation of the full-, half- and quarter-sweep quadrature approximation equations for problem (1), further discussion will be restricted onto Newton-Cotes quadrature method, which is based on interpolation formulas with equally spaced data. In this paper, three different schemes in Newton-Cotes quadrature method such as repeated trapezoidal (RT), repeated Simpson's $\frac{1}{3}$ (RS1) and repeated Simpson's $\frac{3}{8}$ (RS2) schemes will be applied to discretize the problem (1). RT, RS1 and RS2 are first, second and third order schemes respectively. Further discussions on Newton-Cotes quadrature method to solve Fredholm integral equations can be found in Atkinson (1997), and Muthuvalu and Sulaiman (2009).

Based on RT, RS1 and RS2 schemes, numerical coefficients A_j will satisfy following relations respectively.

$$A_j = \begin{cases} \frac{1}{2}ph, & j = 0, n \\ ph, & otherwise \end{cases} \tag{10}$$

$$A_j = \begin{cases} \frac{1}{3}ph, & j = 0, n \\ \frac{4}{3}ph, & j = p, 3p, 5p, \cdots, n-p \\ \frac{2}{3}ph, & otherwise \end{cases} \tag{11}$$

$$A_j = \begin{cases} \frac{3}{8}ph, & j = 0, n \\ \frac{3}{4}ph, & j = 3p, 6p, 9p, \cdots, n-3p \\ \frac{9}{8}ph, & otherwise \end{cases} \quad (12)$$

where the constant step-size, h is defined as follows

$$h = \frac{b-a}{n} \quad (13)$$

and n is the number of subintervals in the interval $[a,b]$. Meanwhile, the value of p, which corresponds to 1, 2 and 4, represents the full-, half- and quarter-sweep cases respectively.

FORMULATION OF THE ITERATIVE METHODS

As afore-mentioned, FSGS, HSGS and QSGS iterative methods will be applied to solve linear system generated from the discretization of the problem (1), as shown in Equation (9). Let matrix M be decomposed into

$$M = D - L - U \quad (14)$$

where D, $-L$ and $-U$ are diagonal, strictly lower triangular and strictly upper triangular matrices respectively. Thus, the general scheme for FSGS, HSGS and QSGS iterative methods can be written as

$$\underset{\sim}{y}^{(k+1)} = (D-L)^{-1}\left(U\,\underset{\sim}{y}^{(k)} + \underset{\sim}{f}\right). \quad (15)$$

Actually, the iterative methods attempt to find a solution by repeatedly solving the linear system using approximations to the vector $\underset{\sim}{y}$ and continue until the solution is within a predetermined acceptable bound on the error. Based on Abdullah (1991) and, Othman and Abdullah (2000), the general algorithm for FSGS, HSGS and QSGS iterative methods to solve problem (1) would be generally described in Algorithm 1.

Algorithm 1: FSGS, HSGS and QSGS methods

For $i = 0, p, 2p, \cdots, n-2p, n-p, n$ and $j = 0, p, 2p, \cdots, n-2p, n-p, n$ calculate

$$y_i^{(k+1)} \leftarrow \begin{cases} \left(f_i + \sum_{j=p}^{n} A_j K_{i,j} y_j^{(k)} \right) \Big/ \lambda - A_i K_{i,i} \,, & i = 0 \\ \left(f_i + \sum_{j=0}^{n-p} A_j K_{i,j} y_j^{(k+1)} \right) \Big/ \lambda - A_i K_{i,i} \,, & i = n \\ \left(f_i + \sum_{j=0}^{i-p} A_j K_{i,j} y_j^{(k+1)} + \sum_{j=i+p}^{n} A_j K_{i,j} y_j^{(k)} \right) \Big/ \lambda - A_i K_{i,i} \,, & i = otherwise \end{cases}$$

NUMERICAL SIMULATIONS

In order to compare the performances of the iterative methods, several experiments were carried out on the following Fredholm integral equations problems.

Example 1 (Wang (2006))

Consider the integral equation

$$y(x) - \int_0^1 \left(4xt - x^2\right) y(t)\,dt = x \tag{16}$$

and the exact solution of problem (16) is given by $y(x) = 24x - 9x^2$.

Example 2 (Polyanin and Manzhirov (1998))

Consider the integral equation

$$y(x) - \int_0^1 (x^2 + t^2)\, y(t)\,dt = x^6 - 5x^3 + x + 10 \,. \tag{17}$$

Exact solution of problem (17) is

$$y(x)=x^6-5x^3+\frac{1045}{28}x^2+x+\frac{2141}{84}.$$

There are three parameters considered in numerical comparison such as number of iterations, execution time and maximum absolute error. Throughout the experiments, the convergence test considered the tolerance error of the $\varepsilon=10^{-10}$. The experiments were carried out on several different mesh sizes such as 240, 480, 960, 1920, 3840 and 7680. Results of numerical simulations, which were obtained from implementations of the FSGS, HSGS and QSGS iterative methods for Examples 1 and 2, have been recorded in Tables 1 and 2 respectively.

TABLE 1: Comparison of a number of iterations, execution time (seconds) and maximum absolute error for the iterative methods for Example 1

		Number of iterations			Execution time (seconds)			Maximum absolute error		
Mesh Size	**Methods**	**FSGS**	**HSGS**	**QSGS**	**FSGS**	**HSGS**	**QSGS**	**FSGS**	**HSGS**	**QSGS**
240	**RT**	193	192	189	0.51	0.20	0.19	2.136 E-3	8.547 E-3	3.425 E-2
	RS1	193	191	188	0.52	0.21	0.21	7.262 E-10	7.488 E-10	7.055 E-10
	RS2	193	192	189	0.54	0.26	0.23	7.376 E-10	6.830 E-10	6.641 E-10
480	**RT**	194	193	192	1.82	0.63	0.24	5.339 E-4	2.136 E-3	8.547 E-3
	RS1	194	193	191	1.84	0.66	0.29	7.156 E-10	7.262 E-10	7.488 E-10
	RS2	194	193	192	1.86	0.73	0.32	7.212 E-10	7.376 E-10	6.830 E-10
960	**RT**	194	194	193	6.72	2.01	0.85	1.335 E-4	5.339 E-4	2.136 E-3
	RS1	194	194	193	6.79	2.02	0.94	7.552 E-10	7.156 E-10	7.262 E-10
	RS2	194	194	193	6.84	2.25	0.97	7.581 E-10	7.212 E-10	7.376 E-10
1920	**RT**	195	194	194	26.24	7.58	2.63	3.337 E-5	1.335 E-4	5.339 E-4
	RS1	195	194	194	26.93	7.61	2.75	6.868 E-10	7.552 E-10	7.156 E-10
	RS2	195	194	194	27.05	7.79	2.77	8.341 E-6	7.581 E-10	7.212 E-10
3840	**RT**	195	195	194	100.09	27.48	8.55	6.961 E-10	3.337 E-5	1.335 E-4
	RS1	195	195	194	101.99	28.08	8.98	6.868 E-10	6.868 E-10	7.552 E-10
	RS2	195	195	194	102.30	29.33	9.09	6.968 E-10	6.881 E-10	7.581 E-10
7680	**RT**	195	195	195	397.81	108.77	31.44	2.085 E-6	8.341 E-6	3.337 E-5
	RS1	195	195	195	402.91	110.17	33.37	7.008 E-10	6.961 E-10	6.868 E-10
	RS2	195	195	195	403.95	111.43	34.09	7.012 E-10	6.968 E-10	6.881 E-10

TABLE 2: Comparison of a number of iterations, execution time (seconds) and maximum absolute error for the iterative methods for Example 2

		Number of iterations			Execution time (seconds)			Maximum absolute error		
Mesh Size	**Methods**	**FSGS**	**HSGS**	**QSGS**	**FSGS**	**HSGS**	**QSGS**	**FSGS**	**HSGS**	**QSGS**
240	**RT**	56	55	55	0.18	0.14	0.09	2.174 E-3	8.697 E-3	3.481 E-2
	RS1	56	55	55	0.20	0.16	0.10	1.124 E-8	3.342 E-7	2.742 E-6
	RS2	56	55	55	0.21	0.17	0.11	2.542 E-8	3.567 E-7	3.102 E-6
480	**RT**	56	56	55	0.56	0.23	0.16	5.435 E-4	2.174 E-3	8.697 E-3
	RS1	56	56	55	0.57	0.27	0.17	5.882 E-10	4.122 E-8	3.342 E-7
	RS2	56	56	55	0.59	0.29	0.19	1.474 E-9	4.264 E-8	3.567 E-7
960	**RT**	56	56	56	2.52	0.58	0.34	1.359 E-4	5.435 E-4	2.174 E-3
	RS1	56	56	56	2.57	0.63	0.38	8.305 E-11	5.108 E-10	4.122 E-8
	RS2	56	56	56	2.65	0.64	0.40	3.887 E-11	5.196 E-9	4.264 E-8
1920	**RT**	56	56	56	8.24	2.71	1.11	3.397 E-5	1.359 E-4	5.435 E-4
	RS1	56	56	56	8.40	2.79	1.17	1.260 E-10	6.410 E-11	5.108 E-10
	RS2	56	56	56	8.71	2.83	1.20	1.227 E-10	6.355 E-11	5.196 E-9
3840	**RT**	56	56	56	31.62	11.04	5.40	8.492 E-6	3.397 E-5	1.359 E-4
	RS1	56	56	56	32.04	11.87	5.77	1.301 E-10	9.188 E-10	6.410 E-11
	RS2	56	56	56	34.01	12.04	5.91	1.299 E-10	9.155 E-10	6.355 E-11
7680	**RT**	56	56	56	122.70	35.16	22.83	2.123 E-6	8.492 E-6	3.397 E-5
	RS1	56	56	56	124.66	36.79	24.89	1.309 E-10	2.291 E-10	9.188 E-10
	RS2	56	56	56	127.33	38.14	25.11	1.310 E-10	2.291 E-10	9.155 E-10

COMPUTATIONAL COMPLEXITY ANALYSIS

In order to measure the computational complexity of the iterative methods, an estimation of the amount of the computational works required for both methods have been conducted. The computational works are estimated by considering the arithmetic operations performed per iteration. Based on Algorithm 1, it can be observed that there are $\left(\frac{n}{p}+1\right)$ additions/subtractions (ADD/SUB) and $2\left(\frac{n}{p}+1\right)$ multiplications/divisions (MUL/DIV) in computing a value for each node point in the solution domain. From the order of the coefficient matrix, M in Equation (9), the total number of arithmetic operations per iteration for the FSGS, HSGS and QSGS iterative methods has been summarized in Table 3.

TABLE 3: Total number of arithmetic operations per iteration for FSGS, HSGS and QSGS methods

Methods	Arithmetic Operation	
	ADD/SUB	MUL/DIV
FSGS	$(n+1)^2$	$2(n+1)^2$
HSGS	$\left(\frac{n}{2}+1\right)^2$	$2\left(\frac{n}{2}+1\right)^2$
QSGS	$\left(\frac{n}{4}+1\right)^2$	$2\left(\frac{n}{4}+1\right)^2$

CONCLUSIONS

In this paper, we present applications of the half- and quarter-sweep iterative methods for solving dense linear systems arising from the discretization of the second kind linear Fredholm integral equations by using three different orders of Newton-Cotes quadrature discretization schemes such as RT, RS1 and RS2 schemes. It has shown that the quadrature approximation equations based on Newton-Cotes schemes can be easily formulated and rewritten in general form as shown in Equation (9).

Through numerical results obtained for both Examples 1 and 2 (refer Tables 1 and 2), it shows that number of iterations for HSGS and QSGS methods are nearly same compared to the FSGS method. Through the observation in Tables 1 and 2, HSGS and QSGS iterative methods reduce the execution time compared to the FSGS method. Computational time for FSGS, HSGS and QSGS iterative methods with RS1 and RS2 schemes are increased compared to the iterative methods with RT scheme. It is due to the computational complexity of the high-order discretization schemes. In terms of accuracy of numerical solutions obtained, RS1 and RS2 schemes are more accurate than the RT scheme. Besides that, applications of second order Lagrange interpolation to compute remaining points managed to overcome the problem discussed in Muthuvalu and Sulaiman (2009).

Overall, the numerical results show that the QSGS method is a better method compared to the FSGS and HSGS methods in the sense of number of iterations and execution time. This is mainly because of computational complexity of the QSGS method which is approximately 50% and 75% less than HSGS and FSGS methods respectively (refer Table 3).

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