Malaysian Journal of Mathematical Sciences 5(1): 111-123 (2011)

The Existence-Uniqueness Theorem for a System of Differential Equations in the Spaces l^2_{r+1}.

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ABSTRACT

We study an infinite system of differential equations of the second order. Some special cases of the system result from application of the decomposition method to some hyperbolic equations. We discuss the existence and uniqueness questions in the space l^2_{r+1}. The proved theorem enables us to investigate some optimal control and differential game problems described by such a system.

Keywords: Differential equation, infinite system, solution, Hilbert space.

INTRODUCTION

Some of the control problems for parabolic and hyperbolic partial differential equations can be reduced to the ones described by infinite systems of ordinary differential equations by using the decomposition method. For example, Chernous'ko (1992), Ibragimov (2003) and Satimov and Tukhtasinov (2007) investigated the control and differential game problems described by the following infinite system of differential equations:

$$\dot{z}_k + \mu_k z_k = w_k, \ k = 1, 2, ..., \tag{1}$$

where $w_k, k = 1, 2, ...,$ are control parameters, $z_k, w_k \in R^1$ and

$\mu_k, 0 < \mu_1 \le \mu_2 \le ... \le \mu_k \le ... \to \infty,$

are eigenvalues of the elliptic operator

$$Az = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial z}{\partial x_j} \right).$$

Different game problems for the system

$$\ddot{z}_k + \mu_k z_k = w_k, \;\; k = 1, 2, ..., \tag{2}$$

have been considered by Satimov and Tukhtasinov (2007). Hence, there is a significant relationship between control problems described by partial differential equations and those described by infinite system of differential equations.

STATEMENT OF PROBLEM

Let $\lambda_1, \lambda_2, ...$ be a sequence of positive numbers, and r be a fixed number. We introduce into the consideration the space

$$l_r^2 = \{\xi = (\xi_1, \xi_2, ...) : \; \sum_{i=1}^{\infty} \lambda_i^r \xi_i^2 < \infty\}$$

with the inner product and norm

$$(\xi, \eta)_r = \sum_{i=1}^{\infty} \lambda_i^r \xi_i \eta_i, \;\; \xi, \eta \in l_r^2, \;\; \| \xi \| = \left(\sum_{i=1}^{\infty} \lambda_i^r \xi_i^2 \right)^{1/2}.$$

Let $L_2(0, T; l_r^2)$ be the space of functions $f(t) = (f_1(t), f_2(t), ...)$ with measurable coordinates $f_k(t) = (f_{k1}(t), f_{k2}(t)), 0 \le t \le T$, subject to

$$\| f(\cdot) \|_{L_2(0,T;l_r^2)} = \sum_{k=1}^{\infty} \lambda_k^r \int_0^T \left(f_{k1}^2(t) + f_{k2}^2(t) \right) dt < \infty,$$

where T is a given positive number.

We consider the following infinite system of differential equations:

$$\begin{cases} \ddot{x}_k = -\alpha_k x_k - \beta_k y_k + w_{1k}(t), & x_k(0) = x_{k0}, \; \dot{x}_k(0) = x_{k1}, \\ \ddot{y}_k = \beta_k x_k - \alpha_k y_k + w_{2k}(t), & y_k(0) = y_{k0}, \; \dot{y}_k(0) = y_{k1}, \quad k = 1, 2, ..., \end{cases} \tag{3}$$

where α_k, β_k are real number

$$x_0 = (x_{10}, x_{20}, ...) \in l^2_{r+1},\ y_0 = (y_{10}, y_{20}, ...) \in l^2_{r+1},\ x_1 = (x_{11}, x_{21}, ...) \in l^2_r,$$

$$y_1 = (y_{11}, y_{21}, ...) \in l^2_r, w_k = (w_{1k}, w_{2k}),\ \ w(\cdot) = (w_1(\cdot), w_2(\cdot), ...) \in L_2(0,T; l^2_r).$$

The system (3) is obtained if we take in (2)

$$z_k = x_k + iy_k,\ \mu_k = \alpha_k - i\beta_k,\ w_k = w_{1k} + iw_{2k}.$$

In the sequel, we consider $\lambda_k = \sqrt{\alpha_k^2 + \beta_k^2}$. Denote

$$z(t) = (z_1(t),\ z_2(t), ...),\ z_k(t) = (x_k(t),\ y_k(t)),\ |z_k| = \sqrt{x_k^2 + y_k^2}\ ,$$

$$\| z \|^2_{l^2_{r+1}} = \sum_{k=1}^{\infty} \lambda_k^{r+1}(x_k^2 + y_k^2),\ z_0 = (z_{10},\ z_{20}, ...) = (x_{10},\ y_{10},\ x_{20},\ y_{20}, ...),$$

$$z_{k0} = (x_{k0}, y_{k0}),\ \| z_0 \|^2_{l^2_{r+1}} = \sum_{k=1}^{\infty} \lambda_k^{r+1}(x_{k0}^2 + y_{k0}^2),$$

$$z_1 = (z_{11},\ z_{21}, ...) = (x_{11},\ y_{11},\ x_{21},\ y_{21}, ...),\ \ z_{k1} = (x_{k1},\ y_{k1}),\ \| z_1 \|^2_{l^2_r} = \sum_{k=1}^{\infty} \lambda_k^r (x_{k1}^2 + y_{k1}^2).$$

$$z_1 = (z_{11},\ z_{21}, ...) = (x_{11},\ y_{11},\ x_{21},\ y_{21}, ...),\ \ z_{k1} = (x_{k1},\ y_{k1}),$$

$$\| z_1 \|^2_{l^2_r} = \sum_{k=1}^{\infty} \lambda_k^r (x_{k1}^2 + y_{k1}^2).$$

Definition. Let $w(\cdot) \in L_2(0,T; l^2_r)$. A function $z(t) = (z_1(t),\ z_2(t), ...)$, $0 \le t \le T$, with continuous coordinates $z_k(t)$ satisfying the initial conditions $z_k(0) = (x_{k0}, y_{k0})$, $\dot{z}_k(0) = (x_{k1}, y_{k1})$, $k = 1, 2, ...,$ is said to be solution of the system (3) if $\ddot{z}_k(t)$ exists almost everywhere on $[0,T]$ and satisfies the system (3) almost everywhere on $[0,T]$.

AUXILIARY CALCULATIONS

Let

$$A_{k1}(t) = e^{r_{1k}t}\begin{pmatrix} \cos(r_{2k}t) & -\sin(r_{2k}t) \\ \sin(r_{2k}t) & \cos(r_{2k}t) \end{pmatrix}, \ A_{k2}(t) = A_{k1}(-t), \ R_k = \begin{pmatrix} r_{1k} & -r_{2k} \\ r_{2k} & r_{1k} \end{pmatrix},$$

$$A_k(t) = \frac{1}{2}\left(A_{k1}(t) + A_{k2}(t)\right), \ B_k(t) = \frac{1}{2}R_k^{-1}\left(A_{k1}(t) - A_{k2}(t)\right), \tag{4}$$

$$r_{1k} = \sqrt{\frac{-\alpha_k + \sqrt{\alpha_k^2 + \beta_k^2}}{2}}, \ r_{2k} = \sqrt{\frac{\alpha_k + \sqrt{\alpha_k^2 + \beta_k^2}}{2}}, \ k = 1, 2, \ldots.$$

Then

$$r_k = \sqrt{r_{1k}^2 + r_{2k}^2} = \sqrt[4]{\alpha_k^2 + \beta_k^2} = \sqrt{\lambda_k}.$$

It can be shown that the matrices $A_{k1}(t)$, $A_{k2}(t)$ have the following properties:

$$A_{k1}(t+h) = A_{k1}(t)A_{k1}(h) = A_{k1}(h)A_{k1}(t), \ | A_{k1}(t)z_k | = | A_{k1}^*(t)z_k | = e^{r_{1k}t} | z_k |, \tag{5}$$

$$A_{k2}(t+h) = A_{k2}(t)A_{k2}(h) = A_{k2}(h)A_{k2}(t), | A_{k2}(t)z_k | = | A_{k2}^*(t)z_k | = e^{-r_{1k}t} | z_k |, \tag{6}$$

where A^* denotes the transpose of the matrix A. Combining (3), (4) ,and (5), we obtain

$$A_k(t)z_k | \leq \frac{1}{2}(e^{r_{1k}T} | z_k | + | z_k |) \leq e^{r_{1k}T} | z_k |. \tag{7}$$

Similarly, we can show that

$$B_k(t)z_k | \ \leq \frac{1}{r_k}e^{r_{1k}T} | z_k |. \tag{8}$$

Also we obtain

$$\| A_{k1}(t) - E_2 \| \ \leq \ \| A_{k1}(t) \| + 1 \ \leq \ e^{r_{1k}T} + 1,$$
$$\| A_{k2}(t) - E_2 \| \ \leq \ \| A_{k2}(t) \| + 1 \leq \ e^{-r_{1k}T} + 1 \leq 2,$$

where $\| A \| = \max_{|x|=1} | Ax |,$ and E2 is the identify 2x2 mariks.

MAIN RESULT

Let $C(0,T;l_r^2)$ be the space of continuous functions $z(t) = (z_1(t), z_2(t), \cdots),\ 0 \le t \le T,$ with the value in the space l_r^2 . The following theorem is true.

Theorem 1. *If* $\{r_{1k}\}_{k\in N}$ *is a bounded above sequence, then the infinite system of differential equations (3) has a unique solution* $z(\cdot) \in C(0,T;l_{r+1}^2)$ *defined by*

$$z_k(t) = A_k(t) z_{k0} + B_k(t) z_{k1} + \int_0^t B_k(t-s) w_k(s) ds. \tag{9}$$

Moreover, $\dot{z}(\cdot) \in C(0,T;l_r^2)$.

Proof: 1^0. *Proof that* $z(t) = (z_1(t), z_2(t), \cdots) \in l_{r+1}^2$ *for each* $t \in [0,T]$.

Let $\gamma = \sup_{k\in N}\{r_{1k}\}$. In accordance with the inequality

$$\left(\sum_{i=1}^n a_i \right)^2 \le n \sum_{i=1}^n a_i^2, \tag{10}$$

with $n = 3$ we obtain from (9) that

$$| z_k(t) |^2 \le 3 \left(| A_k(t) z_{k0} |^2 + | B_k(t) z_{k1} |^2 + \left(\int_0^t | B_k(t-s) w_k(s) | ds \right)^2 \right), t \in [0,T].$$

As by the Cauchy-Schwartz inequality

$$\left(\int_0^t 1. | w_k(s) | ds \right)^2 \le t \int_0^t | w_k(s) |^2 ds,$$

then from the relations (7), (8) we get

$$| z_k(t) |^2 \le 3\left(e^{2\gamma T} | z_{k0} |^2 + \frac{1}{r_k^2} e^{2\gamma T} | z_{k1} |^2 + \frac{1}{r_k^2} e^{2\gamma T} \left(t\int_0^t | w_k(s) |^2 \, ds \right)\right)$$

$$\le 3e^{2\gamma T} \left(| z_{k0} |^2 + \frac{1}{r_k^2} | z_{k1} |^2 + \frac{T}{r_k^2} \int_0^T | w_k(s) |^2 \, ds \right), \ t \in [0,T].$$

Hence

$$| z_k(t) |^2 \ \lambda_k^{r+1} \le 3e^{2\gamma T} \left(| z_{k0} |^2 \ \lambda_k^{r+1} + | z_{k1} |^2 \ \lambda_k^{r} + T\lambda_k^{r} \int_0^T | w_k(s) |^2 \, ds \right), t \in [0,T].$$

Therefore

$$\sum_{k=1}^{\infty} \lambda_k^{r+1} | z_k(t) |^2 \le 3e^{2\gamma T} \left(\sum_{k=1}^{\infty} \lambda_k^{r+1} | z_{k0} |^2 + \sum_{k=1}^{\infty} \lambda_k^{r} | z_{k1} |^2 + T\sum_{k=1}^{\infty} \lambda_k^{r} \int_0^T | w_k(s) |^2 \, ds \right).$$

Then

$$\| z(t) \|_{l_{r+1}^2}^2 \le 3e^{2\gamma T} \left(\| z_0 \|_{l_{r+1}^2}^2 + \| z_1 \|_{l_r^2}^2 + T \| w(\cdot) \|_{L^2(0,T;\, l_r^2)}^2 \right).$$

Thus,

$$z(t) \in l_{r+1}^2 \ \text{ at each } \ t \in [0,T].$$

2^0. *Proof that* $\dot{z}_k(t) = (\dot{z}_1(t),\ \dot{z}_2(t), \ldots .) \in l_r^2$ *for each* $t \in [0,T]$.

The derivative of (9) is

$$\dot{z}_k(t) = R_k^2 B_k(t) z_{k0} + A_k(t) z_{k1} + \int_0^t A_k(t-s) w_k(s)) ds \,. \tag{11}$$

Analysis similar to that in the proof of $z(\cdot) \in C(0,T; l_{r+1}^2)$ shows that

$$\| \dot{z}(t) \|_{l_r^2}^2 \le 3e^{2\gamma T} \left(\| z_0 \|_{l_{r+1}^2}^2 + \| z_1 \|_{l_r^2}^2 + T \| w(\cdot) \|_{L^2(0,T;\, l_r^2)}^2 \right).$$

Thus, $\dot{z}(t) \in l_r^2,\ t \in [0,T]$.

2^0 *Proof that the function* $z(t),\ t \in [0,T]$ *is continuous in* l_{r+1}^2.

A. For $h > 0$, we have

$$z_k(t+h) - z_k(t) = A_k(t+h)z_{k0} + B_k(t+h)z_{k1} + \int_0^{t+h} B_k(t+h-s)w_k(s)ds$$
$$-\left(A_k(t)z_{k0} + B_k(t)z_{k1} + \int_0^t B_k(t-s)w_k(s)ds \right)$$
$$= (A_k(t+h) - A_k(t))z_{k0} + (B_k(t+h) - B_k(t))z_{k1}$$
$$+\int_0^t (B_k(t+h-s) - B_k(t-s))w_k(s)ds + \int_t^{t+h} B_k(t+h-s)w_k(s)ds.$$

According to (4) we get

$$z_k(t+h) - z_k(t) = \frac{1}{2}(A_{k1}(t+h) - A_{k1}(t))z_{k0} + \frac{1}{2}(A_{k2}(t+h) - A_{k2}(t))z_{k0}$$
$$+\frac{1}{2}R_k^{-1}\left(A_{k1}(t+h) - A_{k1}(t)\right)z_{k1} - \frac{1}{2}R_k^{-1}\left(A_{k2}(t+h) - A_{k2}(t)\right)z_{k1}$$
$$+\frac{1}{2}R_k^{-1}\int_0^t (A_{k1}(t+h-s) - A_{k1}(t-s))w_k(s)ds$$
$$-\frac{1}{2}R_k^{-1}\int_0^t (A_{k2}(t+h-s) - A_{k2}(t-s))w_k(s)ds + \int_t^{t+h} B_k(t+h-s)w_k(s)ds.$$

Then employing the relations (5), (6) we obtain

$$z_k(t+h) - z_k(t) = \frac{1}{2}(A_{k1}(h) - E_2)A_{k1}(t)z_{k0} + \frac{1}{2}(A_{k2}(h) - E_2)A_{k2}(t)z_{k0}$$
$$+\frac{1}{2}R_k^{-1}(A_{k1}(h) - E_2)A_{k1}(t)z_{k1} - \frac{1}{2}R_k^{-1}(A_{k2}(h) - E_2)A_{k2}(t)z_{k1}$$

$$+\frac{1}{2}R_k^{-1}\int_0^t (A_{k1}(h)-E_2)A_{k1}(t-s)w_k(s)ds$$

$$-\frac{1}{2}R_k^{-1}\int_0^t (A_{k2}(h)-E_2)A_{k2}(t-s)w_k(s)ds$$

$$+\int_t^{t+h} B_k(t+h-s)w_k(s)ds.$$

According to (10), where $n=7$ we obtain

$$\left|z_k(t+h)-z_k(t)\right|^2 \le 7\left(\left|\frac{1}{2}\left(A_{k1}(h)-E_2\right)A_{k1}(t)z_{k0}\right)\right|^2+\left|\frac{1}{2}\left(A_{k2}(h)-E_2\right)A_{k2}(t)z_{k0}\right)\right|^2$$

$$+\left|\frac{1}{2}\left(R_k^{-1}(A_{k1}(h)-E_2)A_{k1}(t)z_{k1}\right)\right|^2+\left|\frac{1}{2}\left(R_k^{-1}(A_{k2}(h)-E_2)A_{k2}(t)z_{k1}\right)\right|^2$$

$$+\left|\frac{1}{2}\left(R_k^{-1}\int_0^t (A_{k1}(h)-E_2)A_{k1}(t-s)w_k(s)ds\right)\right|^2$$

$$+\left|\frac{1}{2}\left(R_k^{-1}\int_0^t (A_{k2}(h)-E_2)A_{k2}(t-s)w_k(s)ds\right)\right|^2$$

$$+\left|\int_t^{t+h} B_k(t+h-s)w_k(s)ds\right|^2\Bigg).$$

Since $\| R_k^{-1} \| = \frac{1}{r_k},$ it follows from (5), (6), and (8) that

$$\left|z_k(t+h)-z_k(t)\right|^2$$

$$\le 7\left(\frac{1}{4}\| A_{k1}(h)-E_2 \|^2 e^{2\gamma T}\left|z_{k0}\right|^2+\frac{1}{4}\| A_{k2}(h)-E_2 \|^2 e^{2\gamma T}\left|z_{k0}\right|^2\right.$$

$$+\frac{1}{4}\| A_{k1}(h)-E_2 \|^2 \frac{e^{2\gamma T}}{r_k^2}\left|z_{k1}\right|^2+\frac{1}{4}\| A_{k2}(h)-E_2 \|^2 \frac{e^{2\gamma T}}{r_k^2}\left|z_{k1}\right|^2$$

$$+\frac{1}{4}\| A_{k1}(h)-E_2 \|^2 \frac{e^{2\gamma T}}{r_k^2}\left(\int_0^t |w_k(s)| ds\right)^2$$

$$+\frac{1}{4}\| A_{k2}(h)-E_2 \|^2 \frac{e^{2\gamma T}}{r_k^2}\left(\int_0^t |w_k(s)| ds\right)^2$$

$$\left.+\frac{1}{r_k^2}e^{2\gamma T}\left(\int_t^{t+h} | w_k(s) | ds\right)^2\right)$$

$$\leq \frac{7}{4}e^{2\gamma T}\left(\| A_{k1}(h)-E_2 \|^2 + \| A_{k2}(h)-E_2 \|^2\right)$$

$$\times\left(| z_{k0} |^2 + \frac{1}{r_k^2} | z_{k1} |^2 + \frac{1}{r_k^2}\left(\int_0^t | w_k(s) | ds\right)^2\right) + \frac{7}{r_k^2}e^{2\gamma T}\left(\int_t^{t+h} | w_k(s) | ds\right)^2 .$$

Then

$$\sum_{k=1}^{\infty}\lambda_k^{r+1} | z_k(t+h)-z_k(t) |^2 \leq \frac{7}{4}e^{2\gamma T}\sum_{k=1}^{\infty}\left(\| A_{k1}(h)-E_2 \|^2 + \| A_{k2}(h)-E_2 \|^2\right)$$

$$\times\left(\lambda_k^{r+1} | z_{k0} |^2 + \lambda_k^r | z_{k1} |^2 + \lambda_k^r\left(\int_0^t | w_k(s) | ds\right)^2\right) + 7e^{2\gamma T}\sum_{k=1}^{\infty}\lambda_k^r\left(\int_t^{t+h} | w_k(s) | ds\right)^2 .$$

Therefore

$$\| z(t+h)-z(t) \|^2_{l_{r+1}^2} \leq I_1 + I_2 + I_3,$$

where

$$I_1 = \frac{7}{4}e^{2\gamma T}\sum_{k=1}^{N}\left(\| A_{k1}(h)-E_2 \|^2 + \| A_{k2}(h)-E_2 \|^2\right)$$

$$\times\left(\lambda_k^{r+1} | z_{k0} |^2 + \lambda_k^r | z_{k1} |^2 + \lambda_k^r\left(\int_0^t | w_k(s) | ds\right)^2\right),$$

$$I_2 = \frac{7}{4} e^{2\gamma T} \sum_{k=N+1}^{\infty} \left(\| A_{k1}(h) - E_2 \|^2 + \| A_{k2}(h) - E_2 \|^2 \right)$$
$$\times \left(\lambda_k^{r+1} | z_{k0} |^2 + \lambda_k^r | z_{k1} |^2 + \lambda_k^r \left(\int_0^t | w_k(s) | ds \right)^2 \right),$$

$$I_3 = 7e^{2\gamma T} \sum_{k=1}^{\infty} \lambda_k^r \left(\int_t^{t+h} | w_k(s) | ds \right)^2 .$$

Since $e^{\gamma T} \geq 1$, $\| A_{k1}(t) - E_2 \| \leq e^{\gamma T} + 1$ and $\| A_{k2}(t) - E_2 \| \leq 2$, we have

$$\| A_{k1}(h) - E_2 \|^2 + \| A_{k2}(h) - E_2 \|^2 \leq 16 e^{2\gamma T} .$$

Hence,

$$I_2 \leq 28 e^{4\gamma T} \sum_{k=N+1}^{\infty} \left(\lambda_k^{r+1} | z_{k0} |^2 + \lambda_k^r | z_{k1} |^2 + T \int_0^T \lambda_k^r | w_k(s) |^2 \, ds \right).$$

Since the series

$$\sum_{k=1}^{\infty} \lambda_k^{r+1} | z_{k0} |^2, \quad \sum_{k=1}^{\infty} \lambda_k^r | z_{k1} |^2, \quad \sum_{k=1}^{\infty} \lambda_k^r \int_0^T | w_k(s) |^2 \, ds$$

are convergent, therefore for any positive number $\varepsilon > 0$ there exists a positive integer N such that $I_2 < \varepsilon/3$. Now we think about I_1.

Since the sum in I_1 consists of finite number of summands and

$$\left(\| A_{k1}(h) - E_2 \|^2 + \| A_{k2}(h) - E_2 \|^2 \right) \to 0, \quad \text{k=1,...,N},$$

as $h \to 0$, for any positive number $\varepsilon > 0$ there exists $\delta_1 > 0$ such that $I_1 < \varepsilon / 3$ whenever $0 < h < \delta_1$.

As for I_3, using the Cauchy-Schwartz inequality, yields

$$I_3 \le 7e^{2\gamma T}\sum_{k=1}^{\infty}\lambda_k^r\left(h\int_t^{t+h}|w_k(s)|^2\,ds\right)$$

$$\le 7he^{2\gamma T}\sum_{k=1}^{\infty}\lambda_k^r\int_0^T|w_k(s)|^2\,ds \le 7he^{2\gamma T}\,\|w(\cdot)\|_{L_2(0,T;l_r^2)}\,.$$

However, $7e^{2\gamma T}h\|w(\cdot)\|_{L_2(0,T;\,l_r^2)} \to 0$ as $h\to 0$, then we can choose $\delta_2>0$ such that $I_3<\varepsilon/3$ whenever $0<h<\delta_2$.

Thus, $\|z(t+h)-z(t)\|_{l_{r+1}^2}^2$ can be done less than any positive number $\varepsilon>0$ by choosing $\delta=min\{\delta_1,\delta_2\}$ and N. This means $z(t)$ is continuous at each $t\in[0,T]$ from the right.

B. Now consider $\|z(t)-z(t-h)\|_{l_{r+1}^2}^2$, $h>0$. We have

$$z_k(t)-z_k(t-h)=A_k(t)z_{k0}+B_k(t)z_{k1}+\int_0^t B_k(t-s)w_k(s)ds$$

$$-A_k(t-h)z_{k0}-B_k(t-h)z_{k1}-\int_0^{t-h}B_k(t-h-s)w_k(s)ds$$

$$=\frac{1}{2}(A_{k1}(h+(t-h))-A_{k1}(t-h))z_{k0}+\frac{1}{2}(A_{k2}(h+(t-h))$$

$$-A_{k2}(t-h))z_{k0}+\frac{1}{2}R_k^{-1}\left(A_{k1}(h+(t-h))-A_{k1}(t-h)\right)z_{k1}$$

$$-\frac{1}{2}R_k^{-1}\left(A_{k2}(h+(t-h))-A_{k2}(t-h)\right)z_{k1}$$

$$+\frac{1}{2}R_k^{-1}\int_0^{t-h}(A_{k1}(h+t-h-s)-A_{k1}(t-h-s))w_k(s)ds$$

$$-\frac{1}{2}R_k^{-1}\int_0^{t-h}(A_{k2}(h+t-h-s)-A_{k2}(t-h-s))w_k(s)ds$$

$$+\int_{t-h}^{t}B_k(t-s)w_k(s)ds$$

$$
\begin{aligned}
&= \frac{1}{2}(A_{k1}(h) - E_2)A_{k1}(t-h)z_{k0} + \frac{1}{2}(A_{k2}(h) - E_2)A_{k2}(t-h)z_{k0} \\
&+ \frac{1}{2}R_k^{-1}\left(A_{k1}(h) - E_2)A_{k1}(t-h)\right)z_{k1} \\
&- \frac{1}{2}R_k^{-1}\left(A_{k2}(h) - E_2)A_{k2}(t-h)\right)z_{k1} \\
&+ \frac{1}{2}R_k^{-1}\int_0^{t-h}(A_{k1}(h) - E_2)A_{k1}(t-h-s)w_k(s)ds \\
&- \frac{1}{2}R_k^{-1}\int_0^{t-h}(A_{k2}(h) - E_2)A_{k2}(t-h-s)w_k(s)ds + \int_{t-h}^{t}B_k(t-s)w_k(s)ds.
\end{aligned}
$$

By analogy with $\| z(t+h) - z(t) \|^2_{l^2_{r+1}}$, we can show that for any given number $\varepsilon > 0$ there exists $\delta > 0$ such that $\| z(t) - z(t-h) \|^2_{l^2_{r+1}} < \varepsilon$ whenever $0 < h < \delta$. Thus, $z(\cdot) \in C(0,T;l^2_{r+1})$. In the same manner we can see that $\dot{z}(\cdot) \in C(0,T;l^2_r)$. The proof of the theorem is complete.

CONCLUSIONS

In this paper, we have studied an infinite system of differential equations. We have proved Theorem 1 when the function $w(\cdot)$ satisfies an integral constraint. This theorem plays a central role in studying control problems described by such a system, since the system (1) can be considered from the point of view of the Control Theory. For example $w(\cdot)$ can serve as the control function that satisfies the integral constraint. It is clear that this class of control functions contains the functions satisfying the geometric constraint:

$$\sum_{k=1}^{\infty}(w_{1k}^2(s) + w_{2k}^2(s)) \le \rho_0^2.$$

Therefore the obtained result can serve as basis for some problems of the Control Theory and enables us studying the control and differential game problems for such an infinite systems of differential equations.

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