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The Point Symmetric Single-Step Procedure for the Simultaneous Approximation of Polynomial Zeros

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ABSTRACT

The point single-step PS1 procedure established by Alefeld and Herzberger (1974) has *R*-order of convergence greater than 2. This method is modified by using the idea of Aitken (1950). The modified method PSS1 has a faster convergence rate. In this paper, the convergence analysis of the point symmetric single-step PSS1 is shown. The interval version of PSS1 (i.e. the interval symmetric single-step ISS1) is given in Monsi and Wolfe (1988). Computational results indicate that this method is more efficient than the total-step (Kerner (1966)) and the single-step (Alefeld and Herzberger (1974)) methods.

Keywords: Point procedure, simultaneous approximation, simple zeros, *R*-order of convergence.

1. INTRODUCTION

Several point iterative procedures for the simultaneous estimation of simple polynomial zeros exist. See for example, Aberth (1973), Alefeld and Herzberger (1974), Braess and Hadeler (1973), Ehrlich (1967), Farmer and Loizou (1975), Hansen *et al.*(1977), Henrici (1974), Kerner (1966), Milovanovic and Petkovic (1983), Petkovic and Milovanovic (1983), Petkovic and Stefanovic (1986, 1987) and references therein. The point iterative procedures can be very effective but need some sufficient conditions for local convergence. These conditions are usually difficult to verify computationally because they often involve prior knowledge of the zeros themselves. The effectiveness of an algorithm is analyzed by measuring the *R*-order of convergence of the algorithm which is discussed in detail in Ortega and Rheinboldt (1970).

2. THE POINT TOTAL-STEP AND SINGLE-STEP PROCEDURES

Let $p: C^1 \to C^1$ be a polynomial of degree *n* defined by

$$p(x) = \sum_{i=0}^{n} a_i x^i \quad (1)$$

where $a_i \in C^1$ $(i = 1, \dots, n)$ are given. This section contains several point iterative procedures for estimating the *n* simple zeros x_i^* $(i = 1, \dots, n)$ of p simultaneously.

The equation $p(x) = 0$ can be expressed in the form

$$p(x) = \prod_{j=1}^{n} (x - x_j^*) = 0 \quad (2)$$

if $a_n \neq 0$. Therefore it is assumed henceforth that $a_n = 1$, so that

$$p(x) = \prod_{j=1}^{n} \left(x - x_j^*\right). \quad (3)$$

Suppose that, for $j = 1, \dots, n, x_j$ is an estimate of x_j^*, and let $q: C^1 \to C^1$ be defined by

$$q(x) = \prod_{j=1}^{n} (x - x_j). \quad (4)$$

Then

$$q'(x_i) = \prod_{j \neq i} \left(x_i - x_j\right) \quad (i = 1, \dots, n). \quad (5)$$

By (3), if, for $i = 1, \dots, n,$ $x_i \neq x_j$ $(j = 1, \dots, n; j \neq i)$, then

$$x_i^* = x_i - \frac{p(x_i)}{\prod_{j \neq i}(x_i - x_j^*)}. \quad (6)$$

Now $x_j \approx x_j^*$ $(j = 1, \dots, n)$ so by (6),

$$x_i^* \approx x_i - \frac{p(x_i)}{\prod_{j \neq i}(x_i - x_j)} \quad (i = 1, \dots, n) \quad (7)$$

This gives rise to the point total-step procedure PT1 defined by

$$x_i^{(k+1)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j\neq i}(x_i^{(k)} - x_j^{(k)})} \quad (i = 1, \dots, n)(k \geq 0), \tag{8}$$

which has been studied by Kerner (1966) and to the point single-step procedure PS1 defined by

$$x_i^{(k+1)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=1}^{i-1}(x_i^{(k)} - x_j^{(k+1)})\prod_{j=i+1}^{n}(x_i^{(k)} - x_j^{(k)})} \tag{9}$$

$$(i = 1, \dots, n)\ (k \geq 0),$$

which has been studied by Alefeld and Herzberger (1974).

The *R*-order of convergence of an iterative procedure is used in this paper as a measure of the asymptotic convergence rate of the procedure. The concept of *R*-order of convergence is discussed in detail in Ortega and Rheinboldt (1970) and Alefeld and Herzberger (1983).

We now wish to repeat very useful theorem and definitions (Orthega and Rheindboldt (1970)) for evaluation of *R*-order of convergence of an iterative procedure *I*.

Theorem 1

Let I be an iterative procedure and let $\Omega(I, x^*)$ be the set of all sequences $\{x^{(k)}\}$ generated by I which converge to the limit x^*. Suppose that there exists a $p \geq 1$ and a constant γ such that for any $\{x^{(k)}\} \in \Omega(I, x^*)$,

$$\left\|x^{(k+1)} - x^*\right\| \leq \gamma \left\|x^{(k)} - x^*\right\|^p,\ k \geq k_0 = k_0(\{x^{(k)}\}).$$

Then it follows that *R*-order of *I* satisfies the inequality $O_R(I, x^*) \geq p$. ■

Definition 1

If there exists a $p \geq 1$ such that for any null sequence $\{w^{(k)}\}$ generated from $\{x^{(k)}\}$, then the *R*-factor of such sequence is defined to be

$$R_p\left(w^{(k)}\right) = \begin{cases} \lim_{k\to\infty} \sup\left\|w^{(k)}\right\|^{1/k}, & p = 1 \\ \lim_{k\to\infty} \sup\left\|w^{(k)}\right\|^{1/p^k}, & p > 1, \end{cases}$$

where R_p is independent of the norm $\|\cdot\|$. ■

Definition 2

We may now define the R-order of the procedure I in term of R-factor as

$$O_R(I, x^*) = \begin{cases} +\infty \quad if\ R_p(I, x^*) = 0 & \text{for } p \geq 1 \\ \inf\{p \,|\, p \in [1, \infty), R_p(I, x^*) = 1\} & \text{otherwise.} \end{cases}$$

Suppose that $R_p(w^{(k)}) < 1$ then it follows from Ortega and Rheinboldt (1970) that the R-order of I satisfies the inequality $O_R(I, x^*) \geq p$. ■

THE POINT SYMMETRIC SINGLE-STEP PSS1

In this section the symmetric single-step idea of Aitken (1950) is used to derive a symmetric point single-step PSS1 procedure defined by

$$x_i^{(k,0)} = x_i^{(k)} \quad (i = 1, \dots, n) \tag{10a}$$

$$x_i^{(k,1)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=1}^{i-1}(x_i^{(k)} - x_j^{(k,1)}) \prod_{j=i+1}^{n}(x_i^{(k)} - x_j^{(k,0)})} \quad (i = 1, \dots, n), \tag{10b}$$

$$x_i^{(k,2)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=1}^{i-1}(x_i^{(k)} - x_j^{(k,1)}) \prod_{j=i+1}^{n}(x_i^{(k)} - x_j^{(k,2)})} \quad (i = n, \dots, 1), \tag{10c}$$

$$x_i^{(k+1)} = x_i^{(k,2)} \qquad (i = 1, \dots, n), (k \geq 0). \tag{10d}$$

The procedure PSS1 has the following attractive features:

- The values $p\left(x_i^{(k)}\right)$ $(i = 1, \dots, n)$ which are computed for use in (10b) are re-used in (10c).

- The products $\prod_{j=1}^{i-1}(x_i^{(k)} - x_j^{(k,1)})$ $(i = 2, \dots, n)$ which are computed for use in (10b) are re-used in (10c).

- $x_n^{(k,1)} = x_n^{(k,2)}$ $(k \geq 0)$ so that $x_n^{(k,2)}$ need not be computed.

- The R-order of convergence of the point total-step PT1 procedure defined by (8) is at least 2 or $O_R(PT1) \geq 2$. The point single-step PS1 procedure (9) has been studied by Alefeld and Herzberger (1974). The R-order of convergence of PS1 to the set of simple zeros $x^* = (x_1^*, \dots, x_n^*)^T$ is such that $O_R(\mathrm{PS1}, x^*) \geq 1 + \tau > 2$, where $\tau \in (1,2)$ is the unique positive zero of $t^n - t - 1$. As shown subsequently that the corresponding R-order of convergence of PSS1 defined by (10) is at least 3 or $O_R(\mathrm{PSS1}, x^*) \geq 3$.

Lemma 1

If (i) $p: C \to C$ is defined by (3); (ii) $p_i: C \to C$ is defined by

$$p_i(x) = \prod_{m=1}^{i-1}(x - x_m^*) \prod_{m=i+1}^{n}(x - x_m^*) \quad (i = 1, \dots, n) \tag{11}$$

(iii) $q_i = C \to C$ is defined by

$$q_i = \prod_{m=1}^{i-1}(x - \bar{x}_m) \prod_{m=i+1}^{n}(x - \hat{x}_m) \quad (i = 1, \dots, n), \tag{12}$$

where $\bar{x}_j \neq \bar{x}_m$ and $\hat{x}_j \neq \hat{x}_m$ $(j, m = 1, \dots, n; j \neq m)$; (iv) $\varphi_i: C \to C$ is defined by

$$\varphi_i(x) = q_i(x) + \sum_{j=1}^{i-1} \frac{p_i(\bar{x}_j) q_i(x)}{q'_i(\bar{x}_j)(x - \bar{x}_j)} + \sum_{j=i+1}^{n} \frac{p_i(\hat{x}_j) q_i(x)}{q'_i(\hat{x}_j)(x - \hat{x}_j)} \tag{13}$$

$$(i = 1, \dots, n).$$

Then

$$\varphi_i(x) = p_i(x) \quad (\forall x \in C)(i = 1, \dots, n).$$

Proof

By (12), for $j = 1, \dots, i-1$,

$$\frac{q_i(x)}{(x-\bar{x}_j)} = \prod_{\substack{m=1\\ m\neq j}}^{i-1} (x-\bar{x}_m) \prod_{m=i+1}^{n} (x-\hat{x}_m).$$

So by (12), for $j, k = 1, \dots, i-1$,

$$\frac{q_i(\bar{x}_k)}{q_i'(\bar{x}_j)(\bar{x}_k-\bar{x}_j)} = \prod_{\substack{m=1\\ m\neq j}}^{i-1} \left(\frac{\bar{x}_k-\bar{x}_m}{\bar{x}_j-\bar{x}_m}\right) \prod_{m=i+1}^{n} \left(\frac{\bar{x}_k-\hat{x}_m}{\bar{x}_j-\hat{x}_m}\right)$$

$$= \begin{cases} 1 & (j=k) \\ 0 & (j\neq k) \end{cases}.$$

Similarly for $j, k = i+1, \dots, n$,

$$\frac{q_i(\hat{x}_k)}{q_i'(\hat{x}_j)(\hat{x}_k-\hat{x}_j)} = \prod_{m=1}^{i-1} \left(\frac{\hat{x}_k-\bar{x}_m}{\hat{x}_j-\bar{x}_m}\right) \prod_{\substack{m=i+1\\ m\neq j}}^{n} \left(\frac{\hat{x}_k-\hat{x}_m}{\hat{x}_j-\hat{x}_m}\right)$$

$$= \begin{cases} 1 & (j=k) \\ 0 & (j\neq k) \end{cases}.$$

Furthermore, by (12),

$$q_i(\hat{x}_k) = 0 \qquad (i+1 \leq k \leq n),$$

and

$$q_i(\bar{x}_k) = 0 \qquad (1 \leq k \leq i-1).$$

Therefore by (13),

$$\varphi_i(\bar{x}_k) = p_i(\bar{x}_k) \qquad (k = 1, \dots, i-1),$$

and

$$\varphi_i(\hat{x}_k) = p_i(\hat{x}_k) \qquad (k = i+1, \dots, n).$$

Finally $\varphi_i(x)/p_i(x) \rightarrow 1$ $(x \rightarrow \infty)(i = 1, \dots, n)$, so φ_i interpolates p_i at the $n-1$ points $\bar{x}_1, \dots, \bar{x}_{i-1}, \hat{x}_{i+1}, \dots, \hat{x}_n$, and the point at infinity. Therefore by the uniqueness of the Lagrange interpolating polynomial, $\varphi_i(x) = p_i(x)$ $(\forall x \in C)$ $(i = 1, \dots, n)$. □

Lemma 2

If hypotheses (i) – (iv) of Lemma 1 are valid;
(v) $\check{x}_i$ $(i = 1, .., n)$ are such that $p(\check{x}_i) \neq 0$ $(i = 1, \dots, n), \check{x}_i \neq \bar{x}_m$ $(m = 1, \dots, i-1), \check{x}_i \neq \hat{x}_m$ $(m = i+1, \dots, n)$, and

$$\bar{x}_i = \check{x}_i - \frac{p(\check{x}_i)}{\prod_{m=1}^{i-1}(\check{x}_i - \bar{x}_m)\prod_{m=i+1}^{n}(\check{x}_i - \hat{x}_m)} \quad (i = 1, \dots, n); \tag{14}$$

(vi) $\breve{w}_i = \check{x}_i - x_i^*, \hat{w}_i = \hat{x}_i - x_i^*$, and $\bar{w}_i = \bar{x}_i - x_i^*$ $(i = 1, \dots, n)$, then

$$\bar{w}_i = \breve{w}_i \left\{ \sum_{j=1}^{i-1} \bar{\gamma}_{ij} \bar{w}_j + \sum_{j=i+1}^{n} \hat{\gamma}_{ij} \hat{w}_j \right\} \quad (i = 1, \dots, n) \tag{15}$$

where

$$\bar{\gamma}_{ij} = \frac{\prod_{m \neq i,j}(\bar{x}_j - x_m^*)}{q_i'(\bar{x}_j)(\bar{x}_j - \check{x}_i)} \quad (j = 1, \dots, i-1) \tag{16}$$

and

$$\hat{\gamma}_{ij} = \frac{\prod_{m \neq i,j}(\hat{x}_j - x_m^*)}{q_i'(\hat{x}_j)(\hat{x}_j - \check{x}_i)} \quad (j = i+1, \dots, n). \tag{17}$$

Proof

By (12), $q_i(\check{x}_i) \neq 0$ $(i = 1, \dots, n)$. So by (13), Lemma 1, (16), and (17)

$$1 - \frac{p_i(\check{x}_i)}{q_i(\check{x}_i)} = \sum_{j=1}^{i-1} \frac{p_i(\bar{x}_j)}{q_i'(\bar{x}_j)(\bar{x}_j - \check{x}_i)} + \sum_{j=i+1}^{n} \frac{p_i(\hat{x}_j)}{q_i'(\hat{x}_j)(\hat{x}_j - \check{x}_i)}$$

$$= \sum_{j=1}^{i-1} \bar{\gamma}_{ij} \bar{w}_j + \sum_{j=i+1}^{n} \hat{\gamma}_{ij} \hat{w}_j. \tag{18}$$

Also, by (14)

$$\bar{w}_i = \breve{w}_i \left\{ 1 - \frac{p_i(\breve{x}_i)}{q_i(\breve{x}_i)} \right\},$$

whence (15) follows from (18).□

Lemma 3

If hypotheses (i) – (v) of Lemma 2 are valid;
(vi) $|\breve{x}_i - x_i^*| \leq \theta d/(2n-1)$ and $|\hat{x}_i - x_i^*| \leq \theta d/(2n-1)$ $(i = 1, \dots, n)$ where $d = \min\{|x_i^* - x_j^*| | i, j = 1, \dots, n; i \neq j\}$ and $0 < \theta < 1$, then $|\bar{w}_i| \leq \theta |\breve{w}_i|$ $(i = 1, \dots, n)$.

Proof

Now

$$\begin{aligned} |\hat{x}_j - x_m^*| &\geq |x_j^* - x_m^*| - |\hat{x}_j - x_j^*| \\ &\geq d - \frac{\theta}{(2n-1)} d \\ &\geq \left(\frac{2n-2}{2n-1}\right) d \;\; (j, m = 1, \dots, n), \end{aligned}$$

whence

$$\begin{aligned} |\hat{x}_j - \hat{x}_m| &\geq |\hat{x}_j - x_m^*| - |x_m^* - \hat{x}_m| \\ &\geq \left(\frac{2n-2}{2n-1}\right) d - \frac{\theta}{(2n-1)} d \\ &\geq \left(\frac{2n-3}{2n-1}\right) d \quad (j, m = 1, \dots, n). \end{aligned}$$

Therefore

$$\begin{aligned} \frac{|\hat{x}_j - x_m^*|}{|\hat{x}_j - \hat{x}_m|} &\leq 1 + \frac{|\hat{x}_m - x_m^*|}{|\hat{x}_j - \hat{x}_m|} \\ &\leq 1 + \frac{\theta d/(2n-1)}{d(2n-3)/(2n-1)} \\ &\leq 1 + \frac{1}{(2n-3)} \quad (j, m = 1, \dots, n). \end{aligned} \tag{19}$$

Also

$$\left|\hat{x}_j - \check{x}_i\right| \geq \left|\hat{x}_j - x_i^*\right| - \left|x_i^* - \check{x}_i\right|$$

$$\geq \left(\frac{2n-3}{2n-1}\right) d \quad (i, j = 1, \dots, n). \tag{20}$$

So by (17), for $i = 1$,

$$\left|\hat{\gamma}_{1j}\right| = \left|\frac{\prod_{\substack{m=2 \\ m \neq j}} (\hat{x}_j - x_m^*)}{q_1'(\hat{x}_j)(\hat{x}_j - \check{x}_1)}\right| \qquad (j = 2, \dots, n).$$

$$= \frac{\prod_{\substack{m=2 \\ m \neq j}} \left|\hat{x}_j - x_m^*\right|}{\left|q_1'(\hat{x}_j)\right| \left|(\hat{x}_j - \check{x}_1)\right|} \qquad (j = 2, \dots, n)$$

$$\leq \frac{\prod_{\substack{m=2 \\ m \neq j}} \left(1 + \frac{1}{(2n-3)}\right)}{\left(\frac{2n-3}{2n-1}\right) d} \qquad (j = 2, \dots, n)$$

$$= \frac{\left(1 + \frac{1}{(2n-3)}\right)^{n-2}}{d(2n-3)/(2n-1)}. \tag{21}$$

It can easily be shown that

$$\left(1 + \frac{1}{(2n-3)}\right)^{n-2} \leq \frac{2n-3}{n-1} \quad (\forall n \geq 2). \tag{22}$$

Thus by (21) and (22),

$$\left|\hat{\gamma}_{1j}\right| \leq \frac{(2n-1)}{d(n-1)} \quad (j = 2, \dots, n).$$

So by (15), and hypothesis (vi)

$$|\overline{w}_1| \leq |\breve{w}_1| \sum_{j=2}^{n} \frac{1}{(n-1)} \frac{(2n-1)}{d} |\widehat{w}_j|$$

$$\leq |\breve{w}_1| \sum_{j=2}^{n} \frac{1}{(n-1)} \frac{(2n-1)}{d} \frac{\theta d}{(2n-1)}$$

$$= \theta |\breve{w}_1|.$$

Suppose that for some $i \geq 2$, $|\overline{w}_m| \leq \theta |\breve{w}_m|$ $(m = 1, \dots, i-1)$. Then

$$|\bar{x}_j - x_m^*| \geq |x_j^* - x_m^*| - |\bar{x}_j - x_j^*|$$

$$\geq d - \frac{\theta}{(2n-1)} d$$

$$\geq \left(\frac{2n-2}{2n-1}\right) d \quad (j = 1, \dots, i-1; m = 1, \dots, n).$$

So

$$|\bar{x}_j - \bar{x}_m| \geq |\bar{x}_j - x_m^*| - |\bar{x}_m - x_m^*|$$

$$\geq \left(\frac{2n-3}{2n-1}\right) d \quad (j, m = 1, \dots, i-1),$$

whence

$$\frac{|\bar{x}_j - x_m^*|}{|\bar{x}_j - \bar{x}_m|} \leq 1 + \frac{|\bar{x}_m - x_m^*|}{|\bar{x}_j - \bar{x}_m|}$$

$$\leq 1 + \frac{1}{(2n-3)} \quad (j, m = 1, \dots, i-1).$$

Similarly,

$$\frac{|\bar{x}_j - x_m^*|}{|\bar{x}_j - \hat{x}_m|} \leq 1 + \frac{1}{(2n-3)} \quad (j = 1, \dots, i-1; m = 1, \dots, n).$$

Also

$$\left|\bar{x}_j - \check{x}_i\right| \geq \left|\bar{x}_j - x_i^*\right| - \left|x_i^* - \check{x}_i\right|$$

$$\geq \left(\frac{2n-3}{2n-1}\right) d \quad (j = 1, \ldots, i-1; i = 1, \ldots, n).$$

So by (16) and (22),

$$\left|\bar{\gamma}_{ij}\right| = \left|\frac{\prod_{m \neq i,j}(\bar{x}_j - x_m^*)}{q_i^{'}(\bar{x}_j)(\bar{x}_j - \check{x}_i)}\right|$$

$$\leq \frac{1}{(n-1)} \frac{(2n-1)}{d} \qquad (j = 1, \ldots, i-1; i = 1, \ldots, n) \qquad (23)$$

Similarly, by (17) and (22),

$$\left|\hat{\gamma}_{ij}\right| = \left|\frac{\prod_{m \neq i,j}(\hat{x}_j - x_m^*)}{q_i^{'}(\hat{x}_j)(\hat{x}_j - \check{x}_i)}\right|$$

$$\leq \frac{1}{(n-1)} \frac{(2n-1)}{d} \qquad (j = i+1, \ldots, n; i = 1, \ldots, n). \qquad (24)$$

So by (15), (23), (24), and hypothesis (vi),

$$|\overline{w}_i| \leq |\breve{w}_i| \frac{1}{(n-1)} \frac{(2n-1)}{d} \left\{\sum_{j=1}^{i-1} \left|\overline{w}_j\right| + \sum_{j=i+1}^{n} \left|\widehat{w}_j\right|\right\}$$

$$\leq |\breve{w}_i| \frac{1}{(n-1)} \frac{(2n-1)}{d} \left\{\sum_{j=1}^{i-1} \frac{\theta d}{(2n-1)} + \sum_{j=i+1}^{n} \frac{\theta d}{(2n-1)}\right\}$$

$$= \theta |\breve{w}_i|.$$

So by finite induction on i, $|\overline{w}_i| \leq \theta |\breve{w}_i|$ $(i = 1, \ldots, n)$. □

Theorem 2

If (i) $p: C \to C$ defined by (3) has n distinct zeros x_i^* $(i = 1, \dots, n)$; (ii) $\left|x_i^{(0)} - x_i^*\right| \leq \theta d/(2n-1)(i = 1, \dots, n)$ where $0 < \theta < 1$ and $d = \min\left\{\left|x_i^* - x_j^*\right| | i, j = 1, \dots, n; i \neq j\right\}$; (iii) the sequence $\left\{x_i^{(k)}\right\}$ $(i = 1, \dots, n)$ are generated from PSS1 $(i.e.$ from (10)), then $x_i^{(k)} \to x_i^*$ $(k \to \infty)(i = 1, \dots, n)$ and $O_R(\text{PSS1}, x^*) \geq 3$.

Proof

For $i = 1, \dots, n,$ let

$$q_{1,i}(x) = \prod_{m=1}^{i-1}(x - x_m^{(k,1)}) \prod_{m=i+1}^{n}(x - x_m^{(k,0)}), \tag{25}$$

$$q_{2,i}(x) = \prod_{m=1}^{i-1}(x - x_m^{(k,1)}) \prod_{m=i+1}^{n}(x - x_m^{(k,2)}), \tag{26}$$

$$\varphi_{1,i}(x) = q_{1,i}(x) + \sum_{j=1}^{i-1} \frac{p_i(x_j^{(k,1)})q_{1,i}(x)}{q'_{1,i}(x_j^{(k,1)})(x - x_j^{(k,1)})}$$

$$+ \sum_{j=i+1}^{n} \frac{p_i(x_j^{(k,0)})q_{1,i}(x)}{q'_{1,i}(x_j^{(k,0)})(x - x_j^{(k,0)})} , \tag{27}$$

and

$$\varphi_{2,i}(x) = q_{2,i}(x) + \sum_{j=1}^{i-1} \frac{p_i(x_j^{(k,1)})q_{2,i}(x)}{q'_{2,i}(x_j^{(k,1)})(x - x_j^{(k,1)})} +$$

$$+ \sum_{j=i+1}^{n} \frac{p_i(x_j^{(k,2)})q_{2,i}(x)}{q'_{2,i}(x_j^{(k,2)})(x - x_j^{(k,2)})} , \tag{28}$$

where $p_i(x)$ is defined by (11).

By lemma 1 and Lemma 2 with $q_i = q_{1,i}, \check{x}_i = x_i^{(k)}, \hat{x}_i = x_i^{(k,0)}, \overline{x}_i = x_i^{(k,1)}, \varphi_i = \varphi_{1,i} (i = 1, \dots, n)$, it follows that for $i = 1, \dots, n, k \geq 0$,

$$w_i^{(k,1)} = w_i^{(k)} \left\{ \sum_{j=1}^{i-1} \alpha_{ij}^{(k,1)} w_j^{(k,1)} + \sum_{j=i+1}^{n} \alpha_{ij}^{(k,0)} w_j^{(k,0)} \right\}, \tag{29}$$

where

$$w_i^{(k,s)} = x_i^{(k,s)} - x_i^*, \quad (s = 0,1,2),$$

$$\alpha_{ij}^{(k,1)} = \frac{\prod_{m \neq i,j} (x_j^{(k,1)} - x_m^*)}{q_{1,i}^{'}(x_j^{(k,1)})(x_j^{(k,1)} - x_i^{(k)})} \quad (j = 1, \dots, i-1), \tag{30}$$

and

$$\alpha_{ij}^{(k,0)} = \frac{\prod_{m \neq i,j} (x_j^{(k,0)} - x_m^*)}{q_{1,i}^{'}(x_j^{(k,0)})(x_j^{(k,0)} - x_i^{(k)})} \quad (j = i+1, \dots, n). \tag{31}$$

Similarly, By Lemma 1 and Lemma 2, with $q_i = q_{2l,i}, \check{x}_i = x_i^{(k)}, \bar{x}_i = x_i^{(k,1)}, \hat{x}_i = x_i^{(k,2)}, \varphi_i = \varphi_{2,i}$ $(i = 1, \dots, n)$, it follows that for $i = 1, \dots, n, k \geq 0$,

$$w_i^{(k,2)} = w_i^{(k)} \left\{ \sum_{j=1}^{i-1} \beta_{ij}^{(k,1)} w_j^{(k,1)} + \sum_{j=i+1}^{n} \beta_{ij}^{(k,2)} w_j^{(k,2)} \right\}, \tag{32}$$

where

$$\beta_{ij}^{(k,1)} = \frac{\prod_{m \neq i,j} (x_j^{(k,1)} - x_m^*)}{q_{2,i}^{'}(x_j^{(k,1)})(x_j^{(k,1)} - x_i^{(k)})} \quad (j = 1, \dots, i-1), \tag{33}$$

and

$$\beta_{ij}^{(k,2)} = \frac{\prod_{m \neq i,j} (x_j^{(k,2)} - x_m^*)}{q_{2,i}^{'}(x_j^{(k,2)})(x_j^{(k,2)} - x_i^{(k)})} \quad (j = i+1, \dots, n). \tag{34}$$

It follows from (29)-(31) and Lemma 3 that $\left|w_i^{(0,1)}\right| \le \theta \left|w_i^{(0,0)}\right|$ $(i = 1, \dots, n)$, and it follows from (32)-(34) and Lemma 3 that $\left|w_i^{(0,2)}\right| \le \theta^2 \left|w_i^{(0,0)}\right|$ $(i = 1, \dots, n)$, whence $\left|w_i^{(1,0)}\right| \le \theta^2 \left|w_i^{(0,0)}\right|$ $(i = 1, \dots, n)$ follows from (10d). It than follows by induction on k that $\forall k \ge 0$

$$\left|w_i^{(k,0)}\right| \le \theta^{3^k - 1} \left|w_i^{(0,0)}\right| \qquad (i = 1, \dots, n),$$

whence $x_i^{(k)} \to x_i^*$ $(k \to \infty)$, $(i = 1, \dots, n)$. Let

$$h_i^{(k,s)} = \frac{(2n-1)}{d} \left|w_i^{(k,s)}\right| \;\; (i = 1, \dots, n)(s = 0,1,2). \tag{35}$$

Then by (29) - (35), for $i = 1, \dots, n$,

$$h_i^{(k,1)} \le \frac{1}{(n-1)} h_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} h_j^{(k,1)} + \sum_{j=i+1}^{n} h_j^{(k,0)} \right\}, \tag{36}$$

and for $i = n, \dots, 1$,

$$h_i^{(k,2)} \le \frac{1}{(n-1)} h_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} h_j^{(k,1)} + \sum_{j=i+1}^{n} h_j^{(k,2)} \right\}. \tag{37}$$

Let

$$u_i^{(1,1)} = \begin{cases} 2 & (i = 1, \dots, n-1) \\ 3 & (i = n) \end{cases}, \tag{38}$$

$$u_i^{(1,2)} = \begin{cases} 4 & (i = 1) \\ 3 & (i = 2, \dots, n) \end{cases}. \tag{39}$$

For $r = 1,2$, let

$$u_i^{(k+1,r)} = \begin{cases} 3u_i^{(k,r)} + 1 & (i = 1) \\ 3u_i^{(k,r)} & (i = 2, \dots, n) \end{cases}. \tag{40}$$

Then by (38) - (40), for $\forall k \geq 0$,

$$u_i^{(k,1)} = \begin{cases} \frac{5}{2}\left(3^{(k-1)}\right) - \frac{1}{2} & (i = 1) \\ 2(3)^{k-1} & (i = 2, \dots, n-1) \\ 3\left(3^{(k-1)}\right) & (i = n) \end{cases}, \tag{41}$$

$$u_i^{(k,2)} = \begin{cases} \frac{9}{2}(3)^{k-1} - \frac{1}{2} & (i = 1) \\ 3(3)^{(k-1)} & (i = 2, \dots, n) \end{cases}. \tag{43}$$

Suppose, without loss of generality, that

$$h_i^{(0,0)} \leq h < 1 \quad (i = 1, \dots, n). \tag{44}$$

Then by a lengthy inductive argument, it follows from (36) – (44) that for $i = 1, \dots, n, k \geq 0$,

$$h_i^{(k,1)} \leq h^{u_i^{(k+1,1)}},$$

and

$$h_i^{(k,2)} \leq h^{u_i^{(k+1,2)}},$$

whence, by (43) and (10d), ($\forall k \geq 0$)

$$h_i^{(k)} \leq h^{3^k} \quad (i = 1, \dots, n). \tag{45}$$

By (35) for $s = 2$,

$$\left|w_i^{(k,2)}\right| = \frac{d}{(2n-1)} h_i^{(k,2)} \quad (i = 1, \dots, n),$$

then by (10d),

$$\left|w_i^{(k+1)}\right| = \frac{d}{(2n-1)} h_i^{(k+1)} \ (i = 1, \dots, n).$$

So

$$\left|w_i^{(k)}\right| = \frac{d}{(2n-1)} h_i^{(k)} \quad (i = 1, \dots, n)(k \geq 0). \tag{46}$$

Let

$$w^{(k)} = \max_{1 \le i \le n}\{\left|w_i^{(k)}\right|\} \tag{47}$$

and

$$h^{(k)} = \max_{1 \le i \le n}\{h_i^{(k)}\}\,. \tag{48}$$

Then, by (36)-(48)

$$w^{(k)} \le \frac{d}{(2n-1)} h^{3^k} \quad (\forall k \ge 0).$$

So

$$R_3\left(w^{(k)}\right) = \lim_{k\to\infty} \sup\left\{(w^{(k)})^{1/3^k}\right\}$$

$$\le \lim_{k\to\infty} \sup\left\{\left(\frac{d}{2n-1}\right)^{1/3^k} h\right\}$$
$$= h$$
$$< 1.$$

Therefore (Orthega and Rheindboldt (1970)),

$$O_R(\text{PSS1}, x_i^*) \ge 3 \;\; (i = 1, \dots, n). \blacksquare$$

4. CONCLUSION

The above analysis clearly shown that the PSS1 procedure gives better result in term of the rate of convergence, where the R-order of convergence of PSS1 is at least 3 or $O_R(\text{PSS1}, x^*) \ge 3$. On the other hand, the R-order of convergence of PS1 of Alefeld and Herzberger (1974) is greater than 2, that is $O_R(\text{PS1}, x^*) \ge 1 + \tau > 2$, where $\tau \in (1,2)$ is the unique positive zero of $t^n - t - 1$. And also that the R-order of convergence of PT1 of Kerner (1966) is at least 2 or $O_R(\text{PT1}, x^*) \ge 2$.

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