Malaysian Journal of Mathematical Sciences 6(1): 105-112 (2012)

# On the Fundamental Solution of the Cauchy Problem for Time Fractional Diffusion Equation on the Sphere

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## ABSTRACT

Diffusion equation has many applications in sciences, not only in physics but also in biology, astrophysics and etc. Especially interest in diffusion in curved surfaces. The last has an application to biological membranes. One of the problems of biophysics is modeling of the transport of substances in the cell. This process has diffusion character. Diffusion processes in cell are complex phenomenon for modeling. For instant, model can contain fractional derivation by time. In this it is important to determine a fundamental solution of the problem. In this work as a curved surface, we consider *N*-dimensional sphere and study a fundamental solution of the Cauchy problem for time fractional diffusion equation. Fundamental solutions obtained as a series of distributions by spherical harmonics. Convergence of distribution expansions by spherical harmonics in weak topology is considered.

Keywords: Diffusion, sphere, fundamental solution, Cauchy problem.

## 1. INTRODUCTION

It is well known that time fractional diffusion appear on fractal geometry of physical, quant mechanical, and biological engineering. Especially fractional diffusions on smooth surfaces may be observed in cell biology. A solution of such a problems depends not only determination of time fractional derivative but also geometry of the surface where diffusion is actually going on.

As since a fundamental soluition of the problem can be found as a series by spherical harmonics of the Dirac delta functions it is also necessary to study such a functional series in topology of distributions. This is important for existing of fundamental solution. Cauchy problem, in whole space, for pseudo-differential equations with analytic symbols was studied by Dubinskij (1981), Umarov (1986) and Tran Duc Van (1989).

In the fractional diffusion the time fractional derivative may be in the sense of either Riemann-Liouville or Caputo fractional derivative. Before considering a definition of fractional diffusion we define fractional integration.

In the present paper $\alpha \in (0,1)$, however some definitions and statements are true for any $\alpha > 0$ .

**Definition** (Samko *et al.* (1993)).

*Let*

$$\phi(t) \in L_1(0,T), T > 0. \ .$$

*Then integral*

$$I_{0+}^{\alpha}\phi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{\phi(\tau)}{(t-\tau)^{1-\alpha}}d\tau$$

*exists almost everywhere on* $(0,T)$ *and called a fractional integral of function* $\phi$ *of order* $\alpha$.

Fractional differentiation can be introduced as inverse operator for the fractional integration:

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha}}.$$

Determined above fractional integral and derivative is called Riemann-Liouville's fractional integration and differentiation accordantly.

If a function $f(t)$ absolutely continuous, $f \in AC[0,T]$, then it almost everywhere has fractional derivative $D_{0+}^{\alpha} f(t)$ from $L_r(0,T)$, $1 \le r < \frac{1}{\alpha}$. Moreover $D_{0+}^{\alpha} f(t)$ can be represented as

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)}[f(0)t^{-\alpha} + \int_0^t \frac{f'(\tau)d\tau}{(t-\tau)^{\alpha}}]. \tag{1}$$

Note that fractional differentiation and integration has inverse interrelation in following sense: for any summability function $\phi(t)$

$$D_{0+}^{\alpha} I_{0+}^{\alpha} \phi(t) = \phi(t),$$

and for any function $f(t) \in L_1(0,T)$ that has summability fractional derivative $D_{0+}^{\alpha} f(t)$, we have

$$I_{0+}^{\alpha} D_{0+}^{\alpha} f(t) = f(t) - t^{1-\alpha} f_{1-\alpha}(0),$$

where $f_{\beta}(t) = I_{0+}^{\beta} f(t), \beta > 0.$

For description of fractional diffusion processes one can use also fractional derivatives in sense of Caputo, that first computes an derivative of first order followed by a fractional integration:

$$D_{*}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau) d\tau}{(t-\tau)^{\alpha}}.$$

From (1) it is easy to obtain following relationship between two fractional derivatives

$$D_{0+}^{\alpha} f(t) = D_{*}^{\alpha} f(t) + \frac{f(0)}{\Gamma(1-\alpha)} t^{-\alpha}.$$

The time fractional Cauchy problem with Riemann-Lioville fractional derivative requires initial condition $I_{0+}^{1-\alpha} f(0) = b_0$ and with Caputo derivative initial condition $f(0) = b_0$. However, it seems more natural in sense of applications consider $f(0) = b_0$, in the present paper we will use first one.

## 2. TIME FRACTIONAL DIFFUSION EQUATION ON THE SPHERE

Denote by $S^N$ unit sphere in $R^{N+1}$, which is defined as follows

$$S^N = \{ x = (x_1, x_2, \ldots\ldots, x_{N+1}) \in R^{N+1} : \sum_{i=1}^{N+1} x_i^2 = 1 \}.$$

Let $ds^2 = g_{ij}dx_i dx_j$ is Riemannian metrics on $S^N$, where $x = (x_1, x_2, ....., x_{N+1})$ are local coordinates on sphere and $|| g_{ji} ||$ inverse matrix to $|| g_{ij} ||$, $g = det || g_{i,j} ||$. Then we define by $\Delta_s$ Laplace-Beltrami operator on $S^N$ as following

$$\Delta_S = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{N+1} \frac{\partial}{\partial x_i} (\sqrt{g} g_{ij} \frac{\partial}{\partial x_j}).$$

We consider the problem

$$-\Delta_s u(x,t) + D_{0+}^{\alpha} u(x,t) = f(x,t) \qquad (2)$$

$$I_{0+}^{1-\alpha} u(x,0) = \phi(x), \qquad (3)$$

where fractional differentiation and integration operators are acting by time variable $t$, operator $\Delta_S$ is acting by spherical variable $x$, functions $f(x,t)$ and $\phi(x)$ are given, where $x \in S^N$, $t > 0$.

It is well known that the Laplace-Beltrami operator $-\Delta_s$ as a formal differential operator in the space $C^\infty(S^N)$ is nonnegative, symmetric and its closure is selfadjoint operator in $L_2(S^N)$. Its eigenfunctions $Y^k(x)$ are homogenous harmonic polynomials on sphere $S^N$ and called as spherical harmonics. Spherical harmonics of different power are orthogonal, corresponding eigenvalues are $\lambda_k = k(k+N-1)$ with the frequencies $a_k = N_k - N_{k-2}$, where $N_k = \frac{N+k}{N!k!}$ and system of functions $\{Y_j^k(x)\}_{j=1}^{a_k}$ is an orthonormal basis in the space of spherical harmonics of power $k$.

In this section we will obtain representation for the following homogenous time fractional equation on sphere

$$D_{0+}^{\alpha} u(x,t) = \Delta_s u(x,t), x \in S^N, t > 0. \qquad (4)$$

By multiplication the equation (4) to $Y_j^k(x)$ and integration by sphere $S^N$ we obtain

$$D_{0+}^{\alpha} u_{k,j}(t) = -\lambda_k u_{k,j}(t), t > 0. \tag{5}$$

where $u_{k,j}(t) = \int_{S^N} u(x,t) Y_j^k(x) d\sigma x.$.

A solution of the equation (5) can be represented as

$$u_{k,j}(t) = c_{k,j} t^{\alpha} E_{\alpha,\alpha}(\lambda_k t^{\alpha}),$$

where $c_{k,j} = I_{0+}^{1-\alpha} u_{k,j}(t)|_{t=0}$ unkown constants and $E_{\alpha,\beta}(z)$ is the G.M. Mittag-Leffler function

$$E_{\alpha,\beta}(z) = \sum_{\ell=0}^{\infty} \frac{z^{\ell}}{\Gamma(\alpha\ell + \beta)}.$$

In appropriate class of function we can write following representation for the solution of time fractional diffusion equation on sphere

$$u(x,t) = t^{1-\alpha} \sum_{k=1}^{\infty} E_{\alpha,\alpha}(\lambda_k t^{\alpha}) \sum_{j=1}^{a_k} c_{k,j} Y_j^k(x). \tag{6}$$

Note that due to equality $I_{0+}^{1-\alpha}[t^{\alpha-1} E_{\alpha,\alpha}(\lambda_k t^{\alpha})]|_{t=0} = 1$ for any $k$ we have a theorem of uniqueness for the solution of the problem (2) - (3) that has representation (6).

## 3. FUNCTIONAL SPACES OF SOLUTIONS

Let $G$ be a domain in $R^{N+1}$. By $G_{\sigma}$, $\sigma > 0$, we denote $\sigma$ neighborhood of the domain $G$. Let $\beta = \nu + \kappa > 0$, where $\nu$ is an integer and $0 < \kappa \le 1$. By $H_p^{\beta}(G)$, $p \ge 1$, we denote Banach space of functions defined on $G$ and the following norm of them is finite

$$\| f \|_{H_p^{\beta}(G)} = \| f \|_{L_p(G)} + \sum_{|\gamma|=\nu} \sup_{z} z^{-\kappa} \| \omega_z^2 \partial^{\gamma} f(y) \|_{L_p(G_{|z|})},$$

where $\gamma = (\gamma_1, \gamma_2, ......, \gamma_{N+1})$ is a multiindex, $|\gamma| = \gamma_1 + \gamma_2 + ...... + \gamma_{N+1}$ is a length of $\gamma$ and $\partial^\gamma f(y) = \dfrac{\partial^{|\gamma|} f(y)}{\partial y_1^{\gamma_1} \partial y_2^{\gamma_2} \cdots \partial y_{N+1}^{\gamma_{N+1}}}$ and symbol $\omega_z^2 \partial^\gamma f(y)$ means second order difference of a function $\partial^\gamma f$ :

$$\omega_z^2 \partial^\gamma f(y) = \partial^\gamma f(y+z) - 2\partial^\gamma f(y) + \partial^\gamma f(y-z).$$

These classes of functions is well known as Nikol'skii classes. We consider below Nikol'skii classes $H_p^\beta(S^N)$ on sphere $S^N$ that determined by passing to local coordinates. When $p = \infty$ we obtain Zygmund-Holder classes $H_p^\beta(S^N) = C^\gamma(S^N)$.

Consider first homogeneous equation (2): case $f(x,t) = 0$. By separation of variables we reduce the first homogeneous equation into two problems. First is an eigenvalue problem for Laplace-Beltrami operator on sphere $S^N$. Second problem from the separation of variables will be solution of the following ordinary fractional order differential equation

$$D_{0+}^\alpha T(t) + k(k+N-1)T(t) = 0. \tag{7}$$

Equation (7) has a solution

$$T(t) = c_k t^{\alpha-1} E_{\alpha,\alpha}(k(k+N-1)t^\alpha).$$

We will search a solution of the homogeneous Cauchy problem (2)-(3), in the case of $f(x,t) \equiv 0$, in the following series form:

$$u(x,t) = \sum_{k=0}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(k(k+N-1)t^\alpha) \sum_{j=1}^{a_k} c_{kj} Y_j^k(x)..$$

For determination of the coefficients $c_{kj}$ we use initial condition (2)

$$u(x,0) = \sum_{k=0}^{\infty} \sum_{j=1}^{a_k} c_{kj} Y_j^k(x) = \phi(x). \tag{8}$$

Series (8) is called Fourier-Laplace series on a sphere $S^N$. Let $\phi \in H_p^{\beta}(S^N)$, where $1 \le p \le \infty$, $\beta > 0.$. The case $f(x,t) \ne 0$ can be reduced to homogeneous case by using a fractional Duhamel's principles.

If $\beta > max\left\{\frac{N}{p}, \frac{N-1}{2}\right\}$, then series on the left side of (4) is uniformly convergent on sphere $S^N$ Note that the condition $\beta > max\left\{\frac{N}{p}, \frac{N-1}{2}\right\}$ can not be weaken.

## 4. FUNDAMENTAL SOLUTION

From formula (8) we can conclude that in order to consider a fundamental solution of the equation (2) we should introduce and study of Fourier-Laplace series of distributions. Let $\Sigma_n f(x) =< f, \sigma(x, y, n) >$, denotes partial sums Fourier-Laplace series of distribution $f \in D'(S^N)$, where $S^N$ is $N$-dimensional unique sphere, $D'(S^N)$ - space of distributions .

**Theorem 1.** *Let* $f \in D(S^N)$. *Then in topology of the space* $D$ *it is true that*

$$\Sigma_n f \to f. \tag{9}$$

**Theorem 2.** *Let* $f \in D'(S^N)$. *Then equality (9) is valid in topology* $D'$.

Proposition 1 supplies convergence above mentioned series in the space of smooth functions and proposition 2 in space of distributions. Spectral expansions of distribution are investigated by Sh.A. Alimov and A.A. Rakhimov (1996).

Now let $\phi(x) = \delta(x_0)$, where $x_0 \in S^N$. Then from theorem 2 it follows that series (8) for the Dirac delta function will converge in distributional topology and formula (8 ) in this case provides a solution of the problem.

## ACKNOWLEDGEMENT

This paper has been supported by University Putra Malaysia under Research University Grant (RUGS). Project numbers are 05-01-11-1273RU and 05-01-09-0674RU.

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