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Bernstein-Szegö Inequalities in Reproducing Kernel Hilbert Spaces

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ABSTRACT

Let S and T be bounded linear operators from a Hilbert space $\mathcal{H}$ into a reproducing kernel Hilbert space $\mathcal{K}$ of complex or real-valued functions defined on some set X. For each $x \in X$, let $k_x \in \mathcal{K}$ have the property that $\langle g, k_x \rangle = g(x)$ for each $g \in \mathcal{K}$. Using Bessel's inequality, we obtain a sharp estimate relating $Sf(x)$, $Tf(x)$, $S^\star k_x$ and $T^\star k_x$. This estimate is then applied to obtain Bernstein-Szegö inequalities for Fourier multiplier operators on Sobolev spaces in L^2.

Keywords: Reproducing kernel Hilbert space, Bernstein-Szegö inequalities, Fourier transform, Fourier series

1. INTRODUCTION

The classical Bernstein-Szegö inequality states that

$$T'(x)^2 + n^2 T(x)^2 \leq n^2 \max_{y \in \mathbb{R}} |T(y)|^2$$

for each real number x, and for each real trigonometric polynomial T of degree at most n. This was extended to entire functions of exponential type by Duffin and Schaeffer (1937). They showed that for any real number x,

$$f'(x)^2 + \tau^2 f(x)^2 \leq \tau^2 \sup_{t \in \mathbb{R}} |f(t)|^2 \tag{1}$$

if f is an entire function of exponential type τ, and real-valued on the real line. These kinds of estimates have also been extended to rational functions by Borwein *et al.* (1994). For example, given a sequence $\{a_k\}_{k=1}^{2n}$ in $\mathbb{C}$ with $\operatorname{Im} a_k > 0$ and any real trigonometric polynomial T of degree at most n,

$$f'(x)^2 + B_n^2(x) f(x)^2 \leq B_n^2(x) \max_{t \in \mathbb{R}} |f(t)|^2$$

for each $x \in \mathbb{R}$ where

$$f(x) = \frac{T(x)}{\prod_{k=1}^{2n} |\sin((x - a_k)/2)|} \quad \text{and} \quad B_n(x) = \frac{1}{2}\sum_{k=1}^{2n} \frac{1 - |e^{ia_k}|^2}{|e^{ia_k} - e^{ix}|^2}.$$

For this, we refer the reader to Borwein and Erdélyi (1995), which also serves as an excellent survey.

Our goal in this note is to unify L^2 versions of these inequalities by considering abstract bounded linear operators on reproducing kernel Hilbert spaces. These are Hilbert spaces $\mathcal{K}$ consisting of complex or real-valued functions defined on some set X, and such that for each $x \in X$, the mapping $f \to f(x)$ is a continuous function from $\mathcal{K}$ into $\mathbb{C}$. Consequently, for each $x \in X$, there exists a function $k_x \in \mathcal{K}$ satisfying $\langle g, k_x \rangle = g(x)$ for each $g \in \mathcal{K}$. The reader may consult Partington (1997) for a brief introduction to reproducing kernel Hilbert spaces.

In this note, for example, we obtain an L^2-version of Duffin and Schaeffer's inequality (1.1) for operators of the form

$$Sf(x) = \int_{\mathbb{R}} m(t)\hat{f}(t)e^{itx}dt$$

applied to Sobolev spaces in $L^2(\mathbb{R})$, where $\hat{f}$ denotes the Fourier transform of f.

2. THE BASIC ESTIMATE

In what follows, $\mathcal{H}$ and $\mathcal{K}$ shall denote real or complex Hilbert spaces where $\mathcal{K}$ consists of functions $g: X \to \mathbb{C}$, and X is some given set. The inner products and norms in $\mathcal{H}$ and $\mathcal{K}$ will be denoted by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, $||\cdot||_{\mathcal{H}}$, $\langle\cdot,\cdot\rangle_{\mathcal{K}}$ and $||\cdot||_{\mathcal{K}}$ respectively. Furthermore, we shall assume that $\mathcal{K}$ is a reproducing kernel Hilbert space. In other words, for each $x \in X$, there exists $k_x \in \mathcal{K}$ such that $\langle g, k_x \rangle_{\mathcal{K}} = g(x)$, for all $g \in \mathcal{K}$.

We now give the basic estimate.

Lemma 2.1 *Let $S, T: \mathcal{H} \to \mathcal{K}$ be bounded linear operators with adjoints $S^\star$ and $T^\star$ respectively. Fix $f \in \mathcal{H}$ and $x \in X$ such that $S^\star k_x$ and $T^\star k_x$ are linearly independent . Then*

$$\Delta(x) := ||S^\star k_x||_{\mathcal{H}}^2 - ||T^\star k_x||_{\mathcal{H}}^{-2}|(TS^\star k_x)(x)|^2 > 0 \qquad (2)$$

and

$$\frac{|Tf(x)|^2}{||T^\star k_x||_{\mathcal{H}}^2} + \frac{|Sf(x) - G(x)Tf(x)|^2}{\Delta(x)} \leq ||f||_{\mathcal{H}}^2. \tag{3}$$

with

$$G(x) = ||T^\star k_x||_{\mathcal{H}}^{-2} \overline{(TS^\star k_x)(x)} \tag{4}$$

Proof. First of all, linear independence of the vectors $S^\star k_x$ and $T^\star k_x$ implies that the vector $||T^\star k_x||_{\mathcal{H}}^2 S^\star k_x - \langle S^\star k_x, T^\star k_x \rangle_{\mathcal{H}} T^\star k_x$ is nonzero. Moreover, since $\langle S^\star k_x, T^\star k_x \rangle_{\mathcal{H}} = (TS^\star k_x)(x)$ the square of its norm is

$$\begin{aligned} & ||||T^\star k_x||_{\mathcal{H}}^2 S^\star k_x - \langle S^\star k_x, T^\star k_x \rangle_{\mathcal{H}} T^\star k_x||_{\mathcal{H}}^2 \\ = & ||T^\star k_x||_{\mathcal{H}}^4 ||S^\star k_x||_{\mathcal{H}}^2 - |TS^\star k_x(x)|^2 ||T^\star k_x||_{\mathcal{H}}^2 \\ = & ||T^\star k_x||_{\mathcal{H}}^4 \Delta(x), \end{aligned}$$

with $\Delta(x)$ given in (2). Hence, $\Delta(x) > 0$.

The remainder of the proof consists of a simple application of Bessel's inequality to the orthonormal vectors

$$u = \frac{T^\star k_x}{||T^\star k_x||_{\mathcal{H}}} \quad \text{and} \quad v = \frac{||T^\star k_x||_{\mathcal{H}}^2 S^\star k_x - \langle S^\star k_x, T^\star k_x \rangle_{\mathcal{H}} T^\star k_x}{||T^\star k_x||_{\mathcal{H}}^2 \sqrt{\Delta(x)}}.$$

Observe that $\langle f, T^\star k_x \rangle_{\mathcal{H}} = \langle Tf, k_x \rangle_{\mathcal{K}} = Tf(x)$ and therefore

$$|\langle f, u \rangle_{\mathcal{H}}|^2 = \frac{|Tf(x)|^2}{||T^\star k_x||_{\mathcal{H}}^2}. \tag{5}$$

Furthermore, we have

$$\begin{aligned} |\langle f, v \rangle_{\mathcal{H}}|^2 &= \frac{|||T^\star k_x||_{\mathcal{H}}^2 Sf(x) - \overline{(TS^\star k_x)(x)} Tf(x)|^2}{||T^\star k_x||_{\mathcal{H}}^4 \Delta(x)} \\ &= \frac{|Sf(x) - G(x)Tf(x)\ |^2}{\Delta(x)}. \end{aligned} \tag{6}$$

In view of (5) and (6), Bessel's inequality

$$|\langle f, u\rangle_{\mathcal{H}}|^2 + |\langle f, v\rangle_{\mathcal{H}}|^2 \leq ||f||_{\mathcal{H}}^2 \tag{7}$$

gives the desired estimate (3). □

Remark 2.2 Since Bessel's inequality (7) becomes an equality if f is a linear combination of u and v, (3) becomes an equality if f is taken to be a linear combination of $S^\star k_x$ and $T^\star k_x$.

3. BERNSTEIN- SZEGÖ INEQUALITIES FOR NON-ANALYTIC FUNCTIONS

Our main result in this section is an L^2 version of Duffin and Schaeffer's Bernstein-Szegö inequality for a wide class of smooth functions including non-analytic ones.

3.1 Sobolev spaces in L^2 as reproducing kernel Hilbert spaces

For a function $f \in L^1(\mathbb{R})$, we define its Fourier transform $\hat{f}$ by

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-iwx} dx$$

for all $w \in \mathbb{R}$. Accordingly, the inversion formula is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}(w) e^{iwx} dw, \tag{8}$$

valid for almost every $x \in \mathbb{R}$ for instance when $f \in L^1 + L^2$ and $\hat{f} \in L^1$. For example, see Rudin (1974).

In what follows, we shall fix $\phi \in L^2(\mathbb{R})$ such that

$$\hat{\phi} \text{ is real-valued, non-negative, and bounded with } \hat{\phi} \in L^1(\mathbb{R}) \tag{9}$$

and write $\Sigma(\phi) = \{t \in \mathbb{R}: \hat{\phi}(t) \neq 0\}$. Let $\mathcal{H}_\phi(\mathbb{R})$ be the vector space of all functions $f: \mathbb{R} \to \mathbb{C}$ in $L^2(\mathbb{R})$ such that

$$\hat{f}(t) = 0 \text{ whenever } \hat{\phi}(t) = 0 \text{ and}$$

$$||f||_{\phi}^{2} = \frac{1}{\sqrt{2\pi}} \int_{\Sigma(\phi)} |\hat{f}(t)|^{2} \frac{dt}{\hat{\phi}(t)} < \infty. \qquad (10)$$

For $f, g \in \mathcal{H}_{\phi}(\mathbb{R})$, define

$$\langle f, g \rangle_{\phi} = \frac{1}{\sqrt{2\pi}} \int_{\Sigma(\phi)} \hat{f}(t)\overline{\hat{g}(t)} \frac{dt}{\hat{\phi}(t)}.$$

This defines an inner product making $\mathcal{H}_{\phi}(\mathbb{R})$ a Hilbert space. The corresponding norm is that given in (10).

If $f \in \mathcal{H}_{\phi}(\mathbb{R})$, then $\hat{f}$ is integrable and the inversion formula (8) shows that f equals almost everywhere a continuous function. Thus, we shall assume that elements of $\mathcal{H}_{\phi}(\mathbb{R})$ are continuous.

For $x, t \in \mathbb{R}$, define $\phi_x(t) = \phi(t - x)$. The next lemma summarizes some basic properties of the Hilbert space $\mathcal{H}_{\phi}(\mathbb{R})$.

Lemma 3.1 *Let $f \in \mathcal{H}_{\phi}(\mathbb{R})$ and $x \in \mathbb{R}$. Then*

(*a*) $\phi_x \in \mathcal{H}_{\phi}(\mathbb{R})$ and $||\phi_x||_{\phi}^{2} = \phi(0)$.
(*b*) $f(x) = \langle f, \phi_x \rangle_{\phi}$.

3.2 Bernstein-Szegö inequality for non-analytic functions

In this section, we fix a function ϕ in $L^2(\mathbb{R})$ satisfying (9):

$\hat{\phi}$ is real-valued, non-negative, bounded, and integrable on $\mathbb{R}$.

Furthermore, we shall assume $\phi(0) = 1$ and $\hat{\phi}$ is even.

Theorem 3.2 *Let $m: \mathbb{R} \to \mathbb{C}$ be odd, not identically constant on $\Sigma(\phi) = \{t \in \mathbb{R}: \hat{\phi}(t) \neq 0\}$ and such that $\hat{\phi} \cdot m^2 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Suppose either*

$$(A) \inf\{|m(t)|: t \in \Sigma(\phi)\} > 0 \ \textit{ or } \ (B) \sup\{|m(t)|: t \in \Sigma(\phi)\} < \infty. \qquad (11)$$

Let $f \in \mathcal{H}_\phi(\mathbb{R})$, $x \in \mathbb{R}$ *and define*

$$Sf(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} m(t)\hat{f}(t)e^{ixt}dt. \tag{12}$$

Then

$$|f(x)|^2 + \Delta^{-2}|Sf(x)|^2 \leq \frac{1}{\sqrt{2\pi}} \int_{\Sigma(\phi)} |\hat{f}(t)|^2 \frac{dt}{\hat{\phi}(t)} \tag{13}$$

where

$$\Delta^2 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |m(t)|^2 \hat{\phi}(t)dt.$$

Proof. Define $\rho \in L^2(\mathbb{R})$ such that $\hat{\rho} = \hat{\phi}|m|^2$. Trivially, $\Sigma(\rho) \subset \Sigma(\phi)$. Suppose (*A*) holds and let $\mu_A = \inf\{|m(t)|: t \in \Sigma(\phi)\} > 0$. Then $\Sigma(\rho) = \Sigma(\phi)$. Moreover,

$$\mathcal{H}_\phi(\mathbb{R}) \subset \mathcal{H}_\rho(\mathbb{R}). \tag{14}$$

Indeed, let $g \in \mathcal{H}_\phi(\mathbb{R})$. If $\hat{\rho}(t) = 0$, then $\hat{\phi}(t) = 0$ and by (10), $\hat{g}(t) = 0$. Moreover,

$$||g||_\rho^2 = \frac{1}{\sqrt{2\pi}} \int_{\Sigma(\rho)} |\hat{g}(t)|^2 \frac{dt}{\hat{\rho}(t)} \leq \mu_A^{-2}||g||_\phi^2.$$

Thus, (14) holds and, in fact, the inclusion is bounded.

For each $h \in \mathcal{H}_\phi(\mathbb{R})$, we have

$$||Sh||_\rho^2 = \frac{1}{\sqrt{2\pi}} \int_{\Sigma(\rho)} |m(t)\hat{h}(t)|^2 \hat{\rho}(t)^{-1}dt = ||h||_\phi^2.$$

Thus $S: \mathcal{H}_\phi(\mathbb{R}) \to \mathcal{H}_\rho(\mathbb{R})$ is bounded as well. In fact, S is an isometry.

To obtain the desired inequality (13), we shall apply Lemma 2.1 with $\mathcal{H} = \mathcal{H}_\phi(\mathbb{R})$, $\mathcal{K} = \mathcal{H}_\rho(\mathbb{R})$, $S: \mathcal{H}_\phi(\mathbb{R}) \to \mathcal{H}_\rho(\mathbb{R})$ as given in (12) and with $T: \mathcal{H}_\phi(\mathbb{R}) \to \mathcal{H}_\rho(\mathbb{R})$ as the inclusion map.

To find the adjoint $T^{\star}: \mathcal{H}_{\rho}(\mathbb{R}) \to \mathcal{H}_{\phi}(\mathbb{R})$, observe that for $g \in \mathcal{H}_{\rho}(\mathbb{R})$ and $f \in \mathcal{H}_{\phi}(\mathbb{R})$,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_{\Sigma(\phi)} \widehat{T^{\star} g}(t) \overline{\hat{f}(t)} \frac{dt}{\hat{\phi}(t)} &= \langle T^{\star} g, f \rangle_{\phi} \\ &= \langle g, f \rangle_{\rho} \\ &= \frac{1}{\sqrt{2\pi}} \int_{\Sigma(\rho)} \hat{g}(t) \overline{\hat{f}(t)} \frac{dt}{\hat{\rho}(t)}.\end{aligned}$$

This shows that $\widehat{T^{\star} g}(t) = \hat{g}(t) \hat{\phi}(t) \hat{\rho}(t)^{-1}$ whenever $t \in \Sigma(\rho)$ and $\widehat{T^{\star} g}(t) = 0$ otherwise. In particular

$$\widehat{T^{\star} \rho_x}(t) = e^{-itx} \hat{\phi}(t) = \widehat{\phi_x}(t) \tag{15}$$

for all $t \in \mathbb{R}$. Thus, $T^{\star} \rho_x = \phi_x$ and by Lemma 3.1,

$$||T^{\star} \rho_x||_{\phi}^2 = ||\phi_x||_{\phi}^2 = \phi(0) = 1. \tag{16}$$

Next, we compute for the adjoint of S, $S^{\star}: \mathcal{H}_{\rho}(\mathbb{R}) \to \mathcal{H}_{\phi}(\mathbb{R})$.

A straightforward calculation shows that for all $g \in \mathcal{H}_{\rho}(\mathbb{R})$ and $f \in \mathcal{H}_{\phi}(\mathbb{R})$,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_{\Sigma(\phi)} \frac{\widehat{S^{\star} g}(t) \overline{\hat{f}(t)}}{\hat{\phi}(t)} \, dt &= \langle S^{\star} g, f \rangle_{\phi} \\ &= \langle g, Sf \rangle_{\rho} \\ &= \frac{1}{\sqrt{2\pi}} \int_{\Sigma(\rho)} \frac{\hat{g}(t) \overline{m(t) \hat{f}(t)}}{\hat{\rho}(t)} \, dt.\end{aligned}$$

This shows that $\widehat{S^{\star} g}(t) = \hat{g}(t) \overline{m(t)} \hat{\phi}(t) \hat{\rho}(t)^{-1}$ whenever $t \in \Sigma(\rho)$ and $\widehat{S^{\star} g}(t) = 0$ otherwise. In particular,

$$\widehat{S^{\star} \rho_x}(t) = e^{-itx} \overline{m(t)} \hat{\phi}(t) \tag{17}$$

for each $t \in \mathbb{R}$. Hence,

$$||S^{\star}\rho_x||_{\phi}^2 = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |m(t)|^2 \hat{\phi}(t)dt \tag{18}$$

and since $\overline{m}\hat{\phi}$ is an odd function,

$$(S^{\star}\rho_x)(x) = \langle S^{\star}\rho_x, \phi_x\rangle_{\phi} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \overline{m(t)}\hat{\phi}(t)dt = 0.$$

Thus, $G(x)$ as defined in (4) is identically zero. Likewise, in view of (18), $\Delta(x)$ as defined in (2) becomes

$$\Delta(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |m(t)|^2 \hat{\phi}(t)dt.$$

Our hypothesis that m is not identically constant on $\Sigma(\phi)$ together with (15) and (17) shows that $S^{\star}\rho_x$ and $T^{\star}\rho_x$ are linearly independent. In view of (16), the estimate (3) now implies the desired result (13).

Now, suppose (B) holds and let $\mu_B = \sup\{|m(t)|: t \in \Sigma(\phi)\} < \infty$. Then $||Sg||_{\phi}^2 \leq \mu_B^2 ||g||_{\phi}^2$ for each $g \in \mathcal{H}_{\phi}(\mathbb{R})$. Thus, S maps $\mathcal{H}_{\phi}(\mathbb{R})$ boundedly into itself. The desired estimate (13) is obtained by applying Lemma 2.1 with $\mathcal{H} = \mathcal{K} = \mathcal{H}_{\phi}(\mathbb{R})$, $S: \mathcal{H}_{\phi}(\mathbb{R}) \to \mathcal{H}_{\phi}(\mathbb{R})$ as given in (12) and with $T: \mathcal{H}_{\phi}(\mathbb{R}) \to \mathcal{H}_{\phi}(\mathbb{R})$ as the identity. □

A special case of Theorem 3.2 gives an L^2-version of Duffin and Schaeffer's inequality (1) for higher order derivatives.

Corollary 3.3 *Let* $\tau > 0$ *and* k *be an odd positive integer. Let* $f, \phi \in L^2(\mathbb{R})$ *such that* $\hat{f}(t) = 0 = \hat{\phi}(t)$ whenever $|t| > \tau$. *Then*

$$|f(x)|^2 + \tau^{-2k}|f^{(k)}(x)|^2 \leq \frac{1}{\sqrt{2\pi}}\int_{-\tau}^{\tau} \frac{|\hat{f}(t)|^2}{\hat{\phi}(t)}dt$$

provided $\hat{\phi}$ *is real-valued, even,* $\inf\{\hat{\phi}(t): |t| \leq \tau\} > 0$ *and* $\phi(0) = 1$.

Proof. Apply Theorem 3.2 with $m(t) = (it)^k$ and with $\phi \in L^2(\mathbb{R})$ satisfying the hypotheses of the corollary: $\phi(0) = 1$, $\hat{\phi}$ is even, real-valued and 0 on $\mathbb{R}\backslash[-\tau, \tau]$, and $\inf\{\hat{\phi}(t): |t| \leq \tau\} > 0$. Then condition ($B$) in (11) is satisfied. Note that $\mathcal{H}_{\phi}(\mathbb{R})$ is the set of all functions $g \in L^2(\mathbb{R})$ such that $\hat{g}(t) = 0$ whenever $|t| > \tau$. Observe that by the inversion formula (8),

$$\Delta^2(m,\phi) \;=\; \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} |m(t)|^2\hat{\phi}(t)dt \;\le\; \tau^{2k}. \quad \square$$

4. BERNSTEIN-SZEGÖ INEQUALITIES FOR SOBOLEV SPACES IN $L^2(\mathbb{T})$

Here, we shall obtain periodic versions of the results of the preceding section. First, we introduce certain reproducing kernel Hilbert spaces consisting of 2π-periodic continuous functions.

4.1 Reproducing kernel Hilbert spaces of continuous periodic functions

Let $L^2(\mathbb{T})$ be the Hilbert space of all measurable 2π-periodic functions $f:\mathbb{R}\to\mathbb{C}$ with $\int_{-\pi}^{\pi}|f(t)|^2dt<\infty$. For $f\in L^2(\mathbb{T})$ and $n\in\mathbb{Z}$, we define the nth Fourier coefficient of f by

$$\hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt.$$

Let $\theta\in L^2(\mathbb{T})$ such that

$$\hat{\theta}(n)\ge 0 \text{ for each } n\in\mathbb{Z} \text{ and } \textstyle\sum_{n\in\mathbb{Z}}\hat{\theta}(n)<\infty \tag{12}$$

and let $\Sigma(\theta)=\{n\in\mathbb{Z}:\hat{\theta}(n)\neq 0\}$. We then define $\mathcal{H}_\theta(\mathbb{T})$ as the vector space of all $f\in L^2(\mathbb{T})$ such that

$$\hat{f}(n)=0 \text{ whenever } \hat{\theta}(n)=0 \text{ and } ||f||_\theta^2=\sum_{n\in\Sigma(\theta)}\frac{|\hat{f}(n)|^2}{\hat{\theta}(n)}<\infty. \tag{13}$$

$\mathcal{H}_\theta(\mathbb{T})$ is a Hilbert space with the inner product

$$\langle f,g\rangle_\theta \;=\; \sum_{n\in\Sigma(\theta)}\hat{f}(n)\overline{\hat{g}(n)}\hat{\theta}(n)^{-1}$$

with the corresponding norm given in (13).

If $f \in \mathcal{H}_\theta(\mathbb{T})$, then $\sum_{n\in\mathbb{Z}} |\hat{f}(n)| \leq ||f||_\theta \sum_{n\in\mathbb{Z}} \hat{\theta}(n) < \infty$. Hence f equals almost everywhere a continuous function (namely its uniformly convergent Fourier series). Thus, we shall assume that elements of $\mathcal{H}_\theta(\mathbb{T})$ are continuous functions.

We summarize some basic properties of $\mathcal{H}_\theta(\mathbb{T})$ in the following lemma.

Lemma 4.1 *Let $f \in \mathcal{H}_\theta(\mathbb{T})$ and $x \in \mathbb{R}$. Then*

(*a*) $\theta_x \in \mathcal{H}_\phi(\mathbb{T})$ *and* $||\theta_x||_\theta^2 = \theta(0)$. (*Here*, $\theta_x(t) = \theta(t-x)$.)
(*b*) $f(x) = \langle f, \theta_x\rangle_\theta$.

4.2 Bernstein-Szegö inequalities for Fourier multiplier operators on Sobolev spaces in $L^2(\mathbb{T})$

In this section, we shall fix a function θ in $L^2(\mathbb{T})$ satisfying (4.1): $\hat{\theta}(n) \geq 0$ for each $n \in \mathbb{Z}$ and $\sum_{n\in\mathbb{Z}} \hat{\theta}(n) < \infty$. For simplicity of notation, we shall assume that $\theta(0) = 1$ and

$$\hat{\theta}(n) = \hat{\theta}(-n) \text{ for each } n \in \mathbb{Z}. \tag{14}$$

Theorem 4.2 *Let $\{\sigma_n : n \in \mathbb{Z}\}$ be a sequence of complex numbers such that*

$$\{\hat{\theta}(n)\sigma_n^2 : n \in \mathbb{Z}\} \in l^1(\mathbb{Z}) \cap l^2(\mathbb{Z}),$$

and with

$$\sigma_n = -\sigma_{-n} \textit{ for each } n \in \mathbb{Z}. \tag{15}$$

Suppose either

$$(a)\ \inf\{|\sigma_n| : n \in \Sigma(\theta)\} > 0 \quad or \quad (b)\ \sup\{|\sigma_n| : n \in \Sigma(\theta)\} < \infty.$$

Let $f \in \mathcal{H}_\theta(\mathbb{T})$, $x \in \mathbb{R}$ and define

$$Sf(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\sigma_n e^{inx}. \tag{16}$$

Then

$$|f(x)|^2 + \left(\sum_{n\in\mathbb{Z}} \hat{\theta}(n)|\sigma_n|^2\right)^{-1} |Sf(x)|^2 \ \le \ ||f||_\theta^2. \qquad (17)$$

Proof. We shall only give a sketch of the proof as it follows the same lines as that of Theorem 3.2. Define $\psi \in L^2(\mathbb{T})$ such that $\hat{\psi}(n) = \hat{\theta}(n)|\sigma_n|^2$ for each $n \in \mathbb{Z}$.
Suppose (*a*) holds. Then $\mathcal{H}_\theta(\mathbb{T}) \subset \mathcal{H}_\psi(\mathbb{T})$ and that the inclusion is bounded. Moreover, $S: \mathcal{H}_\theta(\mathbb{T}) \to \mathcal{H}_\psi(\mathbb{T})$ is bounded as well. The desired estimate is obtained by applying Lemma 2.1 with $\mathcal{H} = \mathcal{H}_\theta(\mathbb{T})$, $\mathcal{K} = \mathcal{H}_\psi(\mathbb{T})$, S as defined in (16), and $T: \mathcal{H}_\theta(\mathbb{T}) \to \mathcal{H}_\psi(\mathbb{T})$ as the inclusion mapping.

The adjoint $S^\star: \mathcal{H}_\psi(\mathbb{T}) \to \mathcal{H}_\theta(\mathbb{T})$ satisfies $\widehat{S^\star\psi_x}(n) = \hat{\theta}(n)\overline{\sigma_n}e^{-inx}$ for each $n \in \mathbb{Z}$. Thus,

$$||S^\star\psi_x||_\theta^2 = \sum_{n\in\mathbb{Z}} \hat{\theta}(n)|\sigma_n|^2.$$

Moreover, by (14) and (15), we have

$$(S^\star\psi_x)(x) = \sum_{n\in\mathbb{Z}} \hat{\theta}(n)\overline{\sigma_n} = 0. \qquad (18)$$

On the other hand, the adjoint $T^\star: \mathcal{H}_\psi(\mathbb{T}) \to \mathcal{H}_\theta(\mathbb{T})$ satisfies: $T^\star\psi_x = \theta_x$. Thus,

$$||T^\star\psi_x||_\theta^2 = ||\theta_x||_\theta^2 = \theta(0) = 1. \qquad (19)$$

In view of (6), $G(x)$ as given in (4) is identically zero while $\Delta(x)$ as given in (2) becomes

$$\Delta(x) = ||S^\star\psi_x||_\theta^2 = \sum_{n\in\mathbb{Z}} \hat{\theta}(n)|\sigma_n|^2.$$

In view of (19), (3) now implies (17).

Now, suppose (*b*) holds. Then S maps $\mathcal{H}_\theta(\mathbb{T})$ into itself. The desired estimate (17) is obtained similarly as above by applying Lemma 2.1 with $\mathcal{H} = \mathcal{K} = \mathcal{H}_\theta(\mathbb{T})$, $S: \mathcal{H}_\theta(\mathbb{T}) \to \mathcal{H}_\theta(\mathbb{T})$ as given in (16), and T as the identity map on $\mathcal{H}_\theta(\mathbb{T})$. □

A special case of Theorem 4.2 yields an L^2- version of the classical Bernstein-Szegö inequality.

Corollary 4.3 *Let θ and T be trigonometric polynomials of degree at most d. Suppose $\hat{\theta}(n) > 0$ and $\hat{\theta}(n) = \hat{\theta}(-n)$ for each $|n| \leq d$ and $\theta(0) = 1$. Then for any positive odd integer k and $x \in \mathbb{R}$,*

$$|T(x)|^2 + d^{-2k}|T^{(k)}(x)|^2 \leq \sum_{n=-d}^{d} \frac{|\hat{T}(n)|^2}{\hat{\theta}(n)}.$$

Proof. We apply Theorem 4.2 with $\sigma_n = (in)^k$ and θ as given in the statement of the corollary. Note that $\mathcal{H}_\theta(\mathbb{T})$ is precisely the vector space of polynomials of degree at most d. □

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