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Relation between Square and Centered Pentagonal Numbers

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ABSTRACT

Let $s_k(n)$ denote the number of representations of integer n as a sum of k squares and $c_k(n)$ denote the number of representations of integer n as a sum of k centered pentagonal numbers. We derive the relation $s_k\left(\frac{8n-3k}{5}\right) = \alpha_k c_k(n)$ where $\alpha_k = 2^k + 2^{k-1}\binom{k}{4}$ for $1 \le k \le 7$. We give a conjecture on the relation between $s_\lambda(n)$ and $c_\lambda(n)$ is given by $\beta_\lambda c_\lambda(n) = s_\lambda\left(\frac{8n-3k}{5}\right)$ for all integers n and $\lambda = (\lambda_1, \dots, \lambda_m)$ where $\beta_\lambda = 2^m + 2^{m-1}\left(\binom{i_1}{4} + \binom{i_1}{2}\binom{i_2}{1} + \binom{i_1}{1}\binom{i_3}{1}\right)$ and $1 \le k \le 7$. A special case of this conjecture is proved in which $k = 7$ and $\lambda = (3,2,1,1)$.

Keywords: number of representations, sum of squares, centered pentagonal numbers.

1. INTRODUCTION

Let $s_k(n)$ and $t_k(n)$ denote the number of representations of an integer n as a sum of k squares and as a sum of k triangular numbers, respectively. That is, $s_k(n)$ is the number of solutions in integers of the equation

$$x_1^2 + x_2^2 + \dots + x_k^2 = n$$

and $t_k(n)$ is the number of solutions in non-negative integers of the equation

$$\frac{x_1(x_1+1)}{2} + \frac{x_2(x_2+1)}{2} + \dots + \frac{x_k(x_k+1)}{2} = n.$$

Barrucand *et al.* (2002) give a relation between $s_k(n)$ and $t_k(n)$ for any integer n as

$$s_k(8n+k) = \alpha_k t_k(n) \text{ where } \alpha_k = 2^k + 2^{k-1}\binom{k}{4}$$

for $1 \leq k \leq 7$. They have proved by using generating function. Later, a combinatorial proof is given by Cooper and Hirschhorn (2004). It has been proved by Bateman *et al.* (2001) that this result does not hold for any value $k \geq 8$.

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of k. That is, $\lambda_1, \ldots, \lambda_m$ are integers satisfying $\lambda_1 \geq \ldots \geq \lambda_m \geq 1$ and $\lambda_1 + \ldots + \lambda_m = k$. For any non-negative integer n, define $s_\lambda(n)$ is the number of solutions in non-negative integers of the equation

$$\lambda_1 x_1^2 + \ldots + \lambda_m x_m^2 = n \tag{1}$$

and $t_\lambda(n)$ is the number of solutions in non-negative integers of the equation

$$\lambda_1 \frac{x_1(x_1+1)}{2} + \ldots + \lambda_m \frac{x_m(x_m+1)}{2} = n. \tag{2}$$

A solution of Equation (1) and Equation (2) is called a representation of n as a sum of squares induced by λ and representation of n as a sum of triangular numbers induced by λ. For partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $1 \leq k \leq 7$, Adiga *et al.* (2005) give a relation

$$s_\lambda(8n+k) = C_\lambda t_\lambda(n)$$

where $C_\lambda = 2^m + 2^{m-1}\left(\binom{i_1}{4} + \binom{i_1}{2}\binom{i_2}{1} + \binom{i_1}{1}\binom{i_3}{1}\right)$ and i_j denote the number of parts in λ which are equal to j.

Both of the relations have been proved using generating functions and combinatoric method. The above results prompt us to extend this idea to other polygonal numbers consisting of centered pentagonal numbers. This paper is devoted to this objective.

2. RELATION BETWEEN SQUARE AND CENTERED PENTAGONAL NUMBERS

A centered pentagonal number is defined as a centered figurate number that represents a pentagon with a dot in the center and all other dots surrounding the center in successive pentagonal layers. The centered pentagonal number for n is given by the formula

$$\frac{5n^2+5n+2}{2}$$

The first few centered pentagonal numbers are 1, 6, 16, 31, 51, 76, 106, 141,... .

Let $c_k(n)$ denote the number of representations of a positive integer n as a sum of k centered pentagonal numbers. In other words, $c_k(n)$ is the number of solutions in non-negative integers of the equation

$$\sum_{i=1}^{k} \frac{5x_i^2+5x_i+2}{2} = n$$

For example, for $n = 18$ and $k = 3$ we have $18 = 6+6+6 = 16+1+1 = 1+16+1 = 1+1+16$. Thus $(x_1, x_2, x_3) = (1,1,1) = (2,0,0) = (0,2,0) = (0,0,2)$. Then $c_3(18) = 4$.

Let q be an integer and $\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \psi(q) = \sum_{n\geq 0} q^{(n^2+n)/2}$ and $\omega(q) = \sum_{n\geq 0} q^{(5n^2+5n+2)/2}$. As in Barrucand (2002), $s_k(n)$ is defined by the coefficients in the expansion of $\phi^k(q) = \sum_{n\geq 0} s_k(n)q^n$. Also $c_k(n)$ is defined by those in the expansion of $\omega^k(q) = \sum_{n\geq 0} c_k(n)q^n$. In the following theorem, we give the main result of this paper, which gives the relation between $s_k\left(\frac{8n-3k}{5}\right)$ and $c_k(n)$.

Theorem 1. *Let n be a positive integer. Then, for* $1 \leq k \leq 7$,

$$\alpha_k c_k(n) = s_k\left(\frac{8n-3k}{5}\right)$$

where $\alpha_k = 2^k + 2^{k-1}\binom{k}{4}$.

We will prove the following lemma first, which is necessary in the proof of Theorem 1.

Lemma 1. *Let*

$\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$, $\psi(q) = \sum_{n\geq 0} q^{(n^2+n)/2}$, $\omega(q) = \sum_{n\geq 0} q^{(5n^2+5n+2)/2}$

where q is any number. Then, we have the following relations.

(i) $\phi(q) = 2q^{-3/5}\omega(q^{8/5}) + \phi(q^4)$

(ii) $\omega(q) = q\psi(q^5)$.

Proof. The proof will be given according to the sequence above.

(i)
$$\begin{aligned}\phi(q) - \phi(-q) &= 4\sum_{n=0}^{\infty} q^{(2n+1)^2}\\ &= 4\sum_{n=0}^{\infty} q^{4n^2+4n+1}\\ &= 4\sum_{n=0}^{\infty} q^{\frac{8}{5}\left(\frac{5n^2+5n+2}{2}\right)-\frac{3}{5}}\\ &= 4q^{-\frac{3}{5}}\sum_{n=0}^{\infty} q^{\frac{8}{5}\left(\frac{5n^2+5n+2}{2}\right)}\\ &= 4q^{-\frac{3}{5}}\omega(q^{8/5})\end{aligned}$$

By adding this equation with $\phi(q) + \phi(-q) = 2\phi(q^4)$ from Berndt (1991), we have

$$\phi(q) = 2q^{-3/5}\omega(q^{8/5}) + \phi(q^4)$$

(ii) The proof for this part is almost immediate and straight forward.

$$\begin{aligned}\omega(q) &= \sum_{n\geq 0} q^{\frac{5n^2+5n+2}{2}}\\ &= \sum_{n\geq 0} q^{\frac{5(n^2+n)}{2}+1}\\ &= q\psi(q^5)\end{aligned}$$ □

By using Lemma 1, we give the proof of Theorem 1 as follows.

Proof of Theorem 1. Let $s_k(n)$ and $c_k(n)$ be defined by

$$\sum_{n\geq 0} s_k(n)q^n = \phi^k(q), \qquad \sum_{n\geq 0} c_k(n)q^n = \omega^k(q)$$

Then, for $1 \leq k \leq 7$, we have $\phi(q) = 2q^{-3/5}\omega\left(q^{8/5}\right) + \phi(q^4)$ from part (i) of Lemma 1. It follows that

$$\begin{aligned}\phi^k(q) &= \left(2q^{-3/5}\omega(q^{8/5}) + \phi(q^4)\right)^k \\ &= \sum_{r=0}^{k} \binom{k}{r} 2^{k-r}\left(q^{-3/5}\right)^{k-r} \omega^{k-r}(q^{8/5})\phi^r(q^4) \\ &= 2^k\left(q^{-3/5}\right)^k \omega^k\left(q^{8/5}\right) + \ldots + \binom{k}{4} 2^{k-4}\left(q^{-3/5}\right)^{k-4} \omega^{k-4}\left(q^{8/5}\right)\phi^4(q^4) + \ldots\end{aligned}$$

By applying $\phi^2(q) = \phi^2(q^2) + 4q\psi^2(q^4)$ from Barrucand (2002) to $\phi^4(q^4)$, we have

$$\begin{aligned}\phi^k(q) &= 2^k\left(q^{-3/5}\right)^k \omega^k\left(q^{8/5}\right) + \ldots + \binom{k}{4} 2^{k-4}\left(q^{-3/5}\right)^{k-4} \omega^{k-4}\left(q^{8/5}\right) \\ &\quad (\phi^2(q^8) + 4q^4\psi^2(q^{16}))^2 + \ldots \\ &= 2^k\left(q^{-3/5}\right)^k \omega^k\left(q^{8/5}\right) + \ldots + \binom{k}{4} 2^{k-4}\left(q^{-3/5}\right)^{k-4} \omega^{k-4}\left(q^{8/5}\right) \\ &\quad (\phi^4(q^8) + 8q^4\phi^2(q^8)\psi^2(q^{16}) + 16q^8\psi^4(q^{16})) + \ldots\end{aligned}$$

Applying $\phi(q)\psi(q^2) = \psi^2(q)$ from Barrucand (2002) to $\phi^2(q^8)\psi^2(q^{16})$, we obtain

$$\begin{aligned}\phi^k(q) &= 2^k\left(q^{-3/5}\right)^k \omega^k\left(q^{8/5}\right) + \ldots + \binom{k}{4} 2^{k-4}\left(q^{-3/5}\right)^{k-4} \omega^{k-4}\left(q^{8/5}\right) \\ &\quad (\phi^4(q^8) + 8q^4\psi^4(q^8) + 16q^8\psi^4(q^{16})) + \ldots\end{aligned}$$

Using part (ii) of Lemma 1, we have $\psi^4(q^8) = q^{-32/5}\omega^4\left(q^{8/5}\right)$. By applying this equation,

$$\begin{aligned}\phi^k(q) &= 2^k\left(q^{-3/5}\right)^k \omega^k\left(q^{8/5}\right) + \ldots + \binom{k}{4} 2^{k-4}\left(q^{-3/5}\right)^{k-4} \omega^{k-4}\left(q^{8/5}\right) \\ &\quad \left(\phi^4(q^8) + 8\left(q^{-3/5}\right)^4 \omega^4\left(q^{8/5}\right) + 16q^8\psi^4(q^{16})\right) + \ldots\end{aligned}$$

Now, by extracting those terms in which the degrees of q are $\frac{8n-3k}{5}$, we have

$$\begin{aligned}\sum_{n\geq 0} s_k\left(\frac{8n-3k}{5}\right) q^{\frac{8n-3k}{5}} &= 2^k\left(q^{-3/5}\right)^k \omega^k\left(q^{8/5}\right) \\ &\quad + \binom{k}{4} 2^{k-4}\left(q^{-3/5}\right)^{k-4} \omega^{k-4}\left(q^{8/5}\right)\left(8\left(q^{-3/5}\right)^4 \omega^4\left(q^{8/5}\right)\right).\end{aligned}$$

Multiplying both sides of the equation by $\left(q^{3/5}\right)^k$ and replacing $q^{8/5}$ by q, it follows that

$$\sum_{n\geq 0} s_k\left(\frac{8n-3k}{5}\right)q^n = 2^k\omega^k(q) + \binom{k}{4}2^{k-1}\omega^k(q)$$
$$= \left[2^k + \binom{k}{4}2^{k-1}\right]\omega^k(q)$$

By comparing coefficients of q^n, for every n from both sides of the equation, we will obtain

$$s_k\left(\frac{8n-3k}{5}\right) = \alpha_k c_k(n)$$

where $\alpha_k = 2^k + 2^{k-1}\binom{k}{4}$. □

As an illustration, let $n = 19$ and $k = 4$. We have

$$s_4\left(\frac{8(19)-3(4)}{5}\right) = \left(2^4 + 2^{4-1}\binom{4}{4}\right)c_4(19)$$
$$s_4(28) = 24c_4(19)$$
$$192 = 24(8).$$

3. RELATION BETWEEN SQUARE AND CENTERED PENTAGONAL NUMBERS INDUCED BY PARTITIONS

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of k and $c_\lambda(n)$ denote the number of representations of a positive integer n as a sum of centered pentagonal numbers induced by λ. In other words, $c_\lambda(n)$ is the number of solutions in non-negative integers of the equation

$$\lambda_1\frac{5x_1^2+5x_1+2}{2} + \ldots + \lambda_m\frac{5x_m^2+5x_m+2}{2} = n.$$

For example, for $n = 22$ and $\lambda = (4,2,1)$ we have $m = 3$, and $22 = 4(1) + 2(6) + 1(6) = 4(1) + 2(1) + 1(16)$.

Thus $(x_1, x_2, x_3) = (0,1,1) = (0,0,2)$. Then $c_{(4,2,1)}(22) = 2$.

Based on our observation on the relation between $s_\lambda\left(\frac{8n-3k}{5}\right)$ and $c_\lambda(n)$ for integer n and $1 \leq k \leq 7$, we have the following conjecture.

Conjecture . *If* $1 \leq k \leq 7$ *and* $\lambda = (\lambda_1, \ldots, \lambda_m)$ *is a partition of k, then*

$$\beta_\lambda c_\lambda(n) = s_\lambda\left(\frac{8n-3k}{5}\right)$$

for all integers n where $\beta_\lambda = 2^m + 2^{m-1}\left(\binom{i_1}{4} + \binom{i_1}{2}\binom{i_2}{1} + \binom{i_1}{1}\binom{i_3}{1}\right)$
and i_j *denotes the number of parts in* λ *which are equal to j.*

Let $p(n)$ represents the partition function of n. That is $p(n)$ is the number of possible partitions of a natural number n. For $1 \leq k \leq 7$, $\sum_{k=1}^{7} p(k) = 1 + 2 + 3 + 5 + 7 + 11 + 15 = 44$. Hence, in the conjecture, there are 44 cases to be considered. In the following theorem, we prove a special case of the conjecture in which $k = 7$ and $\lambda = (3,2,1,1)$.

Theorem 2. *If* $k = 7$ *and* $\lambda = (3,2,1,1)$ *is a partition of k then*

$$\beta_\lambda c_\lambda(n) = s_\lambda\left(\frac{8n-21}{5}\right)$$

for all integers n where $\beta_\lambda = 40$.

We prove Theorem 2 using generating function method. The generating functions for $s_\lambda(n)$ and $c_\lambda(n)$ are

$$\sum_{n=0}^{\infty} s_\lambda(n) q^n = \phi(q^{\lambda_1})\phi(q^{\lambda_2}) \ldots \phi(q^{\lambda_m})$$

$$\sum_{n=0}^{\infty} c_\lambda(n) q^n = \omega(q^{\lambda_1})\omega(q^{\lambda_2}) \ldots \omega(q^{\lambda_m})$$

To prove Theorem 2 using generating function, we need some properties in Lemma 1 and additional properties in the following lemma.

Lemma 2.

(i) $\phi^2(q) = \phi^2(q^2) + 4q^{-3/5}\omega^2(q^{4/5})$

(ii) $\phi(q^8)\omega(q^{16/5}) = \omega^2(q^{8/5})$

(iii) $\phi(q^{48})\omega(q^{32/5}) + q^{-24/5}\phi(q^{16})\omega(q^{96/5}) = \omega(q^{8/5})\omega(q^{24/5})$

Proof.

(i) From Adiga et al. (2005), we have

$$\begin{aligned}\phi^2(q) &= \phi^2(q^2) + 4q\psi^2(q^4)\\ &= \phi^2(q^2) + 4q\left(q^{-4/5}\omega\left(q^{4/5}\right)\right)^2\\ &= \phi^2(q^2) + 4q^{-3/5}\omega^2\left(q^{4/5}\right)\end{aligned}$$

(ii)

$$\begin{aligned}\phi(q^8)\omega\left(q^{16/5}\right) &= q^{16/5}\phi(q^8)\psi(q^{16})\\ &= q^{16/5}\psi^2(q^8)\\ &= \omega^2\left(q^{8/5}\right)\end{aligned}$$

(iii)

$$\begin{aligned}&\phi(q^{48})\omega\left(q^{32/5}\right) + q^{-24/5}\phi(q^{16})\omega\left(q^{96/5}\right)\\ &= q^{32/5}\phi(q^{48})\psi(q^{32}) + q^{72/5}\phi(q^{16})\psi(q^{96})\\ &= q^{32/5}[\phi(q^{48})\psi(q^{32}) + q^8\phi(q^{16})\psi(q^{96})]\\ &= q^{32/5}\psi(q^8)\psi(q^{24})\\ &= \omega\left(q^{8/5}\right)\omega\left(q^{24/5}\right)\end{aligned}$$

□

Proof of Theorem 2.

$$\begin{aligned}\sum_{n=0}^{\infty} & s_{(3,2,1,1)}(n)q^n\\ &= \phi(q^3)\phi(q^2)\phi(q)^2\\ &= \left[2q^{-9/5}\omega(q^{24/5}) + \phi(q^{12})\right]\left[2q^{-6/5}\omega(q^{16/5}) + \phi(q^8)\right]\\ &\quad \left[2q^{-3/5}\omega(q^{8/5}) + \phi(q^4)\right]^2\\ &= \left[2q^{-9/5}\omega\left(q^{24/5}\right) + \phi(q^{48}) + 2q^{-36/5}\omega(q^{96/5})\right]\left[2q^{-6/5}\omega(q^{16/5})\right.\\ &\quad \left.+\phi(q^8)\right]\left[\phi^2(q^4) + 4q^{-3/5}\phi(q^4)\omega(q^{8/5}) + 4q^{-6/5}\omega^2(q^{8/5})\right]\end{aligned}$$

By applying part (i) of Lemma 2 to $\phi^2(q^4)$, and part (i) of Lemma 1 to $\phi(q^4)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} & s_{(3,2,1,1)}(n)q^n\\ &= \left[2q^{-9/5}\omega\left(q^{24/5}\right) + \phi(q^{48}) + 2q^{-36/5}\omega(q^{96/5})\right]\left[2q^{-6/5}\omega(q^{16/5})\right.\\ &\quad \left.+\phi(q^8)\right][\phi^2(q^8) + 4q^{-12/5}\omega^2(q^{16/5}) + 4q^{-3/5}\\ &\quad \left(\phi(q^{16}) + 2q^{-12/5}\omega(q^{32/5})\right)\omega(q^{8/5}) + 4q^{-6/5}\omega^2(q^{8/5})] .\end{aligned}$$

Now, by extracting those terms in which the degrees of q are $\frac{8n-21}{5}$, we have

$$\sum_{n=0}^{\infty} s_{(3,2,1,1)}\left(\frac{8n-21}{5}\right) q^{\frac{8n-21}{5}}$$
$$= 16q^{-21/5}\phi(q^{48})\omega(q^{32/5})\omega(q^{16/5})\omega(q^{8/5})$$
$$+16q^{-45/5}\phi(q^{16})\omega(q^{96/5})\omega(q^{16/5})\omega(q^{8/5})$$
$$+8q^{-21/5}\phi(q^{8})\omega(q^{24/5})\omega^2(q^{16/5})$$
$$+16q^{-21/5}\omega(q^{24/5})\omega(q^{16/5})\omega^2(q^{8/5})$$
$$= 16q^{-21/5}\left[\phi(q^{48})\omega(q^{32/5}) + q^{-24/5}\phi(q^{16})\omega(q^{96/5})\right]\omega(q^{16/5})$$
$$\omega(q^{8/5}) + 8q^{-21/5}\omega(q^{24/5})\omega(q^{16/5})\left[\phi(q^{8})\omega(q^{16/5})\right]$$
$$+16q^{-21/5}\omega(q^{24/5})\omega(q^{16/5})\omega^2(q^{8/5}).$$

By using part (ii) and (iii) in Lemma 2, we have

$$\sum_{n=0}^{\infty} s_{(3,2,1,1)}\left(\frac{8n-21}{5}\right) q^{\frac{8n-21}{5}}$$
$$= 16q^{-21/5}\omega(q^{24/5})\omega(q^{16/5})\omega^2(q^{8/5})$$
$$+8q^{-21/5}\omega(q^{24/5})\omega(q^{16/5})\omega^2(q^{8/5})$$
$$+16q^{-21/5}\omega(q^{24/5})\omega(q^{16/5})\omega^2(q^{8/5}).$$

Multiplying both sides of the equation by $q^{21/5}$ and replacing $q^{8/5}$ by q, we have

$$\sum_{n=0}^{\infty} s_{(3,2,1,1)}\left(\frac{8n-21}{5}\right) q^{n}$$
$$= 16\omega(q^3)\omega(q^2)\omega^2(q) + 8\omega(q^3)\omega(q^2)\omega^2(q)$$
$$+16\omega(q^3)\omega(q^2)\omega^2(q)$$
$$= 40\omega(q^3)\omega(q^2)\omega^2(q)$$
$$= 40\sum_{n=0}^{\infty} c_{(3,2,1,1)}(n) q^n$$

For $\lambda = (3,2,1,1)$ we have $\mathrm{m} = 4, \mathrm{i}_1 = 2, \mathrm{i}_2 = 1, \mathrm{i}_3 = 1$, so

$$\begin{aligned}\beta_{(3,2,1,1)} &= 2^4 + 2^3\big(0 + (1)(1) + (2)(1)\big)\\ &= 16 + 24\\ &= 40\end{aligned}$$

This proves Theorem 2 for the partition $\lambda = (3,2,1,1)$. □

Remark. For any value of $k \geq 8$, the value of $s_k\left(\frac{8n-3k}{5}\right)/c_k(n)$ is not consistent. For example, if $k = 8$ and $n = 8$, then $s_8(8)/c_8(8) = 9328$. However, if $k = 8$ and $n = 13$, then $s_8(16)/c_8(13) = 9358$. So α_8 is not constant at the value of $k = 8$. For $k = 9$ and $n = 9$ then $s_9(9)/c_9(9) = 34802$. However, if $k = 9$ and $n = 19$ then $s_9(25)/c_9(19) = 34904.5$. So α_9 is not constant at the value of $k = 9$. This pattern of inconsistency is repeated when $k \geq 8$ for different value of *n*. Similar pattern also exists for $s_\lambda\left(\frac{8n-3k}{5}\right)/c_\lambda(n)$ Hence, the assertions of both of the relations do not hold for $k \geq 8$.

4. CONCLUSIONS

In this paper, a general relation between the number of representations of non-negative integer *n* as a sum of *k* squares and as a sum of *k* centered pentagonal numbers is derived. It is given by $s_k\left(\frac{8n-3k}{5}\right) = \alpha_k c_k(n)$ where $\alpha_k = 2^k + 2^{k-1}\binom{k}{4}$ for $1 \leq k \leq 7$. Conjecture on a relation between $s_\lambda(n)$ and $c_\lambda(n)$ is also given as $\beta_\lambda c_\lambda(n) = s_\lambda\left(\frac{8n-3k}{5}\right)$ for all integers *n* and $\lambda = (\lambda_1, \dots, \lambda_m)$ where

$$\beta_\lambda = 2^m + 2^{m-1}\left(\binom{i_1}{4} + \binom{i_1}{2}\binom{i_2}{1} + \binom{i_1}{1}\binom{i_3}{1}\right)$$

and $1 \leq k \leq 7$. There are 44 cases of partition to be considered. We give a proof for a special case of a partition in which $k = 7$ and $\lambda = (3,2,1,1)$. This is an extension of work by earlier authors on finding relationships between squares and triangles as shown by Barrucand *et al.*(2002) and Adiga *et al.* (2005) for example. In our future work, we will examine the relationship between these representations for $k \geq 8$.

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