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Boundary Value Problem for the Higher Order Equation with Fractional Derivative

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ABSTRACT

In this paper we study the boundary value problem for higher order equation with fractional derivative in the sense of Caputo. Existence of the unique solution of this problem and its continuous dependence on the initial data and on the right part of the equation are proved.

Keywords: Boundary value problems, fractional derivative in the sense of Caputo, Volterra integral equation of second kind, Mittag-Leffler type function.

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1. INTRODUCTION

The Cauchy problem for partial pseudo-differential equations of fractional order in the sense of Caputo in semi-space $\{(x,y,t):(x,y)\in R^2, t>0\}$ investigated by Gorenflo, Luchko and Umarov (2000). The boundary value problems for partial differential equations with operators of fractional differentiation or integration in the sense of Riemann-Liouville were studied by Djrbashyan and Nersesyan (1968), Samko, Kilbas and Marichev (1987), Nahushev (2000), Pshu (2005), Virchenko and Ribak (2007), Kilbas and Repin (2010).

The boundary value problem for heat conduction equation with fractional derivative in the sense of Caputo investigated by Kadirkulov and Turmetov(2006). The boundary value problem for higher order partial differential equation with fractional derivative in the sense of Caputo in the case when order of fractional derivative belongs to the interval $(0,1)$ was studied by Amanov (2008). The boundary value problem for the fourth order equation with fractional derivative in the sense of Caputo in the case when order of fractional derivative belongs to the interval $(1,2)$ by Amanov (2009).

In the present paper we study the boundary value problem for the higher order partial differential equation with fractional derivative in the sense of Caputo in the case when the order of fractional derivative belongs to the interval $(1,2)$ in a spatial domain.

2. STATEMENT OF THE PROBLEM

In the domain $\Omega=\{(x,y,t):0<x<p,\ 0<y<q,\ -T<t<0\}$ we consider the following equation

$$\frac{\partial^{2k}u}{\partial x^{2k}}+\frac{\partial^{2k}u}{\partial y^{2k}}-(-1)^k{}_cD_{t0}^{\beta}u=f(x,y,t) \tag{1}$$

where $k(k\geq 1)$ is a fixed integer, $1<\beta<2$, ${}_cD_{t0}^{\beta}$ is the operator of fractional differentiation with respect to t in the sense of Caputo. The boundary value problem for higher order partial differential equation with operator ${}_cD_{0t}^{\alpha}$, $1<\alpha<2$ was studied by Amanov (2009).

Problem 1. Find the solution $u(x,y,t)$ of the equation (1) satisfying the following conditions

$$\frac{\partial^{2m}u(0,y,t)}{\partial x^{2m+1}}=\frac{\partial^{2m}u(p,y,t)}{\partial x^{2m}}=0,\ m=0,1,...,k-1,\ 0\leq y\leq q,\ -T\leq t\leq 0, \tag{2}$$

$$\frac{\partial^{2m}u(x,0,t)}{\partial y^{2m}}=\frac{\partial^{2m}u(x,q,t)}{\partial y^{2m}}=0,\ m=0,1,...,k-1,\ 0\leq x\leq p,\ -T\leq t\leq 0. \tag{3}$$

$$u(x,y,0)=\phi(x,y),\ u_t(x,y,0)=\psi(x,y),\ 0\leq x\leq p,\ 0\leq y\leq q \tag{4}$$

in the domain Ω.

3. CONSTRUCTION OF FORMAL SOLUTION OF PROBLEM 1

We search a solution of Problem 1 in the form of Fourier series

$$u(x,y,t)=\sum_{n,m=1}^{\infty}u_{nm}(t)\upsilon_{nm}(x,y) \tag{5}$$

expanded in complete orthonormal system

$$\upsilon_{nm}(x,y)=\frac{2}{\sqrt{pq}}\sin\frac{n\pi}{p}x\cdot\sin\frac{m\pi}{q}y.$$

Denote

$$\Omega_0=\Omega\cap(t=0),\ \frac{n\pi}{p}=\nu_n,\ \frac{m\pi}{q}=\mu_m,\ \nu_n^{2k}+\mu_m^{2k}=\lambda_{nm}^{2k}.$$

We expand the function $f(x,y,t)$ into the Fourier series by functions $\upsilon_{nm}(x,y)$

$$f(x,y,t)=\sum_{n,m=1}^{\infty}f_{nm}(t)\upsilon_{nm}(x,y) \tag{6}$$

where

$$f_{nm}(t)=\int_0^p\int_0^q f(x,y,t)\upsilon_{nm}(x,y)\,dydx. \tag{7}$$

Substituting (5) and (6) into equation (1) we obtain

$${}_cD_{t0}^{\beta}u_{nm}(t)+(-1)^k\lambda_{nm}^{2k}u_{nm}(t)=f_{nm}(t),\quad -T\le t\le 0. \tag{8}$$

It is known that (Djrbashyan (1966)),

$${}_cD_{t0}^{\beta}u_{nm}(t)=-I_{t0}^{2-\beta}u''_{nm}(t) \tag{9}$$

where

$$I_{t0}^{\beta}f(t)=\frac{1}{\Gamma(\beta)}\int_t^0(\tau-t)^{\beta-1}f(\tau)d\tau$$

is Riemann-Liouville integral of fractional order.

Using (9), equation (8) can be rewritten as

$$I_{t0}^{2-\beta}u_{nm}^{''}(t)+\lambda_{nm}^{2k}u_{nm}(t)=(-1)^k f_{nm}(t),\quad -T\le t\le 0.$$

Acting with operator I_{t0}^{β} to both parts of the last equation we get

$$\begin{aligned} u_{nm}(t) &= -\lambda_{nm}^{2k}\int_t^0 \frac{(\tau-t)^{\beta-1}}{\Gamma(\beta)}u_{nm}(\tau)d\tau+ \\ &+\frac{(-1)^k}{\Gamma(\beta)}\int_t^0 (\tau-t)^{\beta-1} f_{nm}(\tau)d\tau+u_{nm}(0)+tu_{nm}^{'}(0). \end{aligned} \tag{10}$$

This is Volterra integral equation of second kind. We solve it by successive approximations. Denote

$$K_1(\tau,t)=\begin{cases}\dfrac{(\tau-t)^{\beta-1}}{\Gamma(\beta)}, & -T\le t<\tau<0\\ 0, & \tau\le t\le 0\end{cases}$$

and defining further sequence of kernels $\{K_s(\tau,t)\}_1^{\infty}$ by recurrent relations

$$K_{s+1}(\tau,t)=\int_t^{\tau} K_s(\tau,\eta)K_1(\eta,t)d\eta,\qquad s=1,2,\dots\ .$$

By induction with respect to s we find

$$K_{s+1}(\tau,t)=\begin{cases}\dfrac{(\tau-t)^{(s+1)\beta-1}}{\Gamma(s\beta+\beta)}, & -T\le t<\tau<0\\ 0, & \tau\le t\le 0.\end{cases}$$

Hence for the resolvent of equation (10) we have the formula

$$\begin{aligned} R\left(\tau,t,(-1)^k\lambda_{nm}^{2k}\right) &= \sum_{s=0}^{\infty}\left[(-1)^k\lambda_{nm}^{2k}\right]^s K_{s+1}(\tau,t)= \\ &=\begin{cases}(\tau-t)^{\beta-1}E_{\beta,\beta}\left((-1)^k\lambda_{nm}^{2k}(\tau-t)^{\beta}\right), & -T\le t<\tau<0\\ 0, & \tau\le t\le 0,\end{cases} \end{aligned}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler type function (Djrbashyan (1966), Kadirkulov and Turmetov (2006)).

The solution of equation (10) is expressed through resolvent so

$$\begin{aligned} u_{nm}(t) &= u_{nm}(0)\left[1+(-\lambda_{nm}^{2k})\int_t^0 (\tau-t)^{\beta-1} E_{\beta,\beta}\left(-\lambda_{nm}^{2k}(t-\tau)^{\beta}\right)d\tau\right]+ \\ &+u'_{nm}(0)\left[t+(-\lambda_{nm}^{2k})\int_t^0 (\tau-t)^{\beta-1} E_{\beta,\beta}\left(-\lambda_{nm}^{2k}(\tau-t)^{\beta}\right)\tau d\tau\right]+ \\ &+(-\lambda_{nm}^{2k})\frac{1}{\Gamma(\beta)}\int_t^0 (\eta-t)^{\beta-1} E_{\beta,\beta}\left(-\lambda_{nm}^{2k}(\eta-\tau)^{\beta}\right)d\eta\int_\eta^0 (\tau-\eta)^{\beta-1} f_{nm}(\tau)d\tau+ \\ &+\frac{(-1)^k}{\Gamma(\beta)}\int_t^0 (\tau-t)^{\beta-1} f_{nm}(\tau)d\tau. \end{aligned} \tag{11}$$

Applying the following Dirichlet formula

$$\int_a^b dy\int_y^b f(x,y)dx=\int_a^b dx\int_a^x f(x,y)dy$$

to iterated integral we get

$$\begin{aligned} &\frac{1}{\Gamma(\beta)}\int_t^0 (\eta-t)^{\beta-1} E_{\beta,\beta}\left(-\lambda_{nm}^{2k}(\eta-t)^{\beta}\right)d\eta\int_\eta^0 (\tau-\eta)^{\beta-1} f_{nm}(\tau)d\tau= \\ &=\int_t^0 f_{nm}(\tau)d\tau\frac{1}{\Gamma(\beta)}\int_t^\tau (\tau-\eta)^{\beta-1}(\eta-t)^{\beta-1} E_{\beta,\beta}\left(-\lambda_{nm}^{2k}(\eta-t)^{\beta}\right)d\eta. \end{aligned} \tag{12}$$

Denote

$$I_{1nm}(t)=1+(-\lambda_{nm}^{2k})\int_t^0 (\tau-t)^{\beta-1} E_{\beta,\beta}\left(-\lambda_{nm}^{2k}(\tau-t)^{\beta}\right)d\tau,$$

$$I_{2nm}(t)=t+(-\lambda_{nm}^{2k})\int_t^0 (\tau-t)^{\beta-1} E_{\beta,\beta}\left(-\lambda_{nm}^{2k}(\tau-t)^{\beta}\right)\tau d\tau,$$

$$I_{3nm}(t)=\frac{1}{\Gamma(\beta)}\int_t^\tau (\tau-t)^{\beta-1}(\eta-t)^{\beta-1} E_{\beta,\beta}\left(-\lambda_{nm}^{2k}(\eta-t)^{\beta}\right)d\eta.$$

We calculate $I_{1nm}(t)$. We change of variable $\tau - t = s,\ d\tau = ds$ then

$$I_{1nm}(t) = 1 + (-\lambda_{nm}^{2k})\int_0^{-t} s^{\beta-1} E_{\beta,\beta}\left(-\lambda_{nm}^{2k} s^{\beta}\right) ds.$$

Using the formula

$$\frac{1}{\Gamma(\delta)}\int_0^z t^{\mu-1} E_{\alpha,\mu}\left(\lambda t^{\alpha}\right)\left(z-t\right)^{\delta-1} dt = z^{\mu+\delta-1} E_{\alpha,\mu+\delta}\left(\lambda z^{\alpha}\right) \qquad (13)$$

we have

$$I_{1nm}(t) = 1 + (-\lambda_{nm}^{2k})\left(-t\right)^{\beta} E_{\beta,\beta+1}\left(-\lambda_{nm}^{2k}\left(-t\right)^{\beta}\right).$$

Now using the formula

$$\frac{1}{\Gamma(\mu)} + zE_{\alpha,\alpha+\mu}\left(z\right) = E_{\alpha,\mu}\left(z\right) \qquad (14)$$

we obtain

$$I_{1nm}(t) == E_{\beta,1}\left(-\lambda_{nm}^{2k}\left(-t\right)^{\beta}\right). \qquad (15)$$

Analogously we have

$$I_{2nm}(t) = tE_{\beta,1}\left(-\lambda_{nm}^{2k}\left(-t\right)^{\beta}\right) \qquad (16)$$

and

$$I_{3nm}(t) = \left(\tau - t\right)^{2\beta-1} E_{\beta,2\beta}\left(-\lambda_{nm}^{2k}\left(\tau - t\right)^{\beta}\right). \qquad (17)$$

Owing to (12) and substituting (15), (16) and (17) into (11) we get

$$\begin{aligned}u_{nm}(t) &= u_{nm}(0)E_{\beta,1}\left[-\lambda_{nm}^{2k}\left(-t\right)^{\beta}\right] + tu_{nm}^{'}(0)E_{\beta,2}\left[-\lambda_{nm}^{2k}(-t)^{\beta}\right] + \\ &+(-\lambda_{nm}^{2k})\int_t^0 \left(\tau - t\right)^{2\beta-1} E_{\beta,2\beta}\left[-\lambda_{nm}^{2k}(\tau-t)^{\beta}\right] f_{nm}(\tau)d\tau + \\ &+\frac{(-1)^k}{\Gamma(\beta)}\int_t^0 \left(\tau - t\right)^{\beta-1} f_{nm}(\tau)d\tau.\end{aligned}$$

Uniting the last two integrals and using (14) it is received

$$u_{nm}(t)=u_{nm}(0)E_{\beta,1}\left[-\lambda_{nm}^{2k}(-t)^{\beta}\right]+tu_{nm}^{'}(0)E_{\beta,2}\left[-\lambda_{nm}^{2k}(-t)^{\beta}\right]+ \\ +(-1)^{k}\int_{t}^{0}(\tau-t)^{\beta-1}E_{\beta,\beta}\left[-\lambda_{nm}^{2k}(\tau-t)^{\beta}\right]f_{nm}(\tau)d\tau. \tag{18}$$

We expand the functions $\varphi(x,y)$ and $\psi(x,y)$ into Fourier series by functions $\upsilon_{nm}(x,y)$

$$\phi(x,y)=\sum_{n,m=1}^{\infty}\phi_{nm}\upsilon_{nm}(x,y)dydx \quad \psi(x,y)=\sum_{n,m=1}^{\infty}\psi_{nm}\upsilon_{nm}(x,y)dydx$$

where

$$\phi_{nm}=\int_0^p\int_0^q\phi(x,y)\,\upsilon_{nm}(x,y)dydx \tag{19}$$

$$\psi_{nm}=\int_0^p\int_0^q\psi(x,y)\,\upsilon_{nm}(x,y)dydx \tag{20}$$

Owing (19) and (20) from conditions (4) we find $u_{nm}(0)=\varphi_{nm}$, $u_{nm}^{'}(0)=\psi_{nm}$.

Taking into account the last equality the solution (18) has the form

$$u_{nm}(t)=u_{nm}(0)E_{\beta,1}\left[-\lambda_{nm}^{2k}(-t)^{\beta}\right]+t\psi_{nm}E_{\beta,2}\left[-\lambda_{nm}^{2k}(-t)^{\beta}\right]+ \\ +(-1)^{k}\int_{t}^{0}(\tau-t)^{\beta-1}E_{\beta,\beta}\left[-\lambda_{nm}^{2k}(\tau-t)^{\beta}\right]f_{nm}(\tau)d\tau. \tag{21}$$

Substituting (21) into (5) we get formal solution of problem (1).

4. EXISTENCE OF THE UNIQUE SOLUTION OF PROBLEM 1 IN L_2 SPACE.

Lemma 1. Let $\varphi\in L_2(\Omega_0)$, $\psi\in L_2(\Omega_0)$ and $f\in L_2(\Omega)$, then

$$\|u\|_{L_2(\Omega)}\le c\left(\|\phi\|_{L_2(\Omega_0)}+\|\psi\|_{L_2(\Omega_0)}+\|f\|_{L_2(\Omega)}\right), \tag{22}$$

$c=const>0.$

Proof. Using the following estimate (see Kadirkulov *et al.* (2006), p. 136)

$$\left|E_{\alpha,\beta}(z)\right| \le \frac{M}{1+|z|}, \qquad M = const > 0, \quad \operatorname{Re} z < 0 \tag{23}$$

and the Cauchy-Schwarz inequality, (21) gives

$$\left|u_{nm}(t)\right| \le M\left(\left|\varphi_{nm}\right| + T\left|\psi_{nm}\right| + \int_t^0 (\tau - t)^{\beta-1}\left|f_{nm}(\tau)\right| d\tau\right) \le$$

$$\le c_0\left(\left|\varphi_{nm}\right| + T\left|\psi_{nm}\right| + \frac{T^{\beta-\frac{1}{2}}}{(2\beta-1)^{\frac{1}{2}}}\|f\|_{L_2(-T,0)}\right),$$

where $c_0 = \max\left\{M, MT, \dfrac{MT^{\beta-\frac{1}{2}}}{(2\beta-1)^{\frac{1}{2}}}\right\}$.

Further

$$\|u\|^2_{L_2(-T,0)} = \int_{-T}^{0}\left|u_{nm}(t)\right|^2 dt \le 3c_0^2 T\left(\left|\varphi_{nm}\right|^2 + \left|\psi_{nm}\right|^2 + \left\|f_{nm}\right\|^2_{L_2(-T,0)}\right).$$

Using Parseval equality we have

$$\|u\|^2_{L_2(\Omega)} = \left(\sum_{n,m=1}^{\infty} u_{nm}(t)\upsilon_{nm}(x,y), \sum_{i,j=1}^{\infty} u_{ij}(t)\upsilon_{ij}(x,y)\right)_{L_2(\Omega)} =$$

$$= \sum_{n,m=1}^{\infty}\left\|u_{nm}\right\|^2 \le 3c_0^2 T\left(\sum_{n,m=1}^{\infty}\left|\phi_{nm}\right|^2 + \sum_{n,m=1}^{\infty}\left|\psi_{nm}\right|^2 + \sum_{n,m=1}^{\infty}\left\|f_{nm}\right\|^2_{L_2(-T,0)}\right) =$$

$$= c^2\left(\|\phi\|^2_{L_2(\Omega_0)} + \|\psi\|^2_{L_2(\Omega_0)} + \|f\|^2_{L_2(\Omega)}\right),$$

where $c^2 = 3c_0^2 T$.

Lemma 1 is proved.

Corollary 1. From the estimate (22) the convergence of series (5) in $L_2(\Omega)$ and $u \in L_2(\Omega)$ follows.

Corollary 2. From the estimate (22) continuous dependence of the solution (5) on initial data of the Problem 1 and the function $f(x,y,t)$ follows.

Integrating the integrals (7), (13) and (14) by parts with respect to x and y we have

$$\varphi_{nm} = \frac{1}{\nu_n \mu_m}\varphi_{nm}^{(1,1)}, \quad \phi_{nm}^{(1,1)} = \int_0^p \int_0^q \frac{\partial^2 \phi}{\partial x \partial y}\frac{2}{\sqrt{pq}}\cos\nu_n x \cos\mu_m y dy dx, \tag{24}$$

$$\psi_{nm} = \frac{1}{\nu_n \mu_m}\psi_{nm}^{(1,1)}, \quad \psi_{nm}^{(1,1)} = \int_0^p \int_0^q \frac{\partial^2 \psi}{\partial x \partial y}\frac{2}{\sqrt{pq}}\cos\nu_n x \cos\mu_m y dy dx, \tag{25}$$

$$f_{nm} = \frac{1}{\nu_n \mu_m} f_{nm}^{(1,1,0)}(t), \quad f_{nm}^{(1,1,0)}(t) = \int_0^p \int_0^q \frac{\partial^2 f}{\partial x \partial y}\frac{2}{\sqrt{pq}}\cos\nu_n x \cos\mu_m y dy dx, \tag{26}$$

$$f_{nm}(t) = \frac{1}{\nu_n^2 \mu_m^2} f_{nm}^{(2,2,0)}(t), \quad f_{nm}^{(2,2,0)}(t) = \int_0^p \int_0^q \frac{\partial^2 f}{\partial x \partial y}\upsilon_{nm}(x,y) dy dx, \tag{27}$$

Lemma 2. Let $\varphi \in C^1(\bar{\Omega}_0)$, $\varphi_{xy} \in L_2(\Omega_0)$, $\psi \in C^1(\bar{\Omega}_0)$, $\psi_{xy} \in L_2(\Omega_0)$, $f \in C^2(\bar{\Omega})$, $f_{xxy} \in C(\bar{\Omega})$, $f_{xyy} \in C(\bar{\Omega})$, $f_{xxyy} \in C(\bar{\Omega})$, $\varphi = 0$ on $\partial\Omega_0$, $\psi = 0$ on $\partial\Omega_0$, $f = 0$ on $\partial\Omega \times [-T, 0]$, then for any $0 < \varepsilon < 1$ the following estimates hold

$$|u_{nm}(t)| \le c_1 \left(\frac{|\phi_{nm}|}{\nu_n^k \mu_m^k} + \frac{|\psi_{nm}|}{\nu_n^k \mu_m^k} + \frac{1}{\nu_n^{k+1}\mu_m^{k+1}} + \frac{1}{\nu_n^{k+1-\varepsilon}\mu_m^{k+1}} + \frac{1}{\nu_n^{k+1}\mu_m^{k+1-\varepsilon}} \right), \tag{28}$$

$$\lambda_{nm}^{2k}|u_{nm}(t)| \le c_2 \left(\frac{|\phi_{nm}^{(1,1)}|}{\nu_n \mu_m} + \frac{|\psi_{nm}^{(1,1)}|}{\nu_n \mu_m} + \frac{1}{\nu_n^{k+1}\mu_m^{k+1}} + \frac{1}{\nu_n^2 \mu_m^2} + \frac{1}{\nu_n^{2-\varepsilon}\mu_m^2} + \frac{1}{\nu_n^2 \mu_m^{2-\varepsilon}} \right) \tag{29}$$

Proof. From conditions of Lemma 2, it follows that functions $f_{nm}^{(1,1,0)}(t)$ and $f_{nm}^{(2,2,0)}(t)$ are bounded

$$\left|f_{nm}^{(1,1,0)}(t)\right| \le N_1 \ , \quad \left|f_{nm}^{(2,2,0)}(t)\right| \le N_2 \tag{30}$$

where $N_1 = const > 0, \quad N_2 = const > 0.$

Let $t \in [-T, t_0]$, where $t_0 < 0$ is sufficiently minor number.

Further we have $2\nu_n \mu_m < \lambda_{nm}^{2k}$. For sufficiently large values of n and m the following are true

$$\ln \lambda_{nm}^{\varepsilon} < \lambda_{nm}^{\varepsilon} < \nu_n^{\varepsilon} + \mu_m^{\varepsilon}, \quad 1 + \lambda_{nm}^{2k} T^{\beta} < 2\lambda_{nm}^{2k} T^{\beta}. \tag{31}$$

Owing to (23), (26), (30), (31) from (21) we get

$$|u_{nm}(t)| \le M \left(\frac{1}{2(-t_0)^{\beta}} \frac{|\varphi_{nm}|}{\nu_n^k \mu_m^k} + \frac{1}{2(-t_0)^{\beta}} \frac{|\psi_{nm}|}{\nu_n^k \mu_m^k} + \frac{N_1}{2\beta \nu_n^{k+1} \mu_m^{k+1}} \int_t^0 \frac{d\left(1 + \lambda_{nm}^{2k}(\tau - t)^{\beta}\right)}{1 + \lambda_{nm}^{2k}(\tau - t)^{\beta}} \right) \le$$

$$\le M \left(\frac{|\varphi_{nm}|}{2(-t_0)^{\beta} \nu_n^k \mu_m^k} + \frac{|\psi_{nm}|}{2(-t_0)^{\beta-1} \nu_n^k \mu_m^k} + \frac{N_1 \left(\ln 2T^{\beta} + \frac{2k}{\varepsilon} \ln \lambda_{nm}^{\varepsilon} \right)}{2\beta \nu_n^{k+1} \mu_m^{k+1}} \right) \le$$

$$\le c_1 \left(\frac{|\varphi_{nm}|}{\nu_n^k \mu_m^k} + \frac{|\psi_{nm}|}{2(-t_0)^{\beta-1} \nu_n^k \mu_m^k} + \frac{1}{\nu_n^{k+1} \mu_m^{k+1}} + \frac{1}{\nu_n^{k+1-\varepsilon} \mu_m^{k+1}} + \frac{1}{\nu_n^{k+1} \mu_m^{k+1-\varepsilon}} \right),$$

where $c_1 = \max \left\{ \frac{M}{2(-t)^{\beta}}, \frac{M}{2(-t)^{\beta-1}}, \frac{MN_1 \ln 2T^{\beta}}{2\beta}, \frac{kMN_1}{\beta\varepsilon} \right\}.$

The inequality (28) is proved. Now we prove the inequality (29). Owing to (23), (24), (27), (30) and (31) from (21) we find

$$\lambda_{nm}^{2k}\left|u_{nm}(t)\right| \le M\left(\frac{1}{(-t)^{\beta}}\frac{\left|\varphi_{nm}^{(1,1)}\right|}{\nu_n\mu_m}+\frac{1}{(-t)^{\beta-1}}\frac{\left|\psi_{nm}^{(1,1)}\right|}{\nu_n\mu_m}+\frac{N_2\ln 2T^{\beta}}{\beta\nu_n^2\mu_m^2}+\frac{2kN_2}{\beta\varepsilon\nu_n^{2-\varepsilon}\mu_m^2}+\frac{2kN_2}{\beta\varepsilon\nu_n^2\mu_m^{2-\varepsilon}}\right)$$

$$\le c_2\left(\frac{\left|\varphi_{nm}^{(1,1)}\right|}{\nu_n\mu_m}+\frac{\left|\psi_{nm}^{(1,1)}\right|}{\nu_n\mu_m}+\frac{1}{\nu_n^2\mu_m^2}+\frac{1}{\nu_n^{2-\varepsilon}\mu_m^2}+\frac{1}{\beta\varepsilon\nu_n^2\mu_m^{2-\varepsilon}}\right),$$

where $c_2=\max\left\{\frac{M}{(-t)^{\beta}},\frac{M}{(-t)^{\beta-1}},\frac{MN_2\ln 2T^{\beta}}{\beta},\frac{2kMN_2}{\beta\varepsilon}\right\}.$

Lemma 2 is proved.

Theorem. Let the conditions of Lemma 2 hold, then there exists a unique regular solution of Problem 1 and it continuously depends of functions $\varphi(x,y)$, $\psi(x,y)$ and $f(x,y,t)$.

Proof. We have to prove, the uniformly and absolutely convergence of series (5) and

$$\frac{\partial^{2k}u}{\partial x^{2k}}=\sum_{n,m=1}^{\infty}(-1)^k\,\nu_n^{2k}u_{nm}(t)\upsilon_{nm}(x,y), \tag{32}$$

$$\frac{\partial^{2k}u}{\partial y^{2k}}=\sum_{n,m=1}^{\infty}(-1)^k\,\mu_m^{2k}u_{nm}(t)\upsilon_{nm}(x,y), \tag{33}$$

$$-(-1)^k\,{}_cD_{t0}^{\beta}=\sum_{n,m=1}^{\infty}f_{nm}(t)\upsilon_{nm}(x,y)-\sum_{n,m=1}^{\infty}(-1)^k\,\lambda_{nm}^{2k}u_{nm}(t)\upsilon_{nm}(x,y). \tag{34}$$

The series

$$\frac{2}{\sqrt{pq}}\sum_{n,m=1}^{\infty}\left|u_{nm}(t)\right| \tag{35}$$

is majorant for the series (5). The series (35) uniformly converges owing to (28). Indeed,

$$\sum_{n,m=1}^{\infty}|u_{nm}(t)|\le c_1\sum_{n,m=1}^{\infty}\left(\frac{|\varphi_{nm}|+|\psi_{nm}|}{\nu_n^k\mu_m^k}+\frac{1}{\nu_n^{k+1}\mu_m^{k+1}}+\frac{1}{\nu_n^{k+1-\varepsilon}\mu_m^{k+1}}+\frac{1}{\nu_n^{k+1}\mu_m^{k+1-\varepsilon}}\right).$$

Applying the Cauchy-Schwarz inequality for the sum and Parseval equality to the series $\sum_{n,m=1}^{\infty}\frac{|\varphi_{nm}|}{\nu_n^k\mu_m^k}$ we have

$$\sum_{n,m=1}^{\infty}\frac{|\varphi_{nm}|}{\nu_n^k\mu_m^k}\le\left(\sum_{n,m=1}^{\infty}\frac{1}{\nu_n^{2k}}\cdot\frac{1}{\mu_m^{2k}}\right)^{1/2}\left(\sum_{n,m=1}^{\infty}|\varphi_{nm}|^2\right)^{1/2}=$$

$$=\frac{p^kq^k}{\pi^{2k}}\left(\sum_{n,m=1}^{\infty}\frac{1}{n^{2k}}\cdot\sum_{n,m=1}^{\infty}\frac{1}{m^{2k}}\cdot\right)^{1/2}\|\varphi\|_{L_2(\Omega_0)},$$

analogously

$$\sum_{n,m=1}^{\infty}\frac{|\psi_{nm}|}{\nu_n^k\mu_m^k}\le\frac{p^kq^k}{\pi^{2k}}\left(\sum_{n,m=1}^{\infty}\frac{1}{\nu^{2k}}\cdot\sum_{n,m=1}^{\infty}\frac{1}{m^{2k}}\right)^{1/2}\|\varphi\|_{L_2(\Omega_0)}.$$

The series $\sum_{n,m=1}^{\infty}\frac{1}{n^{2k}}$, $\sum_{n,m=1}^{\infty}\frac{1}{m^{2k}}$, uniformly converge for any $k\ge 1$ according to Cauchy integral sign. Further, since $k+1-\varepsilon>k\ge 1$, then the series

$$\sum_{n,m=1}^{\infty}\frac{1}{\nu_n^{k+1}\mu_m^{k+1}},\quad\sum_{n,m=1}^{\infty}\frac{1}{\nu_n^{k+1-\varepsilon}\mu_m^{k+1}},\quad\sum_{n,m=1}^{\infty}\frac{1}{\nu_n^{k+1}\mu_m^{k+1-\varepsilon}}$$

uniformly converge to the same Cauchy integral sign. Consequently the series (5) uniformly and absolutely converges for any $t_0<0$ in the closed domain $\Omega_{t_0}=\{(x,y,t):0\le x\le p,\,0\le y\le q,\,-T\le t\le t_0\}$. The series (5) converges at $t=0$ to $\varphi(x,y)$. Owing to $\nu_n^{2k}<\lambda_{nm}^{2k}$ and $\mu_m^{2k}<\lambda_{nm}^{2k}$ the series

$$\frac{2}{\sqrt{pq}}\sum_{n,m=1}^{\infty}\lambda_{nm}^{2k}|u_{nm}(t)| \tag{36}$$

is majorant for the series (32), (33) and the second series of (34). The series (36) uniformly converges owing to (29).

The series

$$\frac{2}{\sqrt{pq}}\sum_{n,m=1}^{\infty}\left|f_{nm}(t)\right| \tag{37}$$

is majorant for the first series of (34). The series (37) uniformly converges for any $t\in[-T,0]$. Indeed, using (27) we have

$$\sum_{n,m=1}^{\infty}\left|f_{nm}(t)\right|=\sum_{n,m=1}^{\infty}\frac{1}{\nu_n^2\mu_m^2}\left|f_{nm}^{(2,2,0)}(t)\right|\le N_2\sum_{n,m=1}^{\infty}\frac{1}{\nu_n^2}\cdot\frac{1}{\mu_m^2}=$$
$$=N_2\frac{p^2q^2}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^2}\cdot\sum_{m=1}^{\infty}\frac{1}{m^2}=N_2\frac{p^2q^2}{36}.$$

Consequently, the first series of (34) uniformly and absolutely converges in $\bar{\Omega}$.

Adding (32), (33) and (34) we convince that the solution (5) satisfies the equation (1). Based on properties of functions $\upsilon_{nm}(x,y)$, we conclude that the solution (5) satisfies conditions (2) and (3).

Simple calculations show that

$$\lim_{t\to -0}E_{\beta,1}\left[-\lambda_{nm}^{2k}(-t)^{\beta}\right]=1,\quad \lim_{t\to -0}\frac{d}{dt}E_{\beta,1}\left[-\lambda_{nm}^{2k}(-t)^{\beta}\right]=0,$$

$$\lim_{t\to -0}E_{\beta,2}\left[-\lambda_{nm}^{2k}(-t)^{\beta}\right]=1,\quad \lim_{t\to -0}t\frac{d}{dt}E_{\beta,2}\left[-\lambda_{nm}^{2k}(-t)^{\beta}\right]=0.$$

Hence, $\lim\limits_{t\to -0}u_{nm}(t)=\varphi_{nm}$, $\lim\limits_{t\to -0}u'_{nm}(0)=\psi_{nm}$.

The last equalities show that the solution (5) satisfies the conditions (4). Since Lemma 1 is true for regular solution of the Problem 1, then from (22) we conclude that the regular solution of the Problem 1 continuously depends on functions $\varphi(x,y)$, $\psi(x,y)$ and $f(x,y,t)$.

Theorem is proved.

With similar argument, the following problem can be solved.

Problem 2. Find the solution $u(x, y, t)$ of the equation (1) satisfying the conditions (4) and

$$\frac{\partial^{2m+1}}{\partial x^{2m+1}}u(0, y, t) = \frac{\partial^{2m+1}}{\partial x^{2m+1}}u(p, y, t) = 0, \quad m = 0,1,\ldots,k-1, \quad 0 \le y \le q, \quad -T \le t \le 0,$$

$$\frac{\partial^{2m+1}}{\partial y^{2m+1}}u(x, 0, t) = \frac{\partial^{2m+1}}{\partial x^{2m+1}}u(x, q, t) = 0, \quad m = 0,1,\ldots,k-1, \quad 0 \le x \le p, \quad -T \le t \le 0.$$

5. CONCLUSION

In this paper a boundary value problem for higher order equation with fractional derivative in the sense of Caputo is studied in spatial domain in the case, when the order of fractional derivative belongs to the interval $(1,2)$. Solution is constructed in the form of Fourier series. The existence and uniqueness of the regular solution and its continuously dependence on initial data and the right part of equation are proved.

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