Malaysian Journal of Mathematical Sciences 7(2): 273-282 (2013)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

Comparison between Fourier and Corrected Fourier Series Methods

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ABSTRACT

Fourier series is a method that can solve for many problems especially for solving various differential equations of interest in science and engineering. However, Gibbs oscillations will be occurs when using a truncated Fourier series. Thus, we solve the problems by using the corrected Fourier series. Here, we want to compare the results between the solutions that we get from Fourier series method and Corrected Fourier series method. The comparison between these two methods will be our finding.

Keywords: Fourier series, corrected fourier series, Gibbs oscillation.

1. INTRODUCTION

An ordinary differential equation (ODE) that contains one or more derivatives of an unknown function, which we call $y(x)$ for x is the independent variable. An ODE is said to be of order n if the nth derivative of the unknown function y is the highest derivative of y in the equation. We can write them as

$$F(x, y, y') = 0 \tag{1}$$

A partial differential equation (PDE) is a mathematical equation having partial derivatives with respect to more than one variable. Consider a general form of the second-order linear partial differential equation as

$$L\left(a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} + d(x,y)\frac{\partial u}{\partial x} + e(x,y)\frac{\partial u}{\partial y} + f(x,y)u \right) = g(x,y) \tag{2}$$

Then we often need to specify some supplementary conditions, which called boundary value conditions or initial value conditions to solve the problem. But, by applied method of Galerkin with corrected Fourier series as its basis function, we do not need to often specify any supplementary conditions. The function approximation is needed to eliminate the Gibbs phenomenon that occurs when using a truncated Fourier series or other eigenfunction series at a simple discontinuity.

2. FOURIER AND CORRECTED FOURIER SERIES METHOD

Fourier series decomposes a periodic function into a sum of simple oscillating functions, namely sines and cosines. Fourier series are very important to the engineer and physicist because they allow the solution of ODEs in connection with forced oscillations and the approximation of periodic functions, (Kreyszig (2011)). Meanwhile, the corrected Fourier series is a combination of the uniformly convergent Fourier series and the correction function consists of algebraic polynomials and Heaviside step functions and is required by the aperiodicity at the endpoints (i.e., $f(0) \neq f(x_0)$) and the finite discontinuities in between, (Zhang (2007)).

2.1 Fourier series

To define Fourier series, we introduce a function $f(x)$ as a periodic function. Suppose that $f(x)$ is a given function of period 2π , then

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right) \tag{3}$$

where the Fourier coefficients of $f(x)$ given by the Euler formulas

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx\,, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx\,,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx$$

2.2 Corrected Fourier series

Suppose $y(x)$ is any mth quasi-smooth function, and has possible discontinuities at x_j $(j = 1,2,\ldots,J)$. The theorem of the corrected Fourier series in (Zhang (2007)) states that a mth quasi-smooth function can be approximated uniformly by a corrected Fourier series consisting of three parts, which are an mth uniformly convergent Fourier series, a no-more-than $(m+1)$ th-order polynomial, and an mth integral of the Heaviside step functions at the discontinuities. Therefore, $y(x)$ is approximated by the following corrected Fourier series,

$$y(x) = \sum_{|n|<\infty} A_n e^{i\alpha_n x} + \sum_{l=1}^{m+1} a_l \frac{x^l}{l!} + \sum_j b_j \frac{(x - x_j)^m}{m!} H(x - x_j),$$

$$\alpha_n = \frac{2n\pi}{x_0} \tag{4}$$

where A_n is the Fourier projection of $y(x)$ to the basis function $e^{i\alpha_n x}$, that is

$$A_n = \frac{1}{x_0}\int_0^{x_0} y(x)e^{-i\alpha_n x}dx \tag{5}$$

in the interval $[0, x_0]$.

3. CORRECTED FOURIER SERIES METHOD FOR SOLVING TWO UNKNOWNS PROBLEM

At first, we consider the general form of the second order linear PDEs in the region $[0, x_0]\times[0, t_0]$,

$$L\{u(x,t)\} \equiv \begin{pmatrix} p_1(x,t)\dfrac{\partial^2 u(x,t)}{\partial x^2} + p_2(x,t)\dfrac{\partial^2 u(x,t)}{\partial x \partial t} + p_3(x,t)\dfrac{\partial^2 u(x,t)}{\partial t^2} + \\ p_4(x,t)\dfrac{\partial u(x,t)}{\partial x} + p_5(x,t)\dfrac{\partial u(x,t)}{\partial t} + p_6(x,t)u(x,t) \end{pmatrix} = f(x,t) \tag{6}$$

where $p_l(x,t)$ with $l = 1,2,\ldots,6$ and $f(x,t)$ are quasi-smooth functions with two unknowns having the discontinuities at $x_j(j = 1,2,\ldots,J)$ and $x_{j_0}(j_0 = 1,2,\ldots,J_0)$.

Definition 1: *Any function,* $\phi(x,t)$ *on the basis function* $e^{i(\alpha_n x + \beta_m t)}$ *in the region* $[0,x_0]\times[0,t_0]$, *generally written as*

$$F_0\left\langle \phi(x,t)\right\rangle_{nm} = \frac{1}{x_0 t_0}\int_0^{t_0}\int_0^{x_0} \phi(x,t)e^{-i(\alpha_n x+\beta_m t)}dxdt \tag{7}$$

is called the Fourier projection where $\alpha_n = \dfrac{2n\pi}{x_0}$ *and* $\beta_m = \dfrac{2m\pi}{t_0}$.

Lemma 1. *Assume* $\tilde{u}(x,t)$ *is the solution of equation (6). The necessary and sufficient conditions for equation (3) with* $\tilde{u}(x,t)$ *to be equivalent to its Fourier projections*

$$F^{-1}\left\langle L\{u(x,t)\}\right\rangle_{nm} = F^{-1}\left\langle f(x,t)\right\rangle_{nm} \tag{8}$$

that is, $\tilde{u}(x,t)$ *satisfies the consistency conditions of the endpoints and discontinuities* $L\{\tilde{u}(x,t)\}\,|_0^{x_0}\,|_0^{t_0} = f(x,t)\,|_0^{x_0}\,|_0^{t_0}$

and $L\{\tilde{u}(x,t)\}\Big|_{x_j-0}^{x_j+0}\,\Big|_{t_{j_0}-0}^{t_{j_0}+0} = f(x,t)\Big|_{x_j-0}^{x_j+0}\,\Big|_{t_{j_0}-0}^{t_{j_0}+0}$ *with* $j = 1,2,\ldots,J$, $j_0 = 1,2,\ldots,J_0$ *where* x_j *and* t_{j_0} *denote the finite discontinuities in either* $p_l(x,t)$ *and* $f(x,t)$ *and*

$$f(x,t)\Big|_0^{x_0}\Big|_0^{t_0} \equiv f(x_0,t) - f(0,t)\Big|_0^{t_0}$$
$$\equiv (f(x_0,t_0) - f(0,t_0)) - (f(x_0,0) - f(0,0)) \quad (9)$$
$$\equiv f(x_0,t_0) + f(0,0) - f(0,t_0) - f(x_0,0).$$

Satisfying the consistency conditions $L\{\tilde{u}(x,t)\} - f(x,t)$ *is a periodic, quasi-smooth continuous function whose Fourier series is uniformly convergent without Gibbs oscillation. This means that* $L\{\tilde{u}(x,t)\} - f(x,t)$ *is equivalent to its Fourier series. Thus Equation (6)≡Equation (8).*
In order $u(x,t)$ to be a solution of equation (6):

i. $u(x,t)$ and its first derivative, $\frac{\partial u(x,t)}{\partial x}$ and $\frac{\partial u(x,t)}{\partial t}$ must be continuous, and

ii. The second derivative of $u(x,t)$, $\frac{\partial^2 u(x,t)}{\partial x^2}$, $\frac{\partial^2 u(x,t)}{\partial x \partial t}$ and $\frac{\partial^2 u(x,t)}{\partial t^2}$ cannot have any discontinuities other than x_j and t_{j_0}.

Therefore, $u(x,t)$ must be second quasi-smooth function with two variables (x and t) and has possible discontinuities at $x_j (j = 1, 2, \ldots, J_0)$ and $j_0 (j_0 = 1, 2, \ldots, J_0)$.

3.1 Derivative of the corrected Fourier series

According to (Zhang (2007)), Theorem 2.2 state that any m -th quasi-smooth continuous function can be approximated uniformly by the sum of an m -th uniformly convergent Fourier series and a polynomial no more than $(m+1)$ -th order. This theorem has been proof for one unknown. Now, we extend this theorem to the case with two unknowns.

Theorem 1

Any m *-th* 2^{nd} *quasi-smooth continuous function* $u_m(x,t) \in S_m([0,x],[0,t_0])$ *can be approximated uniformly by the sum of an* m *-th uniformly convergent Fourier series and a polynomial no more than* $(m+1)$ *-th order*

$$u(x,t)=\sum_{|n|<\infty}\sum_{|m|<\infty}A_{nm}e^{i(\alpha_n x+\beta_m t)}+\sum_{|m|<\infty}\left(a_{1m}x+a_{2m}\frac{x^2}{2!}+a_{3m}\frac{x^3}{3!}\right)e^{i\beta_m t}$$
$$+\sum_{|n|<\infty}\left(b_{1n}t+b_{2n}\frac{t^2}{2!}+b_{3n}\frac{t^3}{3!}\right)e^{i\alpha_n x}+\sum_{l=1}^{3}\sum_{l_0=1}^{3}d_{ll_0}\frac{x^l}{l!}\frac{t^{l_0}}{l_0!} \tag{10}$$

Proof

At first, the function $u(x,t)$ is a second quasi-smooth continuous function with respect to x. Therefore, for any $t\in[0,t_0]$,

$$u(x,t)=\sum_{|n|<\infty}A_n(t)e^{i\alpha_n x}+a_1(t)x+a_2(t)\frac{x^2}{2!}+a_3(t)\frac{x^3}{3!} \tag{11}$$

In the above equation, $A_n(t)$ and $a_l(t)$ are second quasi-smooth continuous function with respect to t. In equation (11), $A_n(t)$ and $a_l(t)$ where $(l=1,2,3)$ can be further expanded into the following

$$A_n(t)=\sum_{|m|<\infty}A_{nm}e^{i\beta_m t}+b_{1n}t+b_{2n}\frac{t^2}{2!}+b_{3n}\frac{t^3}{3!},$$
$$a_l(t)=\sum_{|m|<\infty}a_{lm}e^{i\beta_m t}+d_{l1}t+d_{l2}\frac{t^2}{2!}+d_{l3}\frac{t^3}{3!}\ ,\ (l=1,2,3); \tag{12}$$

where $\alpha_n=\dfrac{2n\pi}{x_0}$ and $\beta_m=\dfrac{2m\pi}{t_0}$.

Then, by substituting $A_n(t)$ and $a_l(t)$ into equation (8), we have

$$u(x,t)=\sum_{|n|<\infty}\sum_{|m|<\infty}A_{nm}e^{i(\alpha_n x+\beta_m t)}+\sum_{|m|<\infty}\left(a_{1m}x+a_{2m}\frac{x^2}{2!}+a_{3m}\frac{x^3}{3!}\right)e^{i\beta_m t}$$
$$+\sum_{|n|<\infty}\left(b_{1n}t+b_{2n}\frac{t^2}{2!}+b_{3n}\frac{t^3}{3!}\right)e^{i\alpha_n x}+\sum_{l=1}^{3}\sum_{l_0=1}^{3}d_{ll_0}\frac{x^l}{l!}\frac{t^{l_0}}{l_0!}. \tag{13}$$

As in the any other Galerkin methods, the corrected Fourier series will be truncated so that $|n|\le N$ and $|m|\le M$ here after. In Equation (13), nine unknowns d_{ll_0} where $(l, l_0 = 1,2,3)$ are obtained by solving the following linear equations

$$\sum_{l=1}^{3}\sum_{l_0=1}^{3} d_{ll_0} \cdot \frac{x^{l-j}}{(l-j)!} \cdot \frac{t^{l_0-j_0}}{(l_0-j_0)!} H(l-l_0, j-j_0) \Bigg|_0^{x_0} \Bigg|_o^{t_0} = \frac{\partial^{j+j_0} u(x,t)}{\partial x^j \partial t^{j_0}} \Bigg|_0^{x_0} \Bigg|_0^{t_0} \qquad (14)$$

where $(j, j_0 = 0,1,2)$ and it depends on the boundary values of $u(x,t)$ and it first and second derivatives only. We can say here that the first three terms on the right-hand side of the equation (13) are identically zero caused by the periodicity of either $e^{i\alpha_n x}$ or $e^{i\beta_m t}$. Then we arrange the nine unknowns d_{ll_0} into a vector ordered as $(d_{11}, d_{12}, d_{13}, d_{21}, d_{22}, d_{23}, d_{31}, d_{32}, d_{33})$ and the equations ordered as when $j = 0,$ then $j_0 = 0,1,2,$ when $j = 1,$ then $j_0 = 0,1,2,$ and $j = 2,$ $j_0 = 0,1,2,$ so that the coefficient matrix of the linear equations is up-triangular and can be easily inverted.

3.2 Computation of the coefficients

Note that when $a_{lm}, b_{l_0 n}$ and d_{ll_0} are suitable chosen, then Gibbs oscillations which often trouble the regular Fourier series method at endpoints and discontinuities are eliminated. The coefficients A_{nm} are readily obtained by the following Fourier projection:

$$A_{nm} = F_0 \left\langle u(x,t) \right\rangle_{nm} - \begin{pmatrix} \displaystyle\sum_{l=1}^{3} a_{lm} \cdot F_1 \left\langle \frac{x^l}{l!} \right\rangle_n + \sum_{l_0=1}^{3} b_{l_0 n} \cdot F_2 \left\langle \frac{t^{l_0}}{l_0!} \right\rangle_m \\ \displaystyle + \sum_{l=1}^{3}\sum_{l_0=1}^{3} d_{ll_0} \cdot F_1 \left\langle \frac{x^l}{l!} \right\rangle_n \cdot F_2 \left\langle \frac{t^{l_0}}{l_0!} \right\rangle_m \end{pmatrix}. \qquad (15)$$

Being looked at in the another way, with coefficients $a_{lm}(l = 1,2,3),$ $a_{lm}(l = 1,2,3)$, $b_{l_0 n}(l_0 = 1,2,3)$ and $d_{ll_0}(l, l_0 = 1,2,3)$ yet to be determined, Equation (13) represents all possible corrected Fourier series solutions of Equation (3), which are uniformly convergent until their second derivatives. Now we have $2N+1$ and $2M+1$ orthogonal conditions by applying Fourier projection Equation (7) to Equation (3) ($2N+1$ and $2M+1$) times, and $J+1, J_0+1$ consistency conditions (Lemma), (Zhang (2005)). Then, $u(x,t)$ is formally expressed as follows

$$u(x,t) = c_1 u_1(x,t) + c_2 u_2(x,t) + u_0(x,t) \quad (16)$$

where c_1 and c_2 are two constants. Viewing $u_1(x,t)$ and $u_2(x,t)$ as two linearly independent solutions and $u_0(x,t)$ as the specific solution, Equation (16) is a general solutions of Equation (3). No boundary conditions are explicitly introduced when Equation (16) is obtained. It is worthwhile noting that $u_1(x,t)$, $u_2(x,t)$ and $u_0(x,t)$ are Galerkin approximated solutions by using the corrected Fourier series, (Zhang (2007)).

4. NUMERICAL PROCEDURE

We solve the problem by using both method, Fourier and corrected Fourier series method. We show the difference between these two methods by graph. In problem 1, we consider the function in one variable. In problem 2, we are solving the PDEs problem which is heat equation.

Problem 1

Find the Fourier series of the function $f(x) = x^2$.

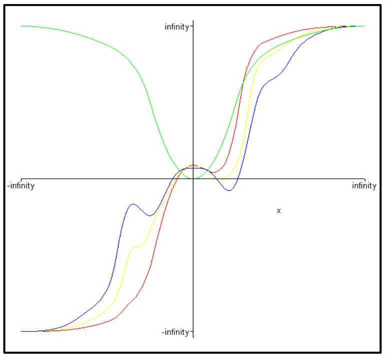

Figure 1

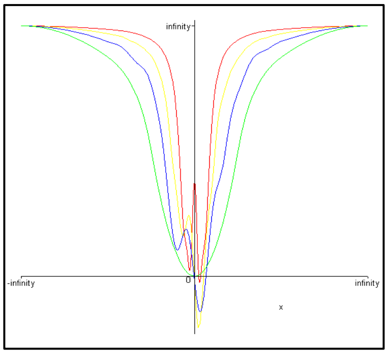

Figure 2

Figure 1 shows the solution of problem 1 by using Fourier series while Figure 2 shows the solution by using Corrected Fourier series. The green color refers to the function of $f(x) = x^2$, blue color refers to the function series after we truncated the series into 3 terms, yellow refers to 5 terms of the series and red color refers to 10 terms of the series.

Problem 2 (PDEs Heat Problem)

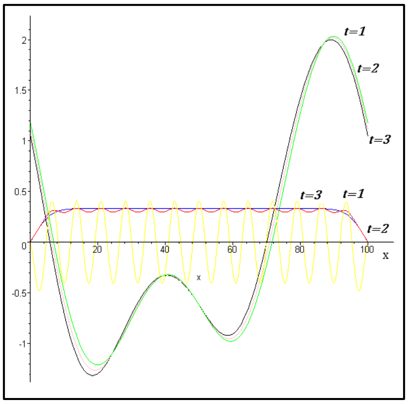

Figure 3

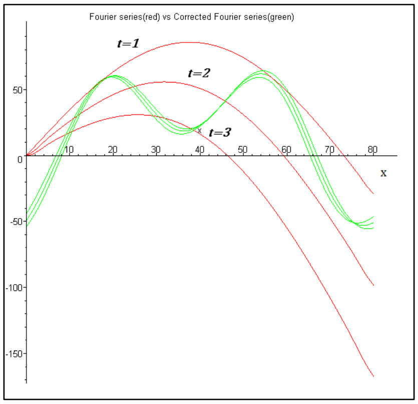

Figure 4

The Figure 3 shows the solution of $u(x,t)$ for $f(x)=0$, $c^2=4\text{cm}^2/\text{sec}$, $x=100$ cm and several values of t. The Figure 4 shows the solution of $u(x,t)$ for $f(x)=100\sin\dfrac{\pi x}{80}$, $c^2=1.158\text{cm}^2/\text{sec}$, 80 cm and several values of t.

We can see here that, by using Fourier series, the difference between graphs with difference value of t is very big rather than by using corrected Fourier.

Problem 3

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=t^2 \text{ in region } [0,80]\times[0,10]$$

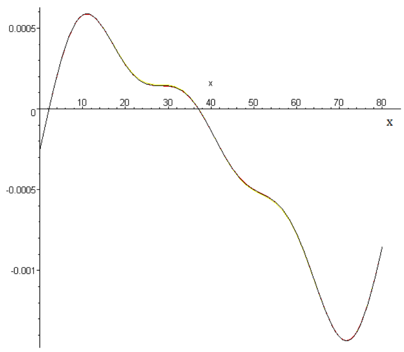

Figure 5

In the Figure 5, we set the value of t equals to 5.

By using corrected Fourier series, and we truncated the series for $n=1, m=1$, $n=2, m=2$ and $n=3, m=3$. From the graph, we can see that there is no difference for the solution. Thus, no Gibbs oscillations appear for this solution.

5. CONCLUSIONS

The corrected Fourier series (CFS) is free of the Gibbs phenomenon, although the quasi-smooth function can be aperiodic and have discontinuities in general. CFS are used to solve the problem that have non-singular coefficients when their exact solution do not always exist. We have such solutions that are being uniformly convergent until its *m*-th derivative in the entire region of the equations by using CFS.

The solutions of the problem that solve by using corrected Fourier series are depends on the value of $\tilde{u}(x,t)$ that we assume in the beginning of the calculations. So, the difference value of $\tilde{u}(x,t)$ will give the difference final solution for each problem.

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